

Using Set
theory in
model theory
Kilpisjärvi
2013

John T.
Baldwin

Context

Absoluteness
of Existence

Set Theoretic
Method

Analytically
Presented
AEC

Almost Galois
 ω -stability and
absoluteness
of \aleph_1 -
categoricity

Models in \aleph_2

Using Set theory in model theory Kilpisjärvi 2013

John T. Baldwin

February 2, 2013

Today's Topics

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2 Absoluteness of Existence

3 Set Theoretic Method

4 Analytically Presented AEC

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New results are from papers by Baldwin/Larson
and
Baldwin/Larson/Shelah
that are on my website.

Using Extensions of ZFC in Model Theory

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A theorem under additional hypotheses is better than no theorem at all.

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A theorem under additional hypotheses is better than no theorem at all.

- 1 The result may guide intuition towards a ZFC result.
- 2 Perhaps the hypothesis is eliminable
 - A The combinatorial hypothesis might be replaced by a more subtle argument.
E.G. Ultrapowers of elementarily equivalent models are isomorphic
 - B The conclusion might be absolute
The elementary equivalence proved in the
Ax-Kochen-Ershov theorem
 - C Consistency may imply truth.

Sacks Dicta

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“... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable.”

Gerald Sacks, 1972

See also the Vaananen article in Model Theoretic Logic volume

Which ‘Central Notions’?

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Chang’s two cardinal theorem (morasses)

‘Vaughtian pair is absolute’

saturation is not absolute

Aside: For aec, saturation is absolute below a categoricity cardinal.

‘elementarily prime model’ is absolute. (countable and atomic)

‘algebraically prime model’ is **open**

Study Theories not logics

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The most fundamental of Shelah's innovations was to shift the focus from properties of logics (completeness, preservation theorem, compactness) to
Classifying theories in a model theoretically fruitful way.
(stable not decidable)

Classification Theory

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Crucial Observation

The stability classification is absolute.

Fundamental Consequence

Crucial properties are provable in ZFC for certain classes of theories.

- 1 All stable theories have full two cardinal transfer.
- 2 There are saturated models exactly in the cardinals where the theory is stable.

But this is for FIRST ORDER theories.

Geography

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Models in \aleph_2

$$L_{\omega,\omega} \subset L_{\omega_1,\omega} \subset \hat{L}_{\omega_1,\omega}(Q) \subset \text{anal. pres. AEC} \subset \text{AEC}.$$

In a central case explained below

Extensions of ZFC are **used** for $L_{\omega_1,\omega}$.

$\hat{L}_{\omega_1,\omega}(Q)$ means only negative occurrences of Q : e.g. Zilber field, counterexample to absoluteness

Extensions of ZFC are **proved necessary** for $L_{\omega_1,\omega}(Q)$.

Two notions of 'use'

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Models in \aleph_2

- 1 Some model theoretic results 'use' extensions of ZFC
- 2 Some model theoretic results are provable in ZFC, using models of set theory.

This Talk

- 1 A quick statement of some results of the first kind
- 2 Discussion of several examples of the second method.
- 3 Some further problems whose 'kind' is unknown.
- 4 Some combinatorial problems that have arisen.

One Completely General Result

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Theorem: $(2^\lambda < 2^{\lambda^+})$ (Shelah)

Suppose $\lambda \geq \text{LS}(\mathbf{K})$ and \mathbf{K} is λ -categorical. For any Abstract Elementary class, if amalgamation fails in λ there are 2^{λ^+} models in \mathbf{K} of cardinality λ^+ .

Is $2^\lambda < 2^{\lambda^+}$ needed?

Is $2^\lambda < 2^{\lambda^+}$ needed?

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Models in \aleph_2

1 $\lambda = \aleph_0$:

a Definitely not provable in ZFC: There are
 $L(Q)$ -axiomatizable examples

- i** Shelah: many models with CH, \aleph_1 -categorical under MA
- ii** Koerwien-Todorcevic: many models under MA,
 \aleph_1 -categorical from PFA.

b Independence Open for $L_{\omega_1, \omega}$

A simple Problem

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Models in \aleph_2

Let ϕ be a sentence of $L_{\omega_1, \omega}$.

Question

Is the property ϕ has an uncountable model absolute?

False Start

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Fact: Easy for complete sentences

If ϕ is a complete sentence in $L_{\omega_1, \omega}$,

ϕ has an uncountable model if and only if there exist
countable $M \not\equiv_{\omega_1, \omega} N$ which satisfy ϕ .

This property is Σ_1^1 and done by Shoenfield absoluteness.

Note: $L_{\omega_1, \omega}$ satisfies downward Löwenheim-Skolem for
sentences but **not** for theories.

Fly in the ointment

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There are uncountable models that have **no**
 $L_{\omega_1, \omega}$ -elementary submodel.

E.g. any uncountable model of the first order theory of
infinitely many independent unary predicates P_i .

So the sentence saying every finite Boolean combination of
the P_i is non-empty has an uncountable model and our
obvious criteria does not work.

Note that if we add the requirement that each type is
realized at most once, then every model has cardinality
 $\leq 2^{\aleph_0}$.

Old argument

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A really basic proof

Karp (1964) had proved completeness theorems for $L_{\omega_1, \omega}$,
and Keisler (late 60's/ early 70's) for $L_{\omega_1, \omega}(Q)$,
Barwise-Kaufmann-Makkai for $L(aa)$ $L_{\omega_1, \omega}(aa)$.

But we look at another argument that introduces some
useful concepts and methods.

Larson's characterization

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Theorem

Given a sentence ϕ of $L_{\omega_1, \omega}(\text{aa})(\tau)$,
the existence of a τ -structure of size \aleph_1 satisfying ϕ
is equivalent to
the existence of a countable model of ZFC° containing
 $\{\phi\} \cup \omega$ which thinks there is a τ -structure of size \aleph_1
satisfying ϕ .

This property is Σ_1^1 and done by Shoenfield absoluteness.

Larger Cardinals

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Models in \aleph_2

It is easy to see that there are sentences of $L_{\omega_1, \omega}$ such that the existence of a model in \aleph_2 depends on the continuum hypothesis.

S. Friedman and M. Koerwien have shown.

Assume GCH (and large cardinals for independence of the Kurepa hypothesis)

- 1 For any $\alpha \in \omega_1 - \{0, 1, \omega\}$ there is a sentence ϕ_α such that the existence of a model in \aleph_α is not absolute.
- 2 For \aleph_3 , there is a complete such sentence.

Method: ‘Consistency implies Truth’

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Models in \aleph_2

Let ϕ be a τ -sentence in $L_{\omega_1, \omega}(Q)$ such that it is consistent that ϕ has a model.

Let \mathcal{A} be the countable ω -model of set theory, containing ϕ , that thinks ϕ has an uncountable model.

Construct \mathcal{B} , an uncountable model of set theory, which is an elementary extension of \mathcal{A} such that \mathcal{B} is correct about uncountability. Then the model of ϕ in \mathcal{B} is actually an uncountable model of ϕ .

How to build \mathcal{B}

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Models in \aleph_2

MT Iterate a theorem of Keisler and Morley (refined by Hutchinson).

ST Iterations of ‘special’ ultrapowers.

ZFC° denotes a sufficient subtheory of ZFC for our purposes.

How to build \mathcal{B}

The main technical tool is the iterated generic elementary embedding induced by the nonstationary ideal on ω_1 , which we will denote by NS_{ω_1} .

The ultrafilter

Forcing with the Boolean algebra $(\mathcal{P}(\omega_1)/\text{NS}_{\omega_1})^M$ over a ZFC model M gives rise to an M -normal ultrafilter U on ω_1^M (i.e., every regressive function on ω_1^M in M is constant on a set in U).

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The Ultrapower

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Given such M and U , we can form the generic ultrapower $\text{Ult}(M, U)$, which consists of all functions in M with domain ω_1^M ,

where for any two such functions f, g , and any relation R in $\{=, \in\}$, fRg in $\text{Ult}(M, U)$ if and only if $\{\alpha < \omega_1^M \mid f(\alpha)Rg(\alpha)\} \in U$.

Nota Bene

If M is countable, $\text{Ult}(M, U)$ is countable.

By convention, we identify the well-founded part of the ultrapower $\text{Ult}(M, U)$ with its Mostowski collapse.

One step in the construction

We use three crucial properties of ZFC° .

- 1 The theory ZFC° holds in every structure of the form $H(\kappa)$ or V_κ , where κ is a regular cardinal greater than $2^{2^{\aleph_1}}$
- 2 If P is a c.c.c. notion of forcing, $2^{|P|^{\aleph_1}} < \theta$ and X is an elementary submodel of $H(\theta)$, then any forcing extension of the transitive collapse of X satisfies ZFC° .
- 3 Whenever M is a model of ZFC° and U is an M -ultrafilter on ω_1^M , M is elementarily embedded in $\text{Ult}(M, U)$.
- 4 If U constructed as above, $\text{Ult}(M, U)$ increases exactly the sets that M thinks are uncountable.

Really distinct iterations

Theorem (Larson)

If M is a countable model of $ZFC^\circ + MA_{\aleph_1}$ and

$$\langle M_\alpha, G_\alpha, j_{\alpha, \gamma} : \alpha \leq \gamma \leq \omega_1, \rangle$$

and

$$\langle M'_\alpha, G'_\alpha, j'_{\alpha, \gamma} : \alpha \leq \gamma \leq \omega_1, \rangle$$

are two distinct iterations of M , then

$$\mathcal{P}(\omega)^{M_{\omega_1}} \cap \mathcal{P}(\omega)^{M'_{\omega_1}} \subset M_\alpha,$$

where α is least such that $G_\alpha \neq G'_\alpha$.

G_α not defined for $\alpha = \omega_1$.

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The Model Theory

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ZFC-Theorem: (Keisler, new proof Larson)

Let F be a countable fragment of $L_{\omega_1, \omega}$ (aa). If there exists a model of cardinality \aleph_1 realizing uncountably many F -types, there exists a 2^{\aleph_1} -sized family of such models, each of cardinality \aleph_1 and pairwise realizing just countably many F -types in common.

(ZFC +CH)-Corollary (Shelah)

If a sentence in $L_{\omega_1, \omega}$ has less than 2^{\aleph_1} models in \aleph_1 then it is (syntactically) ω -stable.

CH used twice.

Sketching New Proof:

Let F be the smallest fragment of $L_{\omega_1, \omega}(Q)$ containing ϕ .

- 1 Choose θ so that $H(\theta)$ contains the model N and θ is large enough for preservation of ZFC° .
- 2 Choose X a countable elementary submodel of $H(\theta)$.
- 3 Let M be the transitive collapse of X , and let N_0 be the image of N under this collapse.
- 4 Let M' be a c.c.c. forcing extension of M satisfying Martin's Axiom. to get really distinct ultrapowers.
- 5 Build a tree of generic ultrapower iterates of M' giving rise to 2^{\aleph_1} many distinct iterations of M' , each of length ω_1 .
- 6 Since F -types can be coded by reals using an enumeration of F in M , the images of N_0 under these iterations will pairwise realize just countably many F -types in common.

ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónsson:

A class of L -structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an abstract elementary class: AEC if both \mathbf{K} and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism plus:

1 If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;

Examples

First order and $L_{\omega_1, \omega}$ -classes

$L(Q)$ classes have Löwenheim-Skolem number \aleph_1 .

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- 1 If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;
- 2 Closure under direct limits of $\prec_{\mathbf{K}}$ -chains;

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- 1 If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;
- 2 Closure under direct limits of $\prec_{\mathbf{K}}$ -chains;
- 3 **Downward Löwenheim-Skolem.**

Examples

First order and $L_{\omega_1, \omega}$ -classes

$L(Q)$ classes have Löwenheim-Skolem number \aleph_1 .

Chains of models in \aleph_1

Let \mathbf{K} be an aec with amalgamation and joint embedding property and Lowenheim-Skolem number \aleph_0 . Lessmann shows that if there exist a pair of countable limit models $(M \prec_{\mathbf{K}} N)$ and a Galois type over a submodel of M that is omitted in N , then there is such a pair of cardinality \aleph_1 . The crux is that all countable ordinals have cofinality ω . Here all models have cardinality μ .

Definition

A (μ, θ) limit model \mathbb{M} is a continuous union $\mathbb{M} = \bigcup_{i \in \theta} \mathbb{M}_i$ where \mathbb{M}_{i+1} is μ -universal over \mathbb{M}_i of size μ .

We say \mathbb{M} is a limit model over \mathbb{N} if there is such a sequence of \mathbb{M}_i where $\mathbb{M}_0 = \mathbb{N}$.

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k -domination

Definition

- Given two limit sequences, $(\mathbb{M}_i)_{i < \alpha}$, $(\mathbb{N}_i)_{i < \beta}$ we say that $(\mathbb{N}_i)_{i < \beta}$ k -dominates $(\mathbb{M}_i)_{i < \alpha}$ if for all $i < \alpha$ there exists $k < \omega$ such that \mathbb{N}_{i+k} is a proper universal extension of \mathbb{M}_i and $\bigcup_{i < \omega} \mathbb{N}_i$ is a proper extension of $\bigcup_{i < \omega} \mathbb{M}_i$. We write $(\mathbb{M}_i)_{i < \alpha} <_L^k (\mathbb{N}_i)_{i < \beta}$.
- Given two limit sequences, $(\mathbb{M}_i)_{i < \alpha}$, $(\mathbb{N}_i)_{i < \beta}$ we say that $(\mathbb{N}_i)_{i < \beta}$ eventually -dominates $(\mathbb{M}_i)_{i < \alpha}$ if for for some k $(\mathbb{M}_i)_{i < \alpha} <_L^k (\mathbb{N}_i)_{i < \beta}$, we write $(\mathbb{M}_i)_{i < \alpha} <_L (\mathbb{N}_i)_{i < \beta}$.

A combinatorial question

Show 'domination' is a property of the filtrations, not the models.

Problem

Find an example of two limit models M and N and respective limit sequences $\langle M_i : i < \alpha \rangle$; $\langle N_i : i < \beta \rangle$ such that $(M_i)_{i < \alpha} <_L^k (N_i)_{i < \beta}$ as well as an additional limit sequence $\langle M'_i : i < \alpha' \rangle$ for M , such that there is no $\langle (N'_i)_{i < \beta'} \rangle$ for N such that for some k' , $(M'_i)_{i < \alpha'} <_L^{k'} (N'_i)_{i < \beta'}$?

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Analytically Presented AEC

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Models in \aleph_2

Definition

An abstract elementary class \mathbf{K} with Löwenheim number \aleph_0 is **analytically presented** if the set of countable models in \mathbf{K} , and the corresponding strong submodel relation $\prec_{\mathbf{K}}$, are both analytic.

Definition

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Fact

An AEC \mathbf{K} is $PC\Gamma(\aleph_0, \aleph_0)$ -presented:

if the models are reducts of models a countable first order theory in an expanded vocabulary which omit a countable family of types and the submodel relation is given in the same way.

AKA:

- 1 Keisler: PC_δ over $L_{\omega_1, \omega}$
- 2 Shelah: $PC(\aleph_0, \aleph_0)$, \aleph_0 -presented

More Precisely

Theorem

If \mathbf{K} is AEC then \mathbf{K} can be analytically presented iff and only if its restriction to \aleph_0 is the restriction to \aleph_0 of a $PC\Gamma(\aleph_0, \aleph_0)$ -AEC.

The following is basically folklore.

Countable case

The countable τ -models of an analytically presented class can be represented as reducts to τ of a sentence in

$L_{\omega_1, \omega}(\tau')$ for appropriate $\tau' \supseteq \tau$.

Moreover the class of countable pairs (M, N) such that $M \prec_{\mathbf{K}} N$ is also a $PC\Gamma(\aleph_0, \aleph_0)$ -class.

Uncountable Case

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Theorem

All τ -models of an analytically presented AEC \mathbf{K} can be represented as reducts to τ of a sentence θ^* in $L_{\omega_1, \omega}(\tau^*)$ for appropriate $\tau^* \supseteq \tau$.

Moreover the class of pairs (M, N) such that $M \prec_{\mathbf{K}} N$ is the class of reducts to τ' of models of θ^* .

Proof combines the countable case with the idea of the proof of the presentation theorem.

Example

Groupable partial orders (Jarden varying Shelah)

Let (\mathbf{K}, \prec) be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order (equivalently admits a group structure) with $M \prec N$ if $M \subseteq N$ and no component is extended.

This AEC is analytically presented.

Add a binary function and say it is a group on each component.

But it has 2^{\aleph_1} models in \aleph_1 and 2^{\aleph_0} models in \aleph_0 .

Recall: this ‘is’ the pseudo-elementary counterexample to Vaught’ s conjecture.

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Galois Types

Let $M \prec_{\mathbf{K}} N_0$, $M \prec_{\mathbf{K}} N_1$, $a_0 \in N_0$ and $a_1 \in N_1$ realize the same **Galois Type** over M iff there exist a structure $N \in \mathbf{K}$ and strong embeddings $f_0: N_0 \rightarrow N$ and $f_1: N_1 \rightarrow N$ such that $f_0|_M = f_1|_M$ and $f_0(a_0) = f_1(a_1)$.

Realizing the same Galois type (over countable models) is an equivalence relation

$$E_M$$

if \mathbf{K}_{\aleph_0} satisfies the amalgamation property.

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The Monster Model

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Models in \aleph_2

If an Abstract Elementary Class has the amalgamation property and the joint embedding property for models of cardinality at most \aleph_0

and has at most \aleph_1 -Galois types over models of cardinality $\leq \aleph_0$

then there is an \aleph_1 -monster model \mathbb{M} for \mathbf{K} and the Galois type of a over a countable M is the orbit of a under the automorphisms of \mathbb{M} which fix M .

So E_M is an equivalence relation on \mathbb{M} .

Another Combinatorial Question

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Question

Let \mathbf{K} be an aec in vocabulary τ of cardinality \aleph_1 with amalgamation and joint embedding property and Lowenheim-Skolem number \aleph_1 .

Suppose there exist a pair of limit models $(M \prec_{\mathbf{K}} N)$ in \aleph_1 and a Galois type over a submodel of M that is omitted in N . Must there be a model of cardinality \aleph_2 omitting p . Can one find a counterexample using some known combinatorial objects?

Some stability notions

Definition

- 1 The abstract elementary class (\mathbf{K}, \prec) is said to be **Galois ω -stable** if for each countable $M \in \mathbf{K}$, E_M has countably many equivalence classes.
- 2 The abstract elementary class (\mathbf{K}, \prec) is **almost Galois ω -stable** if for each countable $M \in \mathbf{K}$, no E_M has a perfect set of equivalence classes.

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Almost Galois Stable

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Models in \aleph_2

Well-orders of type at most \aleph_1 under end-extension are an AEC where countable models have only \aleph_1 Galois types.

Galois equivalence is Σ_1^1

On an analytically presented AEC, having the same Galois type over M is an analytic equivalence relation, E_M . So by Burgess's theorem we have the following trichotomy.

Theorem

An analytically presented abstract elementary class (\mathbf{K}, \prec) is

- 1 Galois ω -stable or
- 2 almost Galois ω -stable or
- 3 has a perfect set of Galois types over some countable model

Again basically folklore.

Keisler for Galois types

Theorem: (B/Larson)

Suppose that

- 1 \mathbf{K} is an analytically presented abstract elementary class;
- 2 N is a \mathbf{K} -structure of cardinality \aleph_1 , and N_0 is a countable structure with $N_0 \prec_{\mathbf{K}} N$;
- 3 P is a perfect set of E_{N_0} -inequivalent members of ω^ω ;
- 4 N realizes the Galois types of uncountably many members of P over N_0 .

Then there exists a family of 2^{\aleph_1} many \mathbf{K} -structures of cardinality \aleph_1 , each containing N_0 and pairwise realizing just countably many P -classes in common.

Fact: Hyttinen-Kesala, Kueker

If a sentence in $L_{\omega_1, \omega}$, satisfying amalgamation and joint embedding, is almost Galois ω -stable then it is Galois ω -stable.

What about analytically presented?

Analytically presented Strictly Almost Galois ω -stable example

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The 'groupable partial order' is almost Galois stable

Let (\mathbf{K}, \prec) be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order with $M \prec N$ if $M \subseteq N$ and no component is extended.

Since there are only \aleph_1 -isomorphism types of components this class is almost Galois ω -stable.

This AEC is analytically presented.

Smallness

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Models in \aleph_2

Definition

- 1 A \mathcal{T} -structure M is L^* -small for L^* a countable fragment of $L_{\omega_1, \omega}(\mathcal{T})$ if M realizes only countably many $L^*(\mathcal{T})$ -types (i.e. only countably many $L^*(\mathcal{T})$ - n -types for each $n < \omega$).
- 2 A \mathcal{T} -structure M is called small or $L_{\omega_1, \omega}$ -small if M realizes only countably many $L_{\omega_1, \omega}(\mathcal{T})$ -types.

Why Smallness matters

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Fact

Each small model satisfies a Scott-sentence, a complete sentence of $L_{\omega_1, \omega}$.

Getting small models I

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Theorem: Shelah

If \mathbf{K} is analytically presented and some model of cardinality \aleph_1 is L^* -small for every countable \mathcal{T} -fragment L^* of $L_{\omega_1, \omega}$, then \mathbf{K} has an $L_{\omega_1, \omega}(\mathcal{T})$ -small model M' of cardinality \aleph_1 .

Getting small models II

Theorem: Baldwin/Shelah/Larson

If \mathbf{K} has a model in \aleph_1 that is not $L_{\omega_1, \omega}(\mathcal{T})$ -small then

- 1 there are at least \aleph_1 complete sentences of $L_{\omega_1, \omega}(\mathcal{T})$ which are satisfied by uncountable models in \mathbf{K} ;
- 2 \mathbf{K} has uncountably many models in \aleph_1 ;
- 3 \mathbf{K} has uncountably many extendible models in \aleph_0 .

Proof: Iterate the previous theorem.

Corollary: Baldwin/Shelah/Larson

Vaught's conjecture is equivalent to Vaught's conjecture for **extendible** models.

A countable model is **extendible** if it has an $L_{\omega_1, \omega}$ -elementary extension.

Complexity of (almost) ω -stability

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Models in \aleph_2

- 1 first order syntactic: Π_1^1
- 2 $L_{\omega_1, \omega}$ -syntactic: Π_1^1
- 3 analytically presented AEC: Galois ω -stable: perhaps boldface Π_4^1
- 4 analytically presented AEC: almost Galois ω -stable: boldface Π_2^1

Absoluteness of \aleph_1 -categoricity

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Models in \aleph_2

1 \aleph_1 -categoricity of a class \mathbf{K} defined in $L_{\omega_1, \omega}$ is absolute between models of set theory that satisfy any one of the following conditions.

- 1** \mathbf{K} is ω -stable;
- 2** \mathbf{K} has arbitrarily large members and \mathbf{K} has amalgamation in \aleph_0 ;
- 3** $2^{\aleph_0} < 2^{\aleph_1}$.

<http://homepages.math.uic.edu/~jrbaldwin/pub/singsep2010.pdf>

2 \aleph_1 -categoricity of an analytically presented AEC \mathbf{K} is absolute between models of set theory in which \mathbf{K} is almost Galois ω -stable, satisfies amalgamation in \aleph_0 , and has an uncountable model.

Why is this absoluteness of \aleph_1 -categoricity true for AEC?

Fact

Suppose that \mathbf{K} is an analytically presented AEC. Then the following statements are equivalent.

- 1 There exist a countable $M \in \mathbf{K}$ and an $N \in \mathbf{K}$ of cardinality \aleph_1 such that:
 - $M \prec_{\mathbf{K}} N$;
 - the set of Galois types over M realized in N is countable;
 - some Galois type over M is not realized in N .
- 2 There is a countable model of ZFC° whose ω_1 is well-founded and which contains trees on ω giving rise to \mathbf{K} , $\prec_{\mathbf{K}}$ and the associated relation \sim_0 , and satisfies statement 1.

Models in \aleph_2

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Models in \aleph_2

Harrington (slides by Marker) <http://homepages.math.uic.edu/~marker/harrington-vaught.pdf>
and Larson have shown that a counterexample, $\phi \in L_A$, to Vaught's conjecture has (models in \aleph_1) of arbitrarily large Scott rank (as sentences in $L_{\omega_2, \omega}$).

The (unrealized) goal is to construct a model in \aleph_2 and thus prove Vaught's conjecture since Hjorth as shown if there is a c.e. there is one with no model in \aleph_2 .

Problem

Find model theoretic properties of these models of high Scott rank. E.G, can we show some subset of these models form an L_A elementary chain?

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Models in \aleph_2

- 1 The set theoretic method provides a uniform method for studying models of various infinitary logics
- 2 We introduced analytically presented AEC and showed:
 - i analytically presented = $PC\Gamma(\aleph_0, \aleph_0)$
 - ii Extended Keisler's few models implies ω -stability theorem to this class
 - iii Assuming countably many models in \aleph_1 :
Almost Galois ω -stable implies Galois ω -stable
 - iv \aleph_1 -categoricity absolute for Almost Galois ω -stable with amalgamation.