Completeness and Categoricity: Formalism as a mathematical tool

John T. Baldwin

November 1, 2012

Outline

Two Questions

Detlefsen asked

(A) Which view is the more plausible—that theories are the better the more nearly they are categorical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?

(B) Is there a single answer to the preceding question? Or is it rather the case that categoricity is a virtue in some theories but not in others? If so, how do we tell these apart, and how do we justify the claim that categoricity is or would be a virtue just in just former?

goals

What is virtue?

I take ‘better’ in this context to mean the property of theories has more mathematical consequences for the theory.

Goals Matter

Two motives of Axiomatization

1. Understand a single significant structure such as $(N, +, \cdot)$ or $(R, +, \cdot)$.
2. Find the common characteristics of a number of structures: theories of the second sort include groups, rings, fields etc.

But the theories of real closed fields and of algebraically closed fields arise from both motives.
CONCLUSION: There is not a single answer to question A.
But we will argue that usually the answer is that it is better to be closer to categorical in power.

Questions

Terminology

A theory T is a collection of sentences in some logic $\mathcal{L}$.

E.G. first order, second order, $L_{\omega_1,\omega}$ and $L_{\omega_1,\omega}(Q)$.)
For simplicity, we will assume that T is consistent (has at least one model) and has only infinite models.

T is categorical if it has exactly one model (up to isomorphism).

T is categorical in power κ if it has exactly one model in cardinality κ.

T is totally categorical if it is categorical in every infinite power.

Complete – the ultimate homonym

A deductive system is complete if for every ϕ

$$\vdash \phi \text{ if and only if } \models \phi$$

A theory T in a logic $\mathcal{L}$ is (semantically) complete if for every sentence $\phi \in \mathcal{L}$

$$T \models \phi \text{ or } T \models \neg\phi$$

Note that for any structure M any logic $\mathcal{L}$,

$$\mathrm{Th}_{\mathcal{L}}(M) = \{\phi \in \mathcal{L} : M \models \phi\}$$

is a complete theory.

Changing the question

I will argue
categoricity of a second order theory does not, by itself, shed any mathematical light on the categorical structure.

But categoricity in power for first order and infinitary logic yields significant structural information about models of theory.

This kind of structural analysis leads to a fruitful classification theory for complete first order theories. Indeed, the fewer the models, the better the structure theory that holds of models of the theory.

Choice of Logic matters

No first order theory is categorical.

There are important categorical second order theories

Second Order Categoricity - Examples

The second order axiom which imposes categoricity also explains the central property of the structure

1. Second order induction guarantees that arithmetic has order type ω.
2. Order completeness of the real numbers is the central point for developing analysis.

Aside: Much of the analysis of polynomials on the reals and complexes can be done on a first order basis.
E.g. Starchenko.

Second Order Categoricity- generalities I

Completeness does not imply categoricity

There are $2^{\aleph_0}$ theories and a proper class of structures.

Second Order Categoricity- generalities I

Completeness does not imply categoricity

There are $2^{\aleph_0}$ theories and a proper class of structures.

Categoricity implies Completeness

Obvious

Second Order Categoricity- generalities I

Completeness does not imply categoricity

There are $2^{\aleph_0}$ theories and a proper class of structures.

Categoricity implies Completeness

Obvious

Categoricity in power does not imply Completeness

The second order sentence 'I am a cardinal' is categorical (in ZFC) in every power.
Some cardinals are regular; some aren't.

Second Order Categoricity- generalities II

Sometimes Completeness implies categoricity

Marek-Magidor/Ajtai (V=L) The second order theory of a countable structure is categorical.

H. Friedman (V=L) The second order theory of a Borel structure is categorical.

Solovay (V=L) A recursively axiomatizable complete 2nd order theory is categorical.

Solovay/Ajtai It is consistent with ZFC that there is a complete finitely axiomatizable second order theory that is not categorical.

Ali Enayat has nicely orchestrated this discussion on FOM and Mathoverflow.

`http://mathoverflow.net/questions/72635/`

Second Order Categoricity- Summary

The specific axiomatization of central mathematical structures that are second order categorical can have important explanatory power.

Second Order Categoricity- Summary

The specific axiomatization of central mathematical structures that are second order categorical can have important explanatory power.

The general theory of categoricity of second order structures

1. doesn't show categoricity yields structural properties or indeed any similarities.
2. is intertwined with set theory.

Second Order Categoricity- Summary

The specific axiomatization of central mathematical structures that are second order categorical can have important explanatory power.

The general theory of categoricity of second order structures

1. doesn't show categoricity yields structural properties or indeed any similarities.
2. is intertwined with set theory.

The close connection of categoricity and completeness for second order logic partially explains the early 20th century difficulty in disentangling those two notions.

First Order Categoricity- generalities

Completeness does not imply categoricity

There are $2^{\aleph_0}$ theories and a proper class of structures.

Categoricity implies Completeness

Obvious

Categoricity in power implies Completeness

Use the upward and downward Löwenheim-Skolem theorems.

Our Argument

1. Categoricity in power implies strong structural properties of each categorical structure.
2. These structural properties can be generalized to all models of certain (syntactically described) complete first order theories.

STRONGLY MINIMAL

$a \in \mathrm{acl}(B)$ if $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.
A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

These two conditions assign a unique dimension which determines each model of T.
The complex field is strongly minimal.

$\aleph_1$-categorical theories

Strongly minimal set are the building blocks of structures whose first order theories are categorical in uncountable power.

Theorem (Morley/ Baldwin-Lachlan/Zilber) TFAE

1. T is categorical in one uncountable cardinal.
2. T is categorical in all uncountable cardinals.
3. T is ω-stable and has no two cardinal models.
4. Each model of T is prime over a strongly minimal set.
5. Each model of T can be decomposed by finite 'ladders'. Classical groups are first order definable in non-trivial categorical theories.

Item 3) implies categoricity in power is absolute.

Bourbaki on Axiomatization

Bourbaki wrote

> *Many of the latter (mathematicians) have been unwilling for a long time to see in axiomatics anything other else than a* futile logical hairsplitting not capable of fructifying any theory whatever.
>
> *This critical attitude can probably be accounted for by a purely historical accident.*
>
> *The first axiomatic treatments and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories,* i.e. *theories which are entirely determined by their complete systems of axioms;*

More Bourbaki

> *for this reason they could not be applied to any theory except the one from which they had been abstracted (quite contrary to what we have seen, for instance, for the theory of groups).*
>
> *If the same had been true of all other structures, the reproach of sterility brought against the axiomatic method, would have been fully justified.*

Bourbaki realizes but then forgets that the hypothesis of this last sentences is false.
Nor do they exploit the distinctions between first and second order logic.

Formalization as a mathematical tool

The study of complete first order theories provides a tool for understanding and proving theorems in everyday mathematics.

This study is enhanced by using syntactic properties to classify theories and find underlying reasons for mathematical theorems.

Formalization

A *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
    1. Specify a class[1] of well formed formulas.
    2. Specify truth of a formula from this class in a structure.
    3. Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

Item 2c) is the least important from our standpoint.

[1]For most logics there are only a set of formulas, but some infinitary languages have a proper class of formulas.

Bourbaki Again

Bourbaki distinguishes between ‘logical formalism’ and the ‘axiomatic method’.

‘We emphasize that it (logical formalism) is but one aspect of this (the axiomatic) method, indeed the least interesting one’.

Bourbaki Again

Bourbaki distinguishes between ‘logical formalism’ and the ‘axiomatic method’.

‘We emphasize that it (logical formalism) is but one aspect of this (the axiomatic) method, indeed the least interesting one’.

We reverse this aphorism:
The axiomatic method is but one aspect of logical formalism.

Bourbaki Again

Bourbaki distinguishes between ‘logical formalism’ and the ‘axiomatic method’.

‘We emphasize that it (logical formalism) is but one aspect of this (the axiomatic) method, indeed the least interesting one’.

We reverse this aphorism:
The axiomatic method is but one aspect of logical formalism.

And the foundational aspect of the axiomatic method is the least important for mathematical practice.

Two roles of formalization

1. Building a piece or all of mathematics on a firm ground specifying the underlying assumptions
2. When mathematics is organized by studying first order (complete) theories, syntactic properties of the theory induce profound similarities in the structures of models. These are tools for mathematical investigation.

Theories are important

The breakthroughs of classification theory as a tool for organizing mathematics come in several steps.

1. (complete) first order theories are important.

Mathematical Applications of Completeness

We give in some detail a striking example, (see the web site of Fields Medalist Terry Tao)
of the role of complete theories and formalization in proving a theorem in algebraic geometry.

Many more examples are in the paper: classification of division algebras over Real closed fields, definition of schemes over fields, Lefschetz principle, foundations of algebraic geometry

The Ax-Grothendieck Theorem

Theorem

Every injective polynomial map on an affine algebraic variety over $\mathcal{C}$ is surjective.

The model theoretic proof:

The condition is axiomatized by a family of 'for all -there exist' first order sentences ϕ_i (one for each pair of a map and a variety).

Such sentences are preserved under direct limit and the ϕ_i are trivially true on all finite fields. So they hold for the algebraic closure of F_p for each p (as it is a direct limit of finite fields).

Ax-Grothendieck proof continued

Note that $T = \mathrm{Th}(\mathcal{C})$, the theory of algebraically closed fields of characteristic 0, is axiomatized by a schema Σ asserting each polynomial has a root and stating for each p that the characteristic is not p.

Since each ϕ_i is consistent with every finite subset of Σ, it is consistent with Σ and so proved by Σ, since the consequences T of Σ form a complete theory.

Kazhdan summary

Kazhdan (Harvard/Hebrew University/ MacArthur Fellow; a mathematician specializing in representation theory) illuminates the key reason to study complete theories:

> *On the other hand, the Model theory is concentrated on gap between an abstract definition and a concrete construction. Let T be a complete theory. On the first glance one should not distinguish between different models of T, since all the results which are true in one model of T are true in any other model. One of main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty.*

Kazhdan continued

Different models of complete theories are of different flavors and support different intuitions. So an attack on a problem often starts which a choice of an appropriate model. Such an approach lead to many non-trivial techniques for constructions of models which all are based on the compactness theorem which is almost the same as the fundamental existence theorem.

On the other hand the novelty creates difficulties for an outsider who is trying to reformulate the concepts in familiar terms and to ignore the differences between models.

Classes of Theories are important

The breakthroughs of classification theory as a tool for organizing mathematics come in several steps.

1. (complete) first order theories are important.

Classes of Theories are important

The breakthroughs of classification theory as a tool for organizing mathematics come in several steps.

1. (complete) first order theories are important.
2. Classes of (complete) first order theories are important.

Mathematical Applications of the stability hierarchy

We quickly sketch the first order stability hierarchy and then

1. Show how it provides a new organization scheme for some mathematics.
2. List a few examples of mathematical applications of these tools.

Bourbaki again

Bourbaki has some beginning notions of combining the 'great mother-structures' (group, order, topology). They write:

'the organizing principle will be the concept of a hierarchy of structures, going from the simple to complex, from the general to the particular.'

But this is a vague vision. We now sketch a realization of a more sophisticated organization of parts of mathematics.

Properties of classes of theories (1970-present)

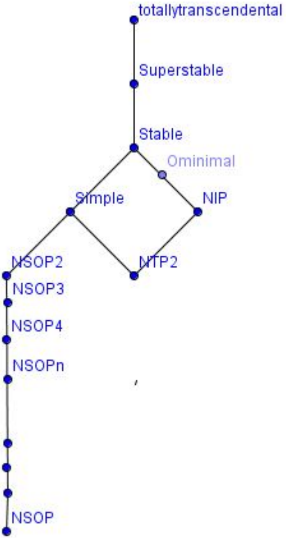
The Stability Hierarchy

Every complete first order theory falls into one of the following 4 classes.

1. ω-stable
2. superstable but not ω-stable
3. stable but not superstable
4. unstable

Crucially these classes are defined by 'syntactic' properties.

The stability hierarchy: diagram



The stability hierarchy: examples

ω-stable

Algebraically closed fields (fixed characteristic), differentially closed fields, complex compact manifolds

strictly superstable

$(\mathbb{Z}, +)$, $(2^\omega, +) = (Z_2^\omega, H_i)_{i<\omega}$,

strictly stable

$(\mathbb{Z}, +)^\omega$, separably closed fields,

Unstable theories: examples

If a first order theory is unstable it has the independence property or the strict order property.

The developing hierarchy of unstability

%beginblockunstable

1. nip: real closed fields, p-adically closed fields, real exponentiation
2. simple: random graph, ACF with automorphisms (ACFA)
3. both ip and strict order property: complex exponentiation,

Consequences: Main Gap

Shelah proved:

Main Gap

For every first order theory T, either

1. Every model of T is decomposed into a tree of countable models with uniform bound on the depth of the tree, or
2. The theory T has the maximal number of models in all uncountable cardinalities.

Consequences in core mathematics

1. o-minimality, Hardy's problem
2. Shelah: uniqueness of differential closures
3. Zilber's classification of 2-transitive groups
4. Hrushovski Mordell-Lang for function fields
5. interaction of '1-based' with arithmetic algebraic geometry.
6. Sela: All free groups on more than two generators are elementarily equivalent.
7. Denef-Van den Dries: rationality of Poincaire series by induction on quantifiers.
8. MacPherson-Steinhorn: asypmtotic classes, understanding the classification of simple groups.

Two roles of formalization

1. Building a piece or all of mathematics on a firm ground specifying the underlying assumptions
2. When mathematics is organized by studying first order (complete) theories, syntactic properties of the theory induce profound similarities in the structures of models. These are tools for mathematical investigation.

Axiom of Infinity and the stability hierarchy

Sentences with only infinite models:

1. infinite linear order
2. $f(x)$ is an injective function; exactly one element does not have a predecessor.
3. $t(x, y)$ is a pairing function

A *complete* axiom of infinity is a first order sentence ϕ such that

- the consequences of ϕ are a complete first order theory
- which has an infinite model.

It is easy to extend 1) linear order to a complete sentence; the others are more difficult.

Categorical Axioms of Infinity

Theorem: Zilber, Cherlin-Harrington-Lachlan

No first order sentence is categorical in all infinite cardinalities.
But such theories are quasi-finitely axiomatizable by a single sentence plus an 'infinity scheme' and there is detailed structure theory for both finite and infinite models.

Theorem:[Peretyatkin]

There is an $\aleph_1$ categorical first order sentence.
Peretyatkin was motivated by trying to capture a tiling problem but his example really seems to capture 'pairing'.

Open Question.

Is there a finitely axiomatizable strongly minimal set?