Perspectives on AEC

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Two Directions in Model Theory

1. Eventual Behavior
2. 'The Lower Infinite'

Two Directions after Morley

1. Classify the countable models of $\aleph_1$-categorical theory. (Baldwin-Lachlan, Zilber, geometric stability theory)

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2. Stability theory developed
   1. abstractly with the stability classification
   2. concretely by finding the stability class of important mathematical theories and using the techniques of the abstract theory.

The absoluteness of fundamental notions such as $\aleph_1$-categoricity and stability liberated first order model theory from set theory.

Two Directions for AEC

AEC???

Kueker's work on definability in infinitary logic.

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AEC???
Kueker's work on definability in infinitary logic.

1. There is much more to do below the Hanf number for categoricity if such exists. (the lower infinite)
2. Superstability, stability spectrum, and applications are open.

Acknowledgements

Thanks to those who organized the meeting: Dave, Alice, Fred's. And to those who came.

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I will interpret or misinterpret the works of many people with only vague and non-uniform specific acknowledgments.

Working with Shelah

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Rule 1

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Rule 2

What is the main idea of the proof?

Hanf's Argument

Theorem

If there is a set of classes of given kind (e.g. aec's) of a given similarity type then for any property $P(\mathbf{K}, \lambda)$ there is a cardinal κ such that if $P(\mathbf{K}, \lambda)$ holds for some $\lambda > \kappa$ then $P(\mathbf{K}, \lambda)$ holds for arbitrarily large λ.

Proof:

$$\mu_{\mathbf{K}} = \sup\{\lambda : P(\mathbf{K}, \lambda) \text{holds if there is such a max}\}$$

$$\kappa = \sup \mu_{\mathbf{K}}$$

as $\mathbf{K}$ ranges over the set of all classes of the given type.

AEC modification

Since there are a proper class of sentences in $L_{\infty,\omega}$, there are a proper class of aec with a given similarity type.

Notation

For any aec **K***, let* $\kappa_{\mathbf{K}} = \sup(|\tau_{\mathbf{K}}|, \mathrm{LS}(\mathbf{K}))$.

Remark

For any cardinal κ, there are only a set aec **K** with $\kappa_{\mathbf{K}} = \kappa$.

Proof sketch: We know (next slide) that the aec is determined by its restriction to models in the Lówenheim number. There are only a set of such restrictions.

Background Trivia

If $(\mathbf{K}_{\leq\kappa}, \prec_{\mathbf{K}})$ is an AEC (but with the union axioms restricted to 'short' unions), there is a unique AEC that restricts to $(\mathbf{K}_{\leq\kappa}, \prec_{\mathbf{K}})$.

Close under arbitrary unions.

AP yields a strong HN for categoricity

Definition

P is downward closed if there is a κ_0 such that if $P(\mathbf{K}, \lambda)$ holds with $\lambda > \kappa_0$, then $P(\mathbf{K}, \mu)$ holds if $\kappa_0 < \mu \leq \lambda$.

Theorem (Shelah 394)

The property $\mathbf{K}$ *is* λ*-categorical is downward closed from successor cardinals for aec with amalgamation and joint embedding.*

So Shelah proved but didn't say:

Corollary

For aec with amalgamation and joint embedding, there is a cardinal $\lambda = H(|\kappa_{\mathbf{K}})|$ *such that if an aec* $\mathbf{K}$ *is categorical in some successor cardinal greater than* λ *then it is categorical in all larger cardinals.*

So what

This falls short of Shelah's categoricity conjecture in two ways:

1. amalgamation and joint embedding are assumed
2. categoricity is only transferred downward from successor.
3. There is no computation of the Hanf number.

Grossberg's question

Hanf's argument also reduces Grossberg's question about the existence of a Hanf number for amalgamation to:

Question

Is amalgamation closed down?

More History

In the late 60's model theory and set theory seemed inextricably intertwined.

And results like Chang's two cardinal theorem seemed to imply that model and axiomatic set theory were inextricably intertwined.

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More History

In the late 60's model theory and set theory seemed inextricably intertwined.

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But the stability classification allowed the study of specific classes of theories where notions became absolute and theorems provable in ZFC.
Tameness, excellence, etc. may play a similar role now.

The goal of Frames

Shelah wants:

If there is a sufficiently strong 'stability theory' on $\mathbf{K}_\kappa$
then the generated AEC is 'controlled' in a strong way by $\mathbf{K}_\kappa$.

He proposes to deduce the existence of such classes from categoricity.
Can we find them 'in nature'?

Three approaches to eventual categoricity

1. Work from the bottom up. Existence of models must be earned. 87a, 87b, Zilber (WGCH) Beginning at ω is crucial.
2. Begin above the Hanf number for existence. Ehrehfeucht-Mostowski models are a powerful tool.
   1. $(ap)^+$ -394,
   2. (with tameness Grossberg-VanDieren and Lessmann)
3. the frame approach: combine the two.

The stability spectrum

For a countable T there are six functions $f_i(\kappa)$ which give all the possible stability spectra.

This might appear 'combinatorial' but stable (i.e. cofinally stable) implies the existence of a well-behaved dependence relation.

The saturation spectrum

The same functions compute the cardinals in which a first order theory has a saturated model.

This can be viewed a property of the group of automorphisms of a monster model.

Some Problems

1. For eventually categorical classes, what happens below.
2. What exactly is the Hanf number for categoricity?
3. Is there one, without assuming ap?
4. Are there Hanf numbers for amalgamation, tameness, compactness . . . ??
5. Use the technology of finitary classes to investigate e.g. Vaught's conjecture.

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What model theoretic of infinitary logic properties are absolute?

The Weak Generalized Continuum Hypothesis

Setting

ZFC is the base theory throughout.

Axiom WGCH: Weak GCH

For every cardinal λ, $2^\lambda < 2^{\lambda^+}$.

Axiom VWGCH: Very Weak GCH

For every cardinal λ with $\lambda < \aleph_\omega$, $2^\lambda < 2^{\lambda^+}$.

Crucial Fact

Weak diamond is the operative form of WGCH.

$2^\lambda < 2^{\lambda^+}$ if and only if Weak-$\diamond$ on λ^+

Model Theoretic Context

For the next section, **K** is the class of models of a sentence ψ in $L_{\omega_1,\omega}$.

We write $M \prec_{\mathbf{K}} N$ where $\prec_{\mathbf{K}}$ is elementary submodel in the smallest fragment L^* containing ψ.

We will sketch how to study this situation as the class of atomic models of a first order theory.

More Background

A model M is *small* if it realizes only countably many $L_{\omega_1,\omega}$-types over the empty set.

M is small if and only M is Karp-equivalent to a countable model.

ϕ is complete for $L_{\omega_1,\omega}$ if for every sentence ψ of $L_{\omega_1,\omega}$, either $\phi \rightarrow \psi$ or $\phi \rightarrow \neg\psi$.

Note that a sentence is complete if and only if it is a Scott sentence; so every model of a complete sentence is small.

Passing to Atomic

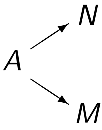
A model M is atomic if every finite sequence from M realizes a principal type over $\emptyset$.

Theorem

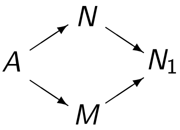
Let ψ be a complete sentence in $L_{\omega_1,\omega}$ in a countable vocabulary τ. Then there is a countable vocabulary τ' extending τ and a complete first order τ'-theory T such that reduct is a 1-1 map from the *atomic* models of T onto the models of ψ.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:



there exists an N_1 such that



Failure of amalgamation yields many models

Theorem (WGCH: Shelah)

If $\mathbf{K}$ is λ-categorical and amalgamation fails in λ there are 2^{λ^+} models in $\mathbf{K}$ of cardinality λ^+.

Upward Löwenheim Skolem

Definition

$\aleph_\alpha$ is characterized by ϕ_α if there is a model of ϕ_α with cardinality $\aleph_\alpha$ but no larger model.

Known

Morley: If ϕ has a model of cardinality at least $\beth_{\omega_1}$, ϕ has arbitrarily large models.

Hjorth: If α is countable $\aleph_\alpha$ is characterizable.

Conjecture

Shelah: If κ is characterized by ϕ, ϕ has 2^λ models in some $\lambda \leq \kappa$.

Few models and smallness

Theorem (Keisler)

If **K** has less than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then every model of **K** realizes only countably many types over the empty set in the countable fragment L^*.

Few models in $\aleph_1$ implies completeness

Theorem (Shelah)

If the $L_{\omega_1,\omega}$-τ-sentence ψ has a model of cardinality $\aleph_1$ which is L^*-small for every countable τ-fragment L^* of $L_{\omega_1,\omega}$, then ψ has a small model of cardinality $\aleph_1$.

κ-Categoricity implies completeness ????

Thus, any $\aleph_1$-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory. ($EC(T, Atomic)$-class) But this result uses properties of $\aleph_1$ heavily.

Question

If the $L_{\omega_1,\omega}$-sentence ψ is κ-categorical, must the model in cardinality κ be small?

Categoricity Transfer in $L_{\omega_1,\omega}$

An atomic class **K** is excellent if it is ω-stable and satisfies certain amalgamation properties for finite systems of models.

ZFC: Shelah 1983

If **K** is an excellent $EC(T, Atomic)$-class then if it is categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.

VWGCH: Shelah 1983

If an $EC(T, Atomic)$-class **K** is categorical in $\aleph_n$ for all $n < \omega$, then it is excellent.

Excellence gained: more precisley

VWGCH: Shelah 1983

An atomic class **K** that has at least one uncountable model and with $I(\mathbf{K}, \aleph_n) \leq 2^{\aleph_{n-1}}$ for each $n < \omega$ is excellent.

Context

K is the class of atomic models (realize only principal types) of a first order theory.
We study $S_{at}(A)$ where $A \subset M \in \mathbf{K}$ and
$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

reprise: Few models and smallness

Theorem (Keisler)

If $\mathbf{K}$ has less than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then every model of $\mathbf{K}$ realizes only countably many types over the empty set in the countable fragment L^*.

ω-stability I

Definition

The atomic class **K** is λ-stable if for every $M \in \mathbf{K}$ of cardinality λ, $|S_{\rm at}(M)| = \lambda$.

Corollary (Shelah) CH

If **K** is $\aleph_1$-categorical and $2^{\aleph_0} < 2^{\aleph_1}$ then **K** is ω-stable.

ω-stability II

Consequences

1. This gets ω-stability without assuming arbitrarily large models.
2. We only demand few types over models, not arbitrary sets; this is crucial.
3. But, apparently uses CH twice! (for amalgamation and type counting)

ω-stability III

Getting ω-stability

1. Assume arbitrarily large models; use Ehrenfeucht-Mostowski models
2. Keisler-Shelah using CH.
3. Diverse classes (Shelah)

Fundamental question

Let ϕ be a sentence of $L_{\omega_1,\omega}$
Are the following properties absolute for cardinal-preserving forcing? :

Almost surely true

ϕ is ω-stable
ϕ is excellent.

e.g. excellence is π^1_2. 'For very configuration of n independent countable models there is a primary extension.'

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Real question

ϕ is $\aleph_1$-categorical

Is WCH is necessary?

Does MA + ¬ CH imply there is a sentence of $L_{\omega_1,\omega}$ that is $\aleph_1$ categorical but

a. is not ω-stable
b. does not satisfy amalgamation even for countable models.

There is such an example in $L_{\omega_1,\omega}(Q)$ but Laskowski showed the example proposed for $L_{\omega_1,\omega}$ by Shelah (and me) fails.

Towards Counterexamples

For any model $M \in \mathbf{K}$,

1. P and Q partition M.
2. E is an equivalence relation on Q.
3. P and each equivalence class of E is denumerably infinite.
4. R is a relation on $P \times Q$ that is extensional on P. That is, thinking of R as the 'element' relation, each member of Q denotes a subset of P.
5. For every set X of n elements X from P and every subset X_0 of X and each equivalence class in Q, there is an element of that equivalence class that is R-related to every element of X_0 and not to any element of $X - X_0$.
6. Similarly, for every set of n elements Y from Q and every subset Y_0 of Y, there is an element of P that is R-related to every element of Y_0 and not to any element of $Y - Y_0$.

An AEC counterexample

Fix the class $\mathbf{K}$ as above and for $M, N \in \mathbf{K}$, define $M \prec_{\mathbf{K}} N$ if $P^M = P^N$ and for each $m \in Q^M$, $\{n \in N : mEn\} = \{n \in M : mEn\}$ (equivalence classes don't expand).

This class does not have finite character.

$L_{\omega_1,\omega}$ vrs $L_{\omega_1,\omega}(Q)$

Without Q -transfer is proved from VWGCH.
With Q, $\aleph_1$-categoricity is not absolute.
The most convincing example (Zilber) is with Q.

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With Q, $\aleph_1$-categoricity is not absolute.
The most convincing example (Zilber) is with Q.
Does finitary capture the difference?

Finitary Classes

1. an independence theory over sets
2. an analysis for arbitrary sentences in $L_{\omega,\omega}$ - Vaught's conjecture.
3. warning: 'uniqueness of monster model'
4. simplicity?
5. the Hart-Shelah example; other examples of Hytinnen.

Trlifaj Every ${}^{\perp}N$ of pure-injective modules is finitary. Conjecturally all ${}^{\perp}N$'s are finitary

More restrictive classes

Properties

Tameness, finitary, amalgamation, Galois-compactness

NEED EXAMPLES

In particular, examples connected to core mathematics:

1. Banach Spaces
2. complex exponentiation and semi-abelian varieties
3. Modules and Abelian groups
4. locally finite groups

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}(Q)$-sentence discovered by the Hrushovski construction.

Done

A. Expand $(\mathcal{C}, +, \cdot)$ by a unary function which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}(Q)$-sentence Σ is categorical and has quantifier elimination.

Very open

B. Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

Questions

1. B is a problem in analysis and number theory
2. If a sentence ψ of $L_{\omega_1,\omega}(Q)$ is categorical up to $\aleph_\omega$, must it be forever categorical?

UNIVERSAL COVERS

When is the exact sequence:

$$0 \rightarrow Z \rightarrow V \rightarrow A \rightarrow 0. \tag{1}$$

categorical where V is a $\mathbb{Q}$ vector space and A is a semi-abelian variety.
Can be viewed as an expansion of V and there is a combinatorial geometry given by:

$$\mathrm{cl}(X) = \ln(\mathrm{acl}(\exp(X)))$$

Contexts

Trivial case. A is $\aleph_1$-categorical in the Abelian group language– e.g. $(Q, \cdot)$
Interesting case. Work with 'field language' on A.
Really interesting case: pseudo-exponentiation and Hrushovski construction.

Semi-abelian varieties

In the Ravello volume Zilber obtains 'arithmetical conditions' on algebraic variety that are equivalent to 'excellence'
Thus he is able to conclude - using Shelah's machinery that under WGCH,
Categoricity up to $\aleph_\omega$ implies 'arithmetical' properties of certain algebraic varieties.
This argument could be shortened if there were a direct algebraic argument for tameness.

Tameness

For countable theories $(\aleph_0, \infty)$-tameness is incredibly powerful. It allows the categoricity transfer from a single small cardinal (Grossberg, VanDieren, Lessmann).

1. Find other sufficient conditions for tameness.
2. What is the relation between tameness and excellence