Using Set theory in model theory
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Today’s Topics
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2. Perhaps the hypothesis is eliminable
   A. The combinatorial hypothesis might be replaced by a more subtle argument.
      E.G. Ultrapowers of elementarily equivalent models are isomorphic.
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   A. The combinatorial hypothesis might be replaced by a more subtle argument.
      E.G. Ultrapowers of elementarily equivalent models are isomorphic
   B. The conclusion might be absolute
      The elementary equivalence proved in the Ax-Kochen-Ershov theorem
Using Extensions of ZFC in Model Theory

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      E.G. Ultrapowers of elementarily equivalent models are isomorphic
   B. The conclusion might be absolute
      The elementary equivalence proved in the Ax-Kochen-Ershov theorem
3. Consistency may imply truth.
“... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable.”

Gerald Sacks, 1972

See also the Vaananen article in Model Theoretic Logic volume
Which ‘Central Notions’?

Chang’s two cardinal theorem (morasses)
‘Vaughtian pair is absolute’
saturation is not absolute
Aside: For aec, saturation is absolute below a categoricity cardinal.
**Classification Theory**

**Crucial Observation**

The stability classification is absolute.

**Fundamental Consequence**

Crucial properties are provable in ZFC for certain classes of theories.

1. All stable theories have full two cardinal transfer.
2. There are saturated models exactly in the cardinals where the theory is stable.

But this is for FIRST ORDER theories.
Geography

$L_{\omega,\omega} \subset L_{\omega_1,\omega} \subset L_{\omega_1,\omega}(Q) \subset \text{anal.pres}.\text{AEC} \subset \text{AEC}$.

In a central case explained below

Extensions of ZFC are used for $L_{\omega_1,\omega}$.

Extensions of ZFC are proved necessary for $L_{\omega_1,\omega}(Q)$.
Two notions of ‘use’

1. Some model theoretic results ‘use’ extensions of ZFC
2. Some model theoretic results are provable in ZFC, using models of set theory.

This Talk

1. A quick statement of some results of the first kind
2. Discussion of several examples of the second method.
Theorem: \((2^{\lambda} < 2^{\lambda^+})\) (Shelah)

Suppose \(\lambda \geq LS(K)\) and \(K\) is \(\lambda\)-categorical. For any Abstract Elementary class, if amalgamation fails in \(\lambda\) there are \(2^{\lambda^+}\) models in \(K\) of cardinality \(\lambda^+\).

Is \(2^{\lambda} < 2^{\lambda^+}\) needed?
Is $2^\lambda < 2^{\lambda^+}$ needed?

1. $\lambda = \aleph_0$:
   a. Definitely not provable in ZFC: There are $L(Q)$-axiomatizable examples
      i. Shelah: many models with CH, $\aleph_1$-categorical under MA
      ii. Koerwien-Todorcevic: many models under MA, $\aleph_1$-categorical from PFA.
   b. Independence Open for $L_{\omega_1,\omega}$

2. Grossberg and VanDieren have announced the AEC analog in larger $\lambda$ using the generalized Martin’s Axiom.
A simple Problem

Let $\phi$ be a sentence of $L_{\omega_1,\omega}$.

Question

Is the property $\phi$ has an uncountable model absolute?
Fact: Easy for complete sentences

If $\phi$ is a complete sentence in $L_{\omega_1,\omega}$, $\phi$ has an uncountable model if and only if there exist countable $M \not\subseteq \omega_1, \omega$ $N$ which satisfy $\phi$.

This property is $\Sigma^1_1$ and done by Shoenfield absoluteness.

Note: $L_{\omega_1,\omega}$ satisfies downward Löwenheim-Skolem for sentences but not for theories.
Fly in the ointment

There are uncountable models that have no $L_{\omega_1,\omega}$-elementary submodel.

E.g. any uncountable model of the first order theory of infinitely many independent unary predicates $P_i$.

So the sentence saying every finite Boolean combination of the $P_i$ is non-empty has an uncountable model and our obvious criteria does not work.

Note that if we add the requirement that each type is realized at most once, then every model has cardinality $\leq 2^{\aleph_0}$. 
Given a sentence $\phi$. Let $L^*$ be the minimal countable fragment of $L_{\omega_1,\omega}$ containing $\phi$.
Suppose $M \prec_{L^*} N$, $M \neq N$.

Does there exist a proper extension $N'$ of $N$ with $N \prec_{L^*} N'$? If so we have an absolute characterization of $\phi$ has a uncountable model.
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Does there exist a proper extension $N'$ of $N$ with $N \prec_{L^*} N'$?

If so we have an absolute characterization of $\phi$ has a uncountable model.

BUT NO! Asserted by Gregory; example found by Johnson, Knight, Ocasio, VanDenDriessche this Fall.
Definition

1. A \( \tau \)-structure \( M \) is \( L^* \)-small for \( L^* \) a countable fragment of \( L_{\omega_1,\omega}(\tau) \) if \( M \) realizes only countably many \( L^*(\tau) \)-types (i.e. only countably many \( L^*(\tau) \)-\( n \)-types for each \( n < \omega \)).

2. A \( \tau \)-structure \( M \) is called small or \( L_{\omega_1,\omega} \)-small if \( M \) realizes only countably many \( L_{\omega_1,\omega}(\tau) \)-types.
Fact

Each small model satisfies a Scott-sentence, a complete sentence of $L_{\omega_1,\omega}$. 
Larson’s characterization

Given a sentence \( \phi \) of \( L_{\omega_1,\omega}(aa) \),
the existence of a model of size \( \aleph_1 \) satisfying \( \phi \)
is equivalent to
the existence of a countable model of \( \text{ZFC}^\circ \) containing
\( \{\phi\} \cup \omega \) which thinks there is a model of size \( \aleph_1 \) with a
member satisfying \( \phi \).

This property is \( \Sigma^1_1 \) and done by Shoenfield absoluteness.
Larger Cardinals

It is easy to see that there are sentences of $L_{\omega_1,\omega}$ such that the existence of a model in $\aleph_2$ depends on the continuum hypothesis. S. Friedman and M. Koerwien have shown.

Assume GCH (and large cardinals for independence of the Kurepa hypothesis)

1. For any $\alpha \in \omega_1 - \{0, 1, \omega\}$ there is a sentence $\phi_\alpha$ such that the existence of a model in $\aleph_\alpha$ is not absolute.

2. For $\aleph_3$, there is a complete such sentence.
Deja vu

The really basic proof

Karp (1964) had proved completeness theorems for $L_{\omega_1, \omega}$, and Keisler (late 60’s/ early 70’s) for $L_{\omega_1, \omega}(Q)$, $L_{\omega_1, \omega}(aa)$. 
Deja vu

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The rest of the talk illustrates the advantages of missing the ‘obvious’ argument.
Let $\phi$ be a $\tau$-sentence in $L_{\omega_1,\omega}(Q)$ such that it is consistent that $\phi$ has a model. Let $\mathcal{A}$ be the countable model of set theory, containing $\phi$, that thinks $\phi$ has an uncountable model.

Construct $\mathcal{B}$, an uncountable model of set theory, which is an elementary extension of $\mathcal{A}$ such that $\mathcal{B}$ is correct about uncountability. Then the model of $\phi$ in $\mathcal{B}$ is actually an uncountable model of $\phi$. 
How to build $B$

**MT** Iterate a theorem of Keisler and Morley (refined by Hutchinson).

**ST** Iterations of ‘special’ ultrapowers.

$\text{ZFC}^{\circ}$ denotes a sufficient subtheory of ZFC for our purposes.
How to build $B$

The main technical tool is the iterated generic elementary embedding induced by the nonstationary ideal on $\omega_1$, which we will denote by $\text{NS}_{\omega_1}$.

**The ultrafilter**

Forcing with the Boolean algebra $(\mathcal{P}(\omega_1)/\text{NS}_{\omega_1})^M$ over a ZFC model $M$ gives rise to an $M$-normal ultrafilter $U$ on $\omega_1^M$ (i.e., every regressive function on $\omega_1^M$ in $M$ is constant on a set in $U$).
The Ultrapower

Given such $M$ and $U$, we can form the generic ultrapower $\text{Ult}(M, U)$, which consists of all functions in $M$ with domain $\omega_1^M$,

where for any two such functions $f$, $g$, and any relation $R$ in $\{=, \in\}$, $fRg$ in $\text{Ult}(M, U)$ if and only if $\{\alpha < \omega_1^M \mid f(\alpha)Rg(\alpha)\} \in U$.

Nota Bene

If $M$ is countable, $\text{Ult}(M, U)$ is countable.

By convention, we identify the well-founded part of the ultrapower $\text{Ult}(M, U)$ with its Mostowski collapse.
The Ultrapower is useful

**Fact**

Suppose that $M$ is a model of $\text{ZFC}^\circ$, and that $j : M \rightarrow \text{Ult}(M, U)$ is an elementary embedding derived from forcing over $M$ with $(\mathcal{P}(\omega_1)/\text{NS}_{\omega_1})^M$. Then for all $x \in M$, $j(x) = j[x]$ if and only if $x$ is countable in $M$.

That is $\text{Ult}(M, U)$ increases exactly the sets that $M$ thinks are uncountable.
Definition

Let $M$ be a model of $\text{ZFC}^\circ$ and let $\gamma$ be an ordinal less than or equal to $\omega_1$. An iteration of $M$ of length $\gamma$ consists of models

$$M_\alpha : (\alpha \leq \gamma),$$

sets

$$G_\alpha : (\alpha < \gamma),$$

and a commuting family of elementary embeddings

$$j_{\alpha\beta} : M_\alpha \rightarrow M_\beta : (\alpha \leq \beta \leq \gamma)$$

such that the successor stages are the ultrapowers just discussed.
What is this good for?

Fact

Suppose that $M$ is a model of $\text{ZFC}^\circ$, and that $M_{\omega_1}$ is the final model of an iteration of $M$ of length $\omega_1$. Then for all $x \in M_{\omega_1}$, $M_{\omega_1} \models \text{“} x \text{ is uncountable} \text{”}$ if and only if $\{ y \mid M_{\omega_1} \models x \in y \}$ is uncountable.

So consistent sentences of $L_{\omega_1,\omega}(Q)$ are provable.

One can also make $M_{\omega_1}$ correct about stationarity, extending the absoluteness results to $L_{\omega_1,\omega}(aa)$. 
Many Iterations

Remark

We emphasize that for any countable model $M$ of $\text{ZFC}^\circ$ there are $2^{\aleph_0}$ many $M$-generic ultrafilters for $(\mathcal{P}(\omega_1)/\text{NS}_{\omega_1})^M$.

It follows that there are $2^{\aleph_1}$ many iterations of $M$ of length $\omega_1$. 
Really distinct iterations

**Theorem (Larson)**

If $M$ is a countable model of $\text{ZFC}^\circ + \text{MA}_{\aleph_1}$ and

$$
\langle M_\alpha, G_\alpha, j_\alpha, \gamma : \alpha \leq \gamma \leq \omega_1, \rangle
$$

and

$$
\langle M'_\alpha, G'_\alpha, j'_\alpha, \gamma : \alpha \leq \gamma \leq \omega_1, \rangle
$$

are two distinct iterations of $M$, then

$$
\mathcal{P}(\omega)^{M_{\omega_1}} \cap \mathcal{P}(\omega)^{M'_{\omega_1}} \subset M_\alpha,
$$

where $\alpha$ is least such that $G_\alpha \neq G'_\alpha$.

$G_\alpha$ not defined for $\alpha = \omega_1$. 
The Model Theory

Theorem: (Keisler, new proof Larson)

Let $F$ be a countable fragment of $L_{\omega_1,\omega}(aa)$. If there exists a model of cardinality $\aleph_1$ realizing uncountably many $F$-types, there exists a $2^{\aleph_1}$-sized family of such models, each of cardinality $\aleph_1$ and pairwise realizing just countably many $F$-types in common.

Corollary (Shelah using ch)

If a sentence in $L_{\omega_1,\omega}$ has less than $2^{\aleph_1}$ models in $\aleph_1$ then it is (syntactically) $\omega$-stable.

CH used twice.
Sketching New Proof:

Constructing many models

Let $N$ be a model of cardinality $\aleph_1$ realizing uncountably many $F$-types, let $X$ be a countable elementary submodel of $H((2^{\aleph_1})^+)\text{containing } \{N\}$ and the transitive closure of $\{F\}$. Let $M$ be the transitive collapse of $X$, and let $N_0$ be the image of $N$ under this collapse.

Build a tree of generic ultrapower iterates of $M'$ giving rise to $2^{\aleph_1}$ many distinct iterations of $M'$, each of length $\omega_1$.

Since $F$-types can be coded by reals using an enumeration of $F$ in $M$, the images of $N_0$ under these iterations will pairwise realize just countably many $F$-types in common.
ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónsson:

A class of $L$-structures, $(K, \preceq_K)$, is said to be an abstract elementary class: AEC if both $K$ and the binary relation $\preceq_K$ are closed under isomorphism plus:

1. If $A, B, C \in K$, $A \preceq_K C$, $B \preceq_K C$ and $A \subseteq B$ then $A \preceq_K B$;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$. 
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2. Closure under direct limits of $\preccurlyeq_K$-chains;

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1. If $A, B, C \in K$, $A \prec_K C$, $B \prec_K C$ and $A \subseteq B$ then $A \prec_K B$;
2. Closure under direct limits of $\prec_K$-chains;
3. Downward Löwenheim-Skolem.

Examples

First order and $L_{\omega_1,\omega}$-classes $L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$. 
Definition

An abstract elementary class $\mathcal{K}$ with Löwenheim number $\aleph_0$ is **analytically presented** if the set of countable models in $\mathcal{K}$, and the corresponding strong submodel relation $\prec_{\mathcal{K}}$, are both analytic.
### Context

**Fact**

Analytically presented $K$ is the same as a $PC_G(\aleph_0, \aleph_0)$ class:

reducts of models a countable first order theory in an expanded vocabulary which omit a countable family of types

**AKA:**

1. Keisler: $PC_\delta$ over $L_{\omega_1,\omega}$
2. Shelah: $PC(\aleph_0, \aleph_0)$, $\aleph_0$-presented
Groupable partial orders (Jarden varying Shelah)

Let \((K, \prec)\) be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order (equivalently admits a group structure) with \(M \prec N\) if \(M \subseteq N\) and no component is extended.

This AEC is analytically presented.
Add a binary function and say it is a group on each component.
But it has \(2^{\aleph_1}\) models in \(\aleph_1\) and \(2^{\aleph_0}\) models in \(\aleph_0\).

Recall: this ‘is’ the pseudo-elementary counterexample to Vaught’s conjecture.
Galois Types

Let $M \prec_K N_0$, $M \prec_K N_1$, $a_0 \in N_0$ and $a_1 \in N_1$ realize the same **Galois Type** over $M$ iff there exist a structure $N \in K$ and strong embeddings $f_0 : N_0 \to N$ and $f_1 : N_1 \to N$ such that $f_0|M = f_1|M$ and $f_0(a_0) = f_1(a_1)$.

Realizing the same Galois type (over countable models) is an equivalence relation

$$E_M$$

if $K_{\aleph_0}$ satisfies the amalgamation property.
The Monster Model

If an Abstract Elementary Class has the amalgamation property and the joint embedding property for models of cardinality at most $\aleph_0$ and has at most $\aleph_1$-Galois types over models of cardinality $\leq \aleph_0$ then there is an $\aleph_1$-monster model $\mathbb{M}$ for $\mathcal{K}$ and Galois type of $a$ over a countable $M$ is the orbit of $a$ under the automorphisms of $\mathbb{M}$ which fix $M$. So $E_M$ is an equivalence relation on $\mathbb{M}$. 
Some stability notions

Definition

1. The abstract elementary class $(\mathcal{K}, \preceq)$ is said to be **Galois $\omega$-stable** if for each countable $M \in \mathcal{K}$, $E_M$ has countably many equivalence classes.

2. The abstract elementary class $(\mathcal{K}, \preceq)$ is **almost Galois $\omega$-stable** if for each countable $M \in \mathcal{K}$, no $E_M$ has a perfect set of equivalence classes.
Almost Galois Stable

Well-orders of type at most $\aleph_1$ under end-extension are an AEC where countable models have only $\aleph_1$ Galois types.
Galois equivalence is $\Sigma^1_1$

On an analytically presented AEC, having the same Galois type over $M$ is an analytic equivalence relation, $E_M$. So by Burgess’s theorem we have the following trichotomy.

**Theorem**

An analytically presented abstract elementary class $(K, \prec)$ is

1. Galois $\omega$-stable or
2. almost Galois $\omega$-stable or
3. has a perfect set of Galois types over some countable model

Known to Shelah but independently rediscovered by Larson/Baldwin
Theorem: (B/Larson)

Suppose that

1. $K$ is an analytically presented abstract elementary class;
2. $N$ is a $K$-structure of cardinality $\aleph_1$, and $N_0$ is a countable structure with $N_0 \prec_K N$;
3. $P$ is a perfect set of $E_{N_0}$-inequivalent members of $\omega^\omega$;
4. $N$ realizes the Galois types of uncountably many members of $P$ over $N_0$.

Then there exists a family of $2^{\aleph_1}$ many $K$-structures of cardinality $\aleph_1$, each containing $N_0$ and pairwise realizing just countably many $P$-classes in common.
Fact: Hyttinen-Kesala, Kueker

If a sentence in $L_{\omega_1,\omega}$, satisfying amalgamation and joint embedding, is almost Galois $\omega$-stable then it is Galois $\omega$-stable.

What about analytically presented?
The ‘groupable partial order’ is almost Galois stable

Let \((K, \prec)\) be the class of partially ordered sets such that each connected component is a countable 1-transitive linear order with \(M \prec N\) if \(M \subseteq N\) and no component is extended. Since there are only \(\aleph_1\)-isomorphism types of components this class is almost Galois \(\omega\)-stable.

This AEC is analytically presented.
Theorem: Shelah

If $K$ is analytically presented and some model of cardinality $\aleph_1$ is $L^*$-small for every countable $\tau$-fragment $L^*$ of $L_{\omega_1,\omega}$, then $K$ has an $L_{\omega_1,\omega}(\tau)$-small model $M'$ of cardinality $\aleph_1$. 
Getting small models II

Theorem: Baldwin/Shelah/Larson

If $K$ has a model in $\aleph_1$ that is not $L_{\omega_1,\omega}(\tau)$-small then

1. there are at least $\aleph_1$ complete sentences of $L_{\omega_1,\omega}(\tau)$ which are satisfied by uncountable models in $K$;
2. $K$ has uncountably many models in $\aleph_1$;
3. $K$ has uncountably many extendible models in $\aleph_0$.

Proof: Iterate the previous theorem.

Corollary: Baldwin/Shelah/Larson

Vaught’s conjecture is equivalent to Vaught’s conjecture for extendible models.

A countable model is extendible if it has an $L_{\omega_1,\omega}$-elementary extension.
Absoluteness of (almost) ω-stability

1. first order syntactic: $\Pi_1^1$
2. $L_{\omega_1,\omega}$-syntactic: $\Pi_1^1$
3. analytically presented AEC: Galois $\omega$-stable: perhaps $\Pi_4^1$
4. analytically presented AEC: almost Galois $\omega$-stable: $\Pi_2^1$
Absoluteness of $\aleph_1$-categoricity

1. $\aleph_1$-categoricity of a class $K$ defined in $L_{\omega_1,\omega}$ is absolute between models of set theory that satisfy any one of the following conditions.
   1. $K$ is $\omega$-stable;
   2. $K$ has arbitrarily large members and $K$ has amalgamation in $\aleph_0$;
   3. $2^{\aleph_0} < 2^{\aleph_1}$.


2. $\aleph_1$-categoricity of an analytically presented AEC $K$ is absolute between models of set theory in which $K$ is almost Galois $\omega$-stable, satisfies amalgamation in $\aleph_0$, and has an uncountable model.
Why is this absoluteness of $\aleph_1$-categoricity true for AEC?

**Fact**

Suppose that $\mathbf{K}$ is an analytically presented AEC. Then the following statements are equivalent.

1. There exist a countable $M \in \mathbf{K}$ and an $N \in \mathbf{K}$ of cardinality $\aleph_1$ such that:
   - $M \prec_\mathbf{K} N$;
   - the set of Galois types over $M$ realized in $N$ is countable;
   - some Galois type over $M$ is not realized in $N$.

2. There is a countable model of $\text{ZFC}^\circ$ whose $\omega_1$ is well-founded and which contains trees on $\omega$ giving rise to $\mathbf{K}$, $\prec_\mathbf{K}$ and the associated relation $\sim_0$, and satisfies statement 1.
Summary

1. The set theoretic method provides a uniform method for studying models of various infinitary logic.

2. We introduced analytically presented AEC and showed:
   i. Extended Keisler’s few models implies $\omega$-stability theorem to this class.
   ii. Assuming countably many models in $\aleph_1$: Almost Galois $\omega$-stable implies Galois $\omega$-stable.
   iii. $\aleph_1$-categoricity absolute for Almost Galois $\omega$-stable with amalgamation.