Inequalities and quadratics

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Class outline

And versus Or

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When is A and B true?
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inclusive or.

Solutions of Absolute Value Inequalities

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$|x| > a$ means
$x > a$ OR $x < -a$.

Homework Problem

$$|3 - 4x| > 8$$

Everyone could draw the solution on the number line. What is the correct way to describe the solution algebraically?

Graphical Methods: Example

Example of the split point method: To solve $|3x - 2| < 7$ by the split point method, graph the two functions $y = |3x - 2|$ and $y = 7$. The solution is the set of numbers on the real line such the value of the first function is less than the value of the second.

The real axis is divided into a finite number of intervals by those x where the lines cross. These are called split points. (In this example the intervals are $(\infty, -\frac{5}{3})$, $(-\frac{5}{3}, 3)$, and $(3, \infty)$. The middle one is where the inequality holds.)

Graphical Methods: General case

Let f and g be polynomials.
To solve: $f(x) < g(x)$.
Graph the two functions. They will cross at finitely many points a_i where $f(a_i) = g(a_i)$.
These are the split points. The solutions are the intervals determined by these split points where the inequality holds.

Two kinds of problems

1. inequalities in one variable. The solution is a union of intervals in the real line – a set of numbers.
2. inequalities in two variable. The solution is a set of points in the plane. The solution will be a shaded set of point in the plane.

Homework reprise

$$\begin{aligned} y &< 3x + 6 \\ -3x &> y - 6 \\ y &< x^2 \end{aligned}$$

Shade the region in the plane that satisfies these inequalities. What are the exact boundary points of the region?

Homework reprise: continued

We must solve the two quadratic equations:

$$\begin{aligned} x^2 &= 3x + 6 \\ x^2 &= -3x + 6 \end{aligned}$$

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What is the difference between an equation and a line.

$y = 2x + 4$
is an equation.
The set of points

$$\{(a, 2a + 4) : a \in \Re\}$$

is a line.
$3y = 6x + 12$ is another equation satisfied by the same line.

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The diagonal of a rectangle measures $\sqrt{205}$ inches.
What are the possible dimensions of the rectangle?

What if I demand that the length and width of the rectangle are integers?
How many solutions are there?

Specific question

The diagonal of a rectangle measures $\sqrt{205}$ inches. The area of the rectangle is 42 square inches.
What are the length and width of the rectangle?

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3. combining: The points $(a, 0)$ from 2.

Methods of Solving Quadratic equations

1. factoring (for contrived problems)
2. completing the square (general method)
3. the quadratic formula (plug and chug)

Clock arithmetic

Definition. For a, b with $0 \leq a, b < 12$ define $a \diamondsuit b$ is the remainder when $a \times b$ is divided by 12.

Examples:

$2 \diamondsuit 3 = ?$

$3 \diamondsuit 5 = ?$

$10 \diamondsuit 11 = ?$

$6 \diamondsuit 4 = ?$

Zero Product Property

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If the product of two (real) numbers is 0, one of them must be 0.
On page 603, it says if and only if. Why?

Factoring

Write out carefully a solution to the equation

$$x^2 + 3x + 2 = 0.$$

Justify your steps.

Solution

If $x^2 + 3x + 2 = 0$ then by the distributive law $(x + 1)(x + 2) = 0$.

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(In fact, each step is reversible so these are the solutions.)

Literal Solutions

Theorem (CME 4.6, page 377)

Given the system

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

the unique solution is

$$(x, y) = \left(\frac{de - b}{ad - bc}, \frac{a - ce}{ad - bc} \right)$$

$ad - bc \neq 0$. *(If $ad - bc = 0$ the two lines are parallel.)*

Work on proving this.

Proof of formula

Multiply first equation by c and second equation by a to get:

$$
\begin{aligned}
cax + cby &= ce \\
acx + ady &= af
\end{aligned}
$$

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Subtract the first equation from the second.

$$(ad - bc)y = af - ce$$

Dividing by $(ad - bc)$, we get the value for y.
Homework: Continue to compute the value for x.

The quadratic formula I

What are the solutions of the quadratic equation

$$ax^2 + bx + c = 0?$$

The quadratic formula II

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$
$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$$
$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$
$$(x + \frac{b}{2a}) = \pm\sqrt(\frac{b^2 - 4ac}{4a^2})$$
$$(x + \frac{b}{2a}) = \frac{\pm\sqrt(b^2 - 4ac)}{2a}$$

$$(x = \frac{-b \pm \sqrt(b^2 - 4ac)}{2a}$$