The Lower Infinite Non-Elementary Classes 2007

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Two Directions

1. Eventual Behavior
2. The Lower Infinite'

Acknowledgements

I will interpret or misinterpret the works of many people with only vague and non-uniform specific acknowledgments.

The end or the beginning?

Shelah's conjecture

There is a κ such that if an AEC is categorical in one cardinal greater than κ then is categorical in all cardinals greater than κ.

Shelah's new result

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Putting this in context

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2. Stability theory developed
   1. abstractly with the stability classification
   2. concretely by finding the stability class of important mathematical theories and using the techniques of the abstract theory.

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2. Superstability, stability spectrum, and applications are open.

More History

In the late 60's model theory and set theory seemed inextricably intertwined.
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Tameness, etc. may play a similar role now.
Or the large cardinals may just be a stalking horse.

Background Trivia

If $(\mathbf{K}_{\leq\kappa}, \prec_{\mathbf{K}})$ is an AEC (but with the union axioms restricted to 'short' unions), there is a unique maximal AEC that restricts to $(\mathbf{K}_{\leq\kappa}, \prec_{\mathbf{K}})$.

Close under arbitrary unions.

The Lesson of Frames

Shelah shows:

If there is a sufficiently strong 'stability theory' on $\mathbf{K}_\kappa$ then the generated AEC is 'controlled' in a strong way by $\mathbf{K}_\kappa$.

He deduces the existence of such classes from categoricity. Can we find them 'in nature'?

Three approaches to eventual categoricity

1. Work from the bottom up. Existence of models must be earned. 87a, 87b, Zilber (WGCH) Beginning at ω is crucial.
2. Begin above the Hanf number for existence. Ehrehfeucht-Mostowski models are a powerful tool.
    1. $(ap)^+$ -394,
    2. (with tameness Grossberg-VanDieren and Lessmann)
3. the frame approach: combine the two.

Ehrenfeucht-Mostowski models

Assume categoricity:

1. needed to deduce stability (pace Keisler)
2. needed to deduce superstability ($\kappa(T)$-form)

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Where the non-uniformity happens.

What is mathematics?

Harvey Friedman:
'At the outer limits normal mathematics is conducted within complete separable metric spaces'

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We will look at examples and defer the definition of 'mathematics'.

The categoricity spectrum

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There is more for $L_{\omega_1,\omega}$.

The stability spectrum

For a countable T there are six functions $f_i(\kappa)$ which give all the possible stability spectra.

This might appear 'combinatorial' but stable (i.e. cofinally stable) implies the existence of a well-behaved dependence relation.

The saturation spectrum

The same functions compute the cardinals in which a first order theory has a saturated model.

This can be viewed a property of the group of automorphisms of a monster model.

Problems in the lower infinite

1. For eventually categorical classes, what happens below.
2. What exactly is the Hanf number for categoricity?
3. Are there Hanf numbers for amalgamation, tameness, compactness ...??
4. Use the technology of finitary classes to investigate e.g. Vaught's conjecture.
5. Stability spectrum of homogeneous classes is known above the Hanf number. What about below?

The Weak Generalized Continuum Hypothesis

Setting

ZFC is the base theory throughout.

Axiom: WGCH

For every cardinal λ, $2^\lambda < 2^{\lambda^+}$.

The Continuum Function

$$f(\kappa) = 2^\kappa$$

By Cantor and König:

$$2^\kappa > \kappa$$

$$\mathrm{cf}(2^\kappa) > \kappa$$

Justifying Axioms

Argument

1. WGCH is consistent. (First, do no harm)
2. WGCH is ‘natural’.
3. WGCH has important consequences.

Analogies

The Axiom of Choice:
The Axiom of Foundation

Combinatorial Content

The combinatorial content of WGCH is the Devlin-Shelah weak diamond.

'Mathematical' Content

ZFC: Shelah 1983

If **K** is an excellent $EC(T, Atomic)$-class then if it is categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.

WGCH: Shelah 1983

If an $EC(T, Atomic)$-class **K** is categorical in $\aleph_n$ for all $n < \omega$, then it is excellent.

More restrictive classes

Properties

Tameness, finitary, amalgamation, Galois-compactness

NEED EXAMPLES

In particular, examples connected to core mathematics:

1. Banach Spaces
2. complex exponentiation and semi-abelian varieties
3. Modules and Abelian groups
4. locally finite groups

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}(Q)$-sentence discovered by the Hrushovski construction.

Done

A. Expand $(\mathcal{C}, +, \cdot)$ by a unary function which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}(Q)$-sentence Σ is categorical and has quantifier elimination.

Very open

B. Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

Questions

1. B is a problem in analysis and number theory
2. If a sentence ψ of $L_{\omega_1,\omega}(Q)$ is categorical up to $\aleph_\omega$, must it be forever categorical?

UNIVERSAL COVERS

When is the exact sequence:

$$0 \to Z \to V \to A \to 0. \tag{1}$$

categorical where V is a $\mathbb{Q}$ vector space and A is a semi-abelian variety.
Can be viewed as an expansion of V and there is a combinatorial geometry given by:

$$\mathrm{cl}(X) = \ln(\mathrm{acl}(\exp(X)))$$

Contexts

Trivial case. A is $\aleph_1$-categorical in the Abelian group language– e.g. $(Q, \cdot)$
Interesting case. Work with 'field language' on A.
Really interesting case: pseudo-exponentiation and Hrushovski construction.

Semi-abelian varieties

In the Ravello volume Zilber obtains ‘arithmetical conditions’ on algebraic variety that are equivalent to ‘excellence’
Thus he is able to conclude - using Shelah’s machinery that under WGCH,
Categoricity up to $\aleph_\omega$ implies ‘arithmetical’ properties of certain algebraic varieties.
This argument could be shortened if there were a direct algebraic argument for tameness.

Tameness

For countable theories $(\aleph_0, \infty)$-tameness is incredibly powerful. It allows the categoricity transfer from a single small cardinal (Grossberg, VanDieren, Lessmann).

1. Find other sufficient conditions for tameness.
2. What is the relation between tameness and excellence

Finitary Classes

1. an independence theory over sets
2. an analysis for arbitrary sentences in $L_{\omega,\omega}$ - Vaught's conjecture.
3. warning: 'uniqueness of monster model'
4. simplicity?
5. the Hart-Shelah example; other examples of Hytinnen.

Superstability

1. Shelah
2. Grossberg-VanDieren-Villaveces
3. JTB
4. Hytinnen-Kesälä

More Topics

1. Compute the possible stability spectra of AEC. (Even assuming tameness).
2. Presentation theorems: Can these be viewed as Lindstrom theorems?
    1. Shelah
    2. Kirby
    3. Kueker