Perspectives on AEC's
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Abstract Elementary Classes

An abstract elementary class is a concrete category.

Grossberg and Shelah noted a category theoretic interpretation in 1983. The formulation that follows is basically due to Kirby.

The Category Str

Objects in str are all structures in a vocabulary τ.
$\mathrm{str}(A, B)$ is the set of τ-embeddings from A into B.

The category $(\mathbf{K}, \mathrm{mod})$

The objects of $\mathbf{K}$ are a class of τ-structures

1. $\mathrm{mod}(A, B) \subseteq \mathrm{str}(A, B)$
2. If $f \in \mathrm{mod}(A, B)$ and $g \in \mathrm{mod}(C, B)$ and $h \in \mathrm{str}(A, C)$ with $f = gh$ then $h \in \mathrm{mod}(A, C)$.
3. mod is closed under direct limit and the mod-direct limit is the str-direct limit.
4. There is a cardinal $\mathrm{LS}(\mathbf{K})$ such if $f \in \mathrm{str}(A, B)$ and $B \in \mathbf{K}$, there is a $C \in \mathbf{K}$ with $|C| \leq |A| + \kappa$ and $g \in \mathrm{str}(A, C)$ and $h \in \mathrm{mod}(C, B)$ with $f = hg$.

Context

This reports works by a number of authors. Detailed proofs are in my monograph:
http://www2.math.uic.edu/ jbaldwin/model.html

Examples

1. π_2 first order theories and submodel
2. first order theories and elementary submodel
3. sentences of $L_{\omega_1,\omega}$ and L^*-elementary submodel.
4. sentences of $L_{\omega_1,\omega}(Q)$ (see next slide)
5. finite diagrams and elementary submodel
6. $^\perp N$ and $\prec_N$

$L_{\omega_1,\omega}(Q)$ as an AEC

Fails union of chains under natural elementary submodel

$\leq^*$

Require that 'small' definable sets do not grow.
Gives an AEC with LN $\aleph_1$.

Weak models

The class of 'weak models' with $\leq^*$ gives an AEC with LN $\aleph_0$.
But we added models and argument showing the existence of many models don't work (they might be weak models).

Approaches to $L_{\omega_1,\omega}(Q)$

1. ad hoc (there is a model in $\aleph_2$.)
2. Q-aec (Coppola)
3. frames
4. deal with sentences that are AEC
    1. for semantic reasons (Zilber/Kirby)
    2. for syntactic reasons (Caicedo)

Closure under direct limits of morphisms

More concrete version.

A3. If $\langle A_i : i < \delta \rangle$ is a continuous $\prec_{\mathbf{K}}$–increasing chain:

1. $\bigcup_{i<\delta} A_i \in \mathbf{K}$;
2. for each $j < \delta$, $A_j \prec_{\mathbf{K}} \bigcup_{i<\delta} A_i$;
3. if each $A_i \prec_{\mathbf{K}} M \in \mathbf{K}$ then $\bigcup_{i<\delta} A_i \prec_{\mathbf{K}} M$.

THE PRESENTATION THEOREM

Every AEC is a PCΓ

More precisely,

Theorem

If K is an AEC with Löwenheim number $\mathrm{LS}(\mathbf{K})$ (in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\mathbf{K})$), there is a vocabulary τ', a first order τ'-theory T' and a set of $2^{\mathrm{LS}(\mathbf{K})}$ τ'-types Γ such that:

$$\mathbf{K} = \{M' \restriction L : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' is an L'-substructure of N' where M', N' satisfy T' and omit Γ then
$M' \restriction L \prec_{\mathbf{K}} N' \restriction L$.

This theorem needs A.3.3.

What's so great about PCΓ

PCΓ gives:

- Ehrehfeucht Mostowski models;
- omitting types (for Galois types);
- can construct non-splitting extensions;
- key to finding showing a sentence of $L_{\omega_1,\omega}(Q)$ that is categorical in $\aleph_1$ has a model in $\aleph_2$.

But $PC\Gamma$ classes may not be closed under unions of chains and there even a PC-class with both the categoricity and non-categoricity spectrum cofinal in the cardinals.

AMALGAMATION PROPERTY

The class **K** satisfies the λ-*amalgamation property* if for any situation with $A, M, N \in \mathbf{K}_\lambda$:

$$\begin{tikzcd} & N \\ A \arrow[ur] \arrow[dr] & \\ & M \end{tikzcd}$$

there exists an N_1 such that

$$\begin{tikzcd} & N \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & N_1 \\ & M \arrow[ur] & \end{tikzcd}$$

AMALGAMATION PROPERTY – variants

K has the amalgamation property if each K_λ-does.
I.e. no cardinality restrictions.

Note that amalgamating over subsets rather than submodels is strictly stronger.

Finite Diagrams and Homogeneous Model Theory

Finite Diagrams

If T is a first order theory $\mathbf{K}$ is the class of models of T that omit a certain set Γ of finite-types over the empty set then $\mathbf{K}$ under first order elementary submodel is called a finite diagram.

Atomic Classes

If Γ is all non-principal types, $\mathbf{K}$ is called an atomic class.

Homogenous Model Theory

is the study of finite diagrams that admit amalgamation over SETS.

Connections to $L_{\omega_1,\omega}$

Every sentence of $L_{\omega_1,\omega}$ can be regarded (class of models is isomorphic) as a finite diagram.

Definition

A sentence ψ in $L_{\omega_1,\omega}$ is called *complete* if for every sentence ϕ in $L_{\omega_1,\omega}$, either $\psi \models \phi$ or $\psi \models \neg\phi$.

Every complete-sentence can be regarded as an atomic class.

Non-homogeneous examples I

There is a finite diagram with:

1. joint embedding,
2. in K_λ for uncountable λ
3. categorical in all uncountable λ
4. but amalgamation of countable models fails.

Under WGCH there can be no such atomic class.

Non-homogeneous examples II

There are $L_{\omega_1,\omega}$ sentences that are categorical in all powers, and do not satisfy set amalgamation.

1. Marcus
2. Knight
3. Zilber: covers of algebraic groups
4. Zilber: psuedoexponentiation ($L_{\omega_1,\omega}(Q)$)

Model Homogeneity

Definition

M is μ-model homogenous if for every $N \prec_{\mathbf{K}} M$ and every $N' \in \mathbf{K}$ with $|N'| < \mu$ and $N \prec_{\mathbf{K}} N'$ there is a $\mathbf{K}$-embedding of N' into M over N.

To emphasize, this differs from the homogenous context because the N must be in $\mathbf{K}$. It is easy to show:

Monster Model

Lemma

(jep) If M_1 and M_2 are μ-model homogenous of cardinality $\mu > \mathrm{LS}(\mathbf{K})$ then
$M_1 \approx M_2$.

Theorem

If $\mathbf{K}$ has the amalgamation property and $\mu *^{<\mu *} = \mu *$ and $\mu * \geq 2^{\mathrm{LS}(\mathbf{K})}$ then there is a model $\mathcal{M}$ of cardinality $\mu *$ which is $\mu *$-model homogeneous.

GALOIS TYPES: Algebraic Form

Suppose **K** has the amalgamation property.

Definition

Let $M \in \mathbf{K}$, $M \prec_{\mathbf{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\mathbf{K}} N \prec_{\mathbf{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Galois vrs Syntactic Types

Syntactic types have certain natural locality properties.

- locality Any increasing chain of types has at most one upper bound;
- tameness two distinct types differ on a finite set;
- compactness an increasing chain of types has a realization.

The translations of these conditions to Galois types do not hold in general.

Tameness

Grossberg and VanDieren focused on the idea of studying 'tame' abstract elementary classes:

Definition

We say $\mathbf{K}$ is (χ, μ)-*tame* if for any $N \in \mathbf{K}$ with $|N| = \mu$ if $p, q, \in \mathbb{S}(N)$ and for every $N_0 \le N$ with $|N_0| \le \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose **K** has the amalgamation property.

K is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\mathbf{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut_N}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut_M}(\mathcal{M})$ with $\alpha(a) = b$.

Consequences of Tameness

Suppose **K** has arbitrarily large models and amalgamation.

Theorem (Grossberg-Vandieren)

If $\lambda > \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is λ^+-categorical and $(\lambda, < \infty)$-tame then $\mathbf{K}$ is categorical in all $\theta \geq \lambda^+$.

Theorem (Lessmann)

If $\mathbf{K}$ with $\mathrm{LS}(\mathbf{K}) = \aleph_0$ is $\aleph_1$-categorical and $(\aleph_0, \infty)$-tame then $\mathbf{K}$ is categorical in all uncountable cardinals

Two Examples that are not tame

1. 'Hiding the zero'
   For each $k < \omega$ a class which is $(\aleph_0, \aleph_{k-3})$-tame but not $(\aleph_{k-3}, \aleph_{k-2})$-tame. Baldwin-Kolesnikov (building on Hart-Shelah)
2. Coding EXT
   A class that is not $(\aleph_0, \aleph_1)$-tame.
   A class that is not $(\aleph_0, \aleph_1)$-tame but is $(2^{\aleph_0}, \infty)$-tame. (Baldwin-Shelah)

Syntactic not Galois

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1,\omega}$ sentence ϕ_k such that:

1. ϕ_k has the disjoint amalgamation property;
2. Syntactic types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
3. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.

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3. But there are syntactic types over models of size $\aleph_{k-3}$ that split into $2^{\aleph_{k-3}}$-Galois types.
4. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
5. ϕ_k is not $\aleph_{k-2}$-Galois stable;
6. But for $m \leq k-3$, ϕ_k is $\aleph_m$-Galois stable;
7. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

Fundamental Construction I

(Baldwin, Lachlan, Marker)

Let G be an expansion of a group and let π map X onto G.

Add to the language a binary function $t : G \times X \to X$ for the fixed-point free action of G on $\pi^{-1}(g)$ for each $g \in G$.

That is, we represent $\pi^{-1}(g)$ as $\{ga : g \in G\}$ for some a with $\pi(a) = g$. This action of G is strictly 1-transitive. This guarantees that each fiber has the same cardinality as G.

π guarantees the number of fibers is the same as $|G|$.

Since there is no interaction among the fibers, categoricity in all uncountable powers follows if G is categorical.

Consequence

This gives us groups which are categorical but not almost strongly minimal or even almost quasi-excellent.

If G is the collection of maps from I into Z_2 with finite support, the structure (I, G, Z_2) is quasiminimal excellent and not first order.

Fundamental Construction II

(Hart-Shelah, Baldwin-Kolesnikov)

vocabulary L': unary predicates I, K, G, G^*, H, H^*;
a binary function e_G taking $G \times K$ to H;
a function π_G mapping G^* to K,
a function π_H mapping H^* to K,
a 4-ary relation t_G on $K \times G \times G^* \times G^*$,
a 4-ary relation t_H on $K \times H \times H^* \times H^*$.

vocabulary L: Add a $(k+1)$-ary relation Q on $(G^*)^k \times H^*$.

Key points

$K = [I]^k$;

G is finite support functions from K to G.

t_G and t_H are the actions on G^* and H^* as before.

The Crux

Q is a $(k+1)$-ary relation on $(G^*)^k \times H^*$.

For all $\gamma_1, \ldots, \gamma_k \in G$ and all $\delta \in H$ we have

$$Q((u_1, x_1), \ldots, (u_k, x_k), (u_{k+1}, x_{k+1}))$$
$$\Leftrightarrow Q((u_1, x_1 + \gamma_1), \ldots, (u_k, x_k + \gamma_k), (u_{k+1}, x_{k+1} + \delta))$$

if and only if $\gamma_1(u_{k+1}) + \cdots + \gamma_k(u_{k+1}) + \delta = 0 \mod 2$.

Tameness gained

Baldwin-Shelah (Goodrick)

Theorem

There is an AEC with the amalgamation property in a countable language with Löwenheim-Skolem number $\aleph_0$ *which is not* $(\aleph_0, \aleph_1)$*-tame but is* $(2^{\aleph_0}, \infty)$*-tame.*

Whitehead Groups

Definition

We say A is a *Whitehead group* if $Ext(A, \mathcal{Z}) = 0$. That is, every short exact sequence

$$0 \rightarrow \mathcal{Z} \rightarrow H \rightarrow A \rightarrow 0,$$

splits or in still another formulation, H is the direct sum of A and $\mathcal{Z}$.

Side question: Under V=L, Whitehead groups are free; hence PCΓ. What about in ZFC?

$^{\perp}N$

Definition

1. $^{\perp}N = \{A : Ext^i(A, N) = 0 : i < \omega\}$
2. For $A \subseteq B$ both in $^{\perp}N$, $A \prec_N B$ if $B/A \in {}^{\perp}N$.

Generalizes the class of Whitehead groups: $Ext(G, \mathcal{Z}) = 0$.

Synergy

Baldwin, Eklof, Trlifaj :

Theorem

1. *For any module N, if the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class then N is a cotorsion module.*
2. *For any R-module N, over a ring R, if N is a pure-injective module then the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class.*
3. *For an abelian group N, (module over a Dedekind domain R), the class $({}^{\perp}N, \prec_N)$ is an abstract elementary class if and only if N is a cotorsion module.*

An Interesting detail

We do not know exactly the rings for which the hypothesis of N cotorsion is sufficient for **A.3**(3). It is sufficient when R is a Dedekind domain:

Lemma

Let R be a Dedekind domain and N a module. Then the following are equivalent:

1. N is cotorsion;
2. $^{\perp}N = {}^{\perp}PE(N)$ where $PE(N)$ denotes the pure–injective envelope of N;
3. $^{\perp}N$ is closed under direct limits;
4. **A3**(3) holds for $(^{\perp}N, \prec_N)$.

3) concludes closure under limits of all homomorphisms.

What causes tameness?

1. $\mathrm{ga}-\mathrm{tp}(a/M) = \mathrm{tp}(\mathbf{a}/M)$ for a 'countable' **a**. ($^{\perp}N$, Abelian groups under pure substructure)
2. excellence (more precisely, the existence of a 'nonforking notion with stationary types and extension (Grossberg-Kolesnikov))
3. ????

Geometric Model Theory in $L_{\omega_1,\omega}$

Zilber showed:

Theorem

If in an $\aleph_1$-categorical theory T some (every) strongly minimal set has trivial geometry then T is almost strongly minimal?

Is the analog true in $L_{\omega_1,\omega}$ with almost quasiminimal excellence replacing almost strongly minimal ?

Problems on the 'Lower Infinite'

lower infinite = below $\beth_{\omega_1}$

1. Around Vaught's conjecture
2. Bounds on existence
3. Bounds on categoricity

Around Vaught's conjecture

The number of models in $\aleph_1$: $L_{\omega,\omega}$

Theorem

If a first order theory is a counterexample to the Vaught conjecture then it has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

Proof outline

This is easy from two difficult theorems:

Theorem (Shelah)

If a first order T is not ω-stable T has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

This argument uses many descriptive set theoretic techniques. See Shelah's book [?] or Baldwin's paper [?].

Theorem (Shelah)

An ω-stable first order theory satisfies Vaught's conjecture.

Does the previous theorem extend to $L_{\omega_1,\omega}$?

Keisler showed:

Theorem

For any $L_{\omega_1,\omega}$-sentence ψ and any fragment L^* containing ψ, if ψ has fewer than $2^{\aleph_1}$ models of cardinality $\aleph_1$ then for any $M \models \psi$ of cardinality $\aleph_1$, M realizes only countably many L^*-types over the empty set

Shelah observed that Theorem **??** implies:

Fact

($2^{\aleph_0} < 2^{\aleph_1}$) If a complete sentence $\psi \in L_{\omega_1,\omega}$ is not ω-stable it has $2^{\aleph_1}$ models of cardinality $\aleph_1$.

Develop ω-stability for finite diagrams

Even assuming $(2^{\aleph_0} < 2^{\aleph_1})$:

Few models in $\aleph_1$ does not imply amalgamation in $\aleph_0$.

Does few models in $\aleph_1$ imply ω-stability in any reasonable sense.

Does VC hold for ω-stable sentences in $L_{\omega_1,\omega}$? For excellent classes?

This is meaningless for complete sentences. But it fits the context of Hytinnen-Kesala.

Characterizing $\aleph_1$

Theorem (Hjorth)

For every $\alpha < \omega_1$ there is a sentence in $L_{\omega_1,\omega}$ whose maximal model has cardinality $\aleph_\alpha$.

Proof outline for $\aleph_1$

vocabulary: S_n binary; R_k is $k+2$-ary.
A set of universal sentences guarantee that every model M satisfies:

1. The S_n are symmetric and partition $M^{[2]}$.
2. For all a, b for some n, $S_n(a, b)$.
3. $$\bigwedge_m [R_k(a, b, c_1, \ldots c_k) \rightarrow$$ $$(S_m(d, a_0) \wedge S_m(d, a_1) \rightarrow d \in \{c_1, \ldots c_k\})]$$

$f(a, b) = n$ if $S_n(a, b)$ maps M^2 into ω.
In the generic model for each a, b there is finite C:
$f(c, a) = f(c, b)$ iff $c \in C$.

Consequences

No model in $\aleph_2$

If $|N| = \aleph_2$ and $M \prec N$, $|M| = \aleph_1$:

For each $a, b \in M$ there is finite $C \subset M$: $f(c, a) = f(c, b)$ iff $c \in C$ (for all $c \in N$!).

So if $e \in N - M$, for every $a, b \in M$, $f(e, a) \neq f(e, b)$. That is $f(e, -)$ is a 1-1 map from N into ω.

Fraïssé construction and $L_{\omega_1,\omega}$

A model in $\aleph_1$

The basic Fraïssé construction yields an atomic model; so the atomic models of its first order theory are axiomatized in $L_{\omega_1,\omega}$. Marker (mainly) and I observed:

For any Fraïssé construction with disjoint amalgamation: The generic has a proper atomic elementary extension and so there is an uncountable atomic model.

Prove or Disprove

Theorem (Shelah technique)

If an atomic class is ω-stable and has a model in $\aleph_1$ it has one in $\aleph_2$.

Modify Hjorth's construction by replacing Fraïssé with Hrushovski to get:

An ω-stable atomic class characterizing $\aleph_2$.

Downward Categoricity

Theorem (Shelah)

If **K** *is categorical in* λ^+ *and satisfies amalgamation and joint embedding with* $\lambda \geq H_2$ *then* **K** *is categorical on* $[H_2, \lambda)$.

What about non-successor cardinals for the hypothesis?

What is the best lower bound?
Is it $\aleph_1, \aleph_\omega, \beth_{\omega_1}, H_2 = \beth_{(\beth_{\omega_1})^+}$?

Necessity of hypotheses

Fact. If $\mathbf{K}$ has amalgamation and is categorical in κ above H_1, then $\mathbf{K}_{\geq\kappa}$ has jep.
But it is easy to construct sentences ϕ_α of $L_{\omega_1,\omega}$ that are categorical in κ iff $\kappa \geq \beth_a lpha$.