Why the weak GCH is true!
ASL-APA Spring Meeting 2007

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Outline

The Weak Generalized Continuum Hypothesis

Setting

ZFC is the base theory throughout.

Axiom: WGCH

For every cardinal λ, $2^\lambda < 2^{\lambda^+}$.

The Continuum Function

$$f(\kappa) = 2^\kappa$$

By Cantor and König:

$$2^\kappa > \kappa$$

$$\mathrm{cf}(2^\kappa) > \kappa$$

Acknowledgements

This is primarily an exposition of work of Shelah and Zilber.
And even the exposition depends on ideas of Maddy, Foreman, and Eklof.
Detailed proof of most of the results here are given in my monograph: Categoricity (available on line).
The only thing new is the actual thesis.

Realism

If you prefer the title

Reasons why one should add the weak continuum hypothesis to ZFC

That's fine

Nothing in this talk is concerned with 'Platonism' except that very naive realism is an easier way to speak.

Justifying Axioms

Argument

1. WGCH is consistent. (First, do no harm)
2. WGCH is 'natural'.
3. WGCH has important consequences.

Analogies

The Axiom of Choice:

The Axiom of Foundation

Context

Fact 1

By Gödel, WGCH is relatively consistent with ZFC.

Fact 2

By Easton forcing, the only constraints on the continuum function for regular cardinals are $2^\kappa > \kappa$ and $\mathrm{cf}(2^\kappa) > \kappa$

Conclusion

The negation of WGCH is relatively consistent with ZFC.

Combinatorialism

Maddy calls combinatorialism the doctrine of Bernays: "Modern analysis ... abstracts from the possibility of giving definitions of sets sequences and functions. These notions are used in a 'quasicombinatorial sense, by which I mean in the sense of an analogy of the infinite to the finite. ..."

Analogy to the finite supports WGCH

An easy consequence

If the continuum function is increasing on successors, it is increasing on all cardinals.

Theorem (WGCH)

If $\mu < \lambda$, $2^\mu < 2^\lambda$.

Proof Sketch

Easy cardinal induction using $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$ and $\mathrm{cf}(2^{<\kappa}) = \mathrm{cf}(\kappa)$ to show $2^\kappa > 2^{<\kappa}$ if κ is a limit cardinal.

Prediction Principles

A prediction principle allows one to make a construction by predicting in advance all or enough obstructions so they can be avoided.

Dushnik-Miller

Theorem

There is a dense subset X of R which has no non-trivial order isomorphism.

Proof Sketch

1. Any order automorphism of X has a unique extension to $\Re$.
2. Any order automorphism ϕ of $\Re$ is determined by $\phi \restriction \mathbb{Q}$.
3. Thus, each order automorphism of X is among $\{\phi_i \restriction X : i < 2^{\aleph_0}\}$, ϕ_i is an order automorphism of $\Re$.

Proof Sketch continued

Construct a sequence of subsets $\langle A_i, B_i \rangle$ for $i < 2^{\aleph_0}$ of $\Re$ insisting that $A_i \subseteq X$ and $B_i \cap X = \emptyset$. At stage i add a_i to A_i and $\phi_i(a_i)$ to B_i.

In this construction we listed all the possible obstructions ϕ_i and killed them inductively. In general, the induction may not be long enough. But stronger conditions can solve this.

Diamond [$\diamond_\lambda$]

$\diamond_\lambda$ is the proposition:

There is a sequence $\{W_\alpha : \alpha \in \lambda\}$ such that for each $\alpha \in \lambda$, $W_\alpha \subseteq \alpha$ and for any $X \subseteq \lambda$:

$$\{\alpha : X \cap \alpha = W_\alpha\}$$

is stationary in λ.

For every $X \subset \lambda$ and $\alpha < \lambda$
$\diamond$ predicts the membership of $X \cap \alpha$.

$[\diamond_\lambda]$ too strong

Fact

If $\lambda = \kappa^+$, $\diamond_\lambda$ implies $2^\kappa = \kappa^+$.

(Aside: Gregory and Shelah have proved that GCH implies $\diamond_\lambda$ for every successor λ except $\aleph_1$.)

Definition: Devlin-Shelah Weak Diamond

Φ_λ is the proposition:

For any function $F : 2^{<\lambda} \to 2$ there exists $g \in 2^\lambda$ such that for every $f \in 2^\lambda$ the set

$$\{\delta < \lambda : F(f \restriction \delta) = g(\delta)\}$$

is stationary.

For every $X \subset \lambda$ and $\alpha < \lambda$,
Weak-$\diamond$ predicts whether $X \cap \alpha$ is in one side or another of a partition of $\mathcal{P}(\alpha)$.

Crucial Fact

$2^\lambda < 2^{\lambda^+}$ if and only if Weak-$\diamond$ on λ^+ (Φ_{λ^+})

So weak diamond is the operative form of WGCH.

Weak Diamond Variant $[\Theta_\lambda]$

Θ_λ is the following proposition:

For any collection of functions $\langle f_\eta : \eta \in 2^\lambda \rangle$, with $f_\eta \in \lambda^\lambda$ and any cub $C \subset \lambda$ there are $\delta \in C$ and η, ν such that:

1. $\eta \restriction \delta = \nu \restriction \delta$,
2. $\eta(\delta) \neq \nu(\delta)$,
3. $f_\eta \restriction \delta = f_\nu \restriction \delta$,

Φ_λ implies Θ_λ

ABSTRACT ELEMENTARY CLASSES

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an *abstract elementary class: AEC* if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

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2. Closure under direct limits of $\prec_{\mathbf{K}}$-chains;
3. Downward Löwenheim-Skolem.

Examples

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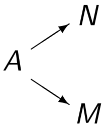
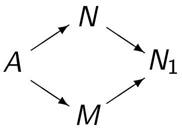
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

AMALGAMATION PROPERTY

The class **K** satisfies the *amalgamation property* if for any situation with $A, M, N \in \mathbf{K}$:



there exists an N_1 such that



No Amalgamation implies no universal model

Lemma

Suppose $\lambda \geq \mathrm{LS}(\mathbf{K})$, $2^{\lambda} < 2^{\lambda^{+}}$, *and* $\mathbf{K}$ *is* λ*-categorical. If amalgamation fails in* λ *there are models in* $\mathbf{K}$ *of cardinality* λ^{+} *but no universal model of cardinality* λ^{+}.

Proof Sketch

Proof. Let $N_0 \prec_{\mathbf{K}} N_1, N_2$ witness the failure of amalgamation. Define a tree of models M_ρ with universe $\lambda(1 + \ell(\rho))$ for $\rho \in \lambda^{\leq \lambda^{+}}$, so that the failure of amalgamation is replicated at each node.

Suppose for contradiction that there is a model M of cardinality κ which is universal. Let f_η be the embedding of M_η into M. The set C of $\delta < \kappa$ of the form: $\delta = \lambda(1 + \delta)$ contains a cub. Applying Θ_{λ^+}, we find $\delta \in C$ and distinct $\eta, \nu \in 2^\kappa$ which agree only up to δ.
Denoting $\eta \wedge \nu$ by $\rho = \eta \restriction \delta$, we have that f_η and f_ν map $M_{\rho\frown 0}$ and $M_{\rho\frown 1}$ into M over M_ρ.

We have amalgamated an isomorphic copy of N_0, N_1, N_2 in K_λ. This contradiction yields the theorem.

Failure of amalgamation yields many models

A considerably more complicated arguments shows:

Theorem (WGCH)

If **K** is λ-categorical and amalgamation fails in λ there are 2^{λ^+} models in **K** of cardinality λ^+.

Categoricity

1. Why study categoricity?
2. A warm-up: quasiminimal excellence.
3. (ZFC) Excellence implies categoricity transfers.
4. (WGCH) Enough categoricity implies excellence.

First Order Categorical Structures

I. $(\mathcal{C}, =)$

II. $(\mathcal{C}, +, =)$ vector spaces over Q.

III. $(\mathcal{C}^*, \times, =)$

IV. $(\mathcal{C}, +, \times, =)$ Algebraically closed fields - Steinitz

MORLEY'S THEOREM

Theorem

If a countable first order theory is categorical in one uncountable cardinal it is categorical in all uncountable cardinals.

Zilber's Precept

Fundamental canonical mathematical structures like I-IV should admit logical descriptions that are categorical in power.

Another Canonical Structure

COMPLEX EXPONENTIATION

Consider the structure $(C, +, \cdot, e^x, 0, 1)$.

The integers are defined as $\{a : e^{2\pi a} = 1\}$.
This makes the first order theory unstable, provides a two cardinal model .... The theory is clearly not categorical.

Thus first order axiomatization can not determine categoricity.

ZILBER'S INSIGHT

Maybe Z is the source of all the difficulty.
Fix Z by adding the axiom:

$$(\forall x) e^x = 1 \rightarrow \bigvee_{n \in Z} x = 2n\pi.$$

QUASIMINIMAL EXCELLENCE

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal excellent* if it admits a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

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1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. and the 'excellence condition' which follows.

Notation

In the following definition it is essential that $\subset$ be understood as *proper* subset.

Definition

1. For any Y, $\mathrm{cl}^{-}(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.
2. We call C (the union of) *an n-dimensional* cl*-independent system* if $C = \mathrm{cl}^{-}(Z)$ and Z is an independent set of cardinality n.

Formal Quasiminimal Excellence

Types over independent systems are finitely based

Let $G \subseteq H \in \mathbf{K}$ with G empty or in $\mathbf{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^{-}(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that $\mathrm{tp}_{\mathrm{qf}}(X/C)$ is is determined by its restriction to C_0.

Quasiminimal Excellence implies Categoricity

Quasiminimal excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of each finite set is countable.

Categoricity for Quasiminimal classes

Theorem Suppose the quasiminimal excellent class **K** is axiomatized by a sentence Σ of $L_{\omega_1,\omega}$, and the relations $y \in \mathrm{cl}(x_1, \ldots x_n)$ are $L_{\omega_1,\omega}$-definable.

Then, for any infinite κ there is a unique structure in **K** of cardinality κ which satisfies the countable closure property.

NOTE BENE: The categorical class could be axiomatized in $L_{\omega_1,\omega}(Q)$. But, the categoricity result does not depend on any such axiomatization.

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}(Q)$-sentence discovered by the Hrushovski construction.

Done

A. Expand $(\mathcal{C}, +, \cdot)$ by a unary function which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1,\omega}(Q)$-sentence Σ is categorical and has quantifier elimination.

Very open

B. Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

Löwenheim-Skolem in $L_{\omega_1,\omega}$

Upward

For every $\alpha < \beth_{\omega_1}$ there is an $L_{\omega_1,\omega}$ sentence ϕ_α that has a model cardinality $\beth_\alpha$, but no model of cardinality bigger than $\beth_\alpha$.

Downward

Each sentence of $L_{\omega_1,\omega}$ has a countable model.
But a theory in $L_{\omega_1,\omega}$ need not have a countable model.

MODEL THEORETIC CONTEXT

Any κ-categorical sentence of $L_{\omega_1,\omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory. ($EC(T, Atomic)$-class)

Shelah defined a notion of excellence; Zilber's is the 'rank one' case.

Categoricity Transfer in $L_{\omega_1,\omega}$

ZFC: Shelah 1983

If **K** is an excellent $EC(T, Atomic)$-class then if it is categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.

WGCH: Shelah 1983

If an $EC(T, Atomic)$-class **K** is categorical in $\aleph_n$ for all $n < \omega$, then it is excellent.

Context

K is the class of atomic models (realize only principal types) of a first order theory.
We study $S_{at}(A)$ where $A \subset M \in \mathbf{K}$ and
$p \in S_{at}(A)$ means Aa is atomic if a realizes p.

Two Examples

T_1 is the theory of an infinite set under equality. $M \models T$. p asserts $x \neq m$ for every $m \in M$. Then $p \in S_{at}(A)$.

(Marcus): There is a model M which is atomic, minimal and contains an infinite indiscernible set.
Every $p \in S_{at}(M)$ is realized in M.

ω-stability I

Definition

The atomic class **K** is λ-stable if for every $M \in$ **K** of cardinality λ, $|S_{\text{at}}(M)| = \lambda$.

Theorem (Keisler)

If **K** is $\aleph_1$-categorical and $2^{\aleph_0} < 2^{\aleph_1}$ then **K** is ω-stable.

Shelah made remark on WCH.

ω-stability II

Two facts

1. This gets ω-stability without assuming arbitrarily large models.
2. We only demand few types over models, not arbitrary sets; this is crucial.
3. WCH is necessary; MA $+ \neg$ CH implies there is a sentence of $L_{\omega_1,\omega}$ that is $\aleph_1$ categorical but
   a. is not ω-stable
   b. does not satisfy amalgamation even for countable models.

The makes MA unappealing from a model theoretic standpoint

Splitting as Independence

Splitting

A complete type p over A *splits* over $B \subset A$ if there are $\mathbf{b}, \mathbf{c} \in A$ which realize the same type over B and a formula $\phi(\mathbf{x}, \mathbf{y})$ with $\phi(\mathbf{x}, \mathbf{b}) \in p$ and $\neg\phi(\mathbf{x}, \mathbf{c}) \in p$.

Independence

$A \underset{M}{\perp} B$ if for every $\mathbf{a} \in A$, there is finite $C \subset M$ such that $\mathrm{tp}(\mathbf{a}/MB)$ does not split over B.

This notion of independence has many good properties of non-forking: extension *of types over models*, symmetry, finite based, monotonicity. But the base of types is a constant concern.

The General Case

Quasiminimality is the rank one case

Any geometry has a notion of independent n-system.

In the more general setting

Splitting gives an analogous notion of independent n-system.

Goodness

Definition

A set A is *good* if the isolated types are dense in $S_{at}(A)$.

For countable A, this is the same as $|S(A)| = \aleph_0$.

Eventually, one shows that there are prime models over good sets.

Excellence

Definition

1. **K** is (λ, n)-*good* if for any independent n-system $\mathcal{S}$ (of models of size λ), the union of the nodes is good.

   That is, there is a prime model over any countable independent n-system.

2. **K** is *excellent* if it is $(\aleph_0, n)$-good for every $n < \omega$.

Excellence implies large models

Theorem

If an atomic class **K** is excellent and has an uncountable model then it has models of arbitrarily large cardinality.

The proof shows by induction on m that **K** is $(\aleph_m, n)$-good for all $m, n < \aleph_0$.

(Very) Few Models

Definition

1. **K** has *few* models in power λ if $I(\mathbf{K}, \lambda) < 2^{\lambda}$.
2. **K** has *very few* models in power λ if $I(\mathbf{K}, \lambda) \leq \lambda$.

(Very) Few Models implies Excellence

WGCH: Shelah 1983

An atomic class **K** that has at least one uncountable model and that has very few models in $\aleph_n$ for each $n < \omega$ is excellent.

It is open whether WGCH suffices to delete 'very'.

Show by induction:

Very few models in $\aleph_n$ implies $(\aleph_0, n-2)$-goodness.

Methodology

Weak-$\diamond$ is used three times in different induction steps.

Consequences

From conditions on countable models and WGCH below $\aleph_\omega$, we make conclusions on models of arbitrary size.

1. There are arbitrarily large models.
2. *n*-Amalgamation holds in all cardinalities.
3. Categoricity below $\aleph_\omega$ implies categoricity in all cardinalities.

Necessity of categoricity up to $\aleph_\omega$

Theorem.[Hart-Shelah, Baldwin-Kolesnikov] For each $k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
2. ϕ_k is not $\aleph_{k-2}$-Galois stable;
3. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$;
4. ϕ_k has the disjoint amalgamation property;

Further consequences of WGCH

In ZFC

We describe some equivalences between 'arithmetic properties of algebraic groups'.

+ WGH

categoricity up $\aleph_\omega$ of certain classes associated with the group is equivalent to these arithmetic properties

Covers of Algebraic Groups

Definition A cover $\mathcal{A}^*$ of a commutative algebraic group $A(\mathcal{C})$ is a short exact sequence

$$0 \rightarrow Z^N \rightarrow V \rightarrow A(\mathcal{C}) \rightarrow 1. \qquad (1)$$

where V is a $\mathbb{Q}$ vector space and A is an algebraic group with the full structure imposed by $(\mathcal{C}, +, \cdot)$.

Can be viewed as an expansion of V and there is a combinatorial geometry given by:

$$\mathrm{cl}(X) = \ln(\mathrm{acl}(\exp(X)))$$

Axiomatizing Covers

Let A be a commutative algebraic group over an algebraically closed field F.

Let T_A be the first order theory asserting:

1. $(V, +, f_q)_{q\in\mathbb{Q}}$ is a $\mathbb{Q}$-vector space.
2. The complete first order theory of A in a language with a symbol for each F-definable variety.
3. ex is a group homomorphism from $(V, +)$ to $(A, \cdot)$.

$T_A + \Lambda = \mathcal{Z}^N$ asserts the kernel of ex is standard.

Categoricity of Covers

When is the theory $T_A + \Lambda = \mathcal{Z}^N$ of the cover:

$$0 \rightarrow Z^N \rightarrow V \rightarrow A \rightarrow 1. \tag{2}$$

categorical?

Zilber has proved categoricity for $\mathcal{A} = (\mathcal{C}^{\times}, \cdot)$.
In general categoricity, WGCH implies categoricity is equivalent to 'arithmetic' properties of $\mathcal{A}$.

Algebraic Formulations of Excellence

Excellence is equivalent in this context to algebraic properties of the groups $\mathcal{A}(k)$ for various countable k. Here are some sample properties.

Let $\mathcal{S} = \{F_s : s \subset n\}$ be an independent n-system of algebraically closed fields contained in a suitable monster $\mathcal{M}$. Denote the subfield of $\mathcal{M}$ generated by $(\bigcup_{s \subset n} F_s)$ as k.

Canonical completions

$$\mathcal{A}(k) = A^n \oplus \prod_{s \subset n} \mathcal{A}(F_s)$$

where A^n is a free Abelian group.

Choosing Roots

Definition

A *multiplicatively closed divisible subgroup* associated with $a \in \mathcal{C}^*$, is a **choice** of a multiplicative subgroup isomorphic to $\mathbb{Q}$ containing a .

Definition

$b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, *determine the isomorphism type* of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ over F if given subgroups of the form $c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ and ϕ_m such that

$$\phi_m : F(b_1^{\frac{1}{m}} \ldots b_\ell^{\frac{1}{m}}) \rightarrow F(c_1^{\frac{1}{m}} \ldots c_\ell^{\frac{1}{m}})$$

is a field isomorphism it extends to

$$\phi_\infty : F(b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}) \rightarrow F(c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}}).$$

A finiteness condition

Thumbtack Lemma

For any $b_1, \ldots b_\ell \subset \mathcal{C}^*$, there exists an m such that $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, *determine the isomorphism type* of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ over F.

Zilber

The Thumbtack Lemma is equivalent to $T_A + \Lambda = \mathcal{Z}^N$ is quasiminimal excellent.

Geometry from Model Theory

The 'Thumbtack Lemma' is an 'arithmetic' statement about $(\mathcal{C}^*, \cdot)$. Analogous conditions can be formulated for other algebraic groups.

Similar formulations can be made for a covers of other algebraic groups.

While, this statement is true for $(\mathcal{C}^*, \cdot)$, there are a semiabelian varieties which are known not to satisfy the conditions.
And there are a semiabelian varieties for which these conditions are an open question.

Summarizing Algebraic/Model Theoretic Consequences

Assume WCGH

$T_A + \Lambda = \mathcal{Z}^N$ is categorical in all $\aleph_n$
iff
A satisfies these algebraic formulation (like thumbtack).

WGCH versus ZFC

ZFC

Quasiminimal-excellence gives upward categoricity transfer.
So algebraic conditions imply categoricity.
Excellence gives full categoricity transfer.

WGCH

Categoricity up to $\aleph_\omega$ implies excellence.
Thus categoricity implies algebraic conditions.

Summary

We have argued that WGCH

1. accords with our intuitions about cardinal arithmetic
2. implies that countable models control behavior in arbitrary cardinalities
3. has mathematically important consequences

(It is worth noting that conditions 2) and 3) only use WGCH below $\aleph_\omega$.)