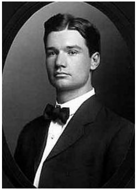
Completeness and Categoricity: Formalism as a mathematical tool

John T. Baldwin

April 24, 2013

Outline

American Postulate Theorists:

Huntington  E.H. Moore  R.L. Moore 

PROC AMS 1902

A COMPLETE SET OF POSTULATES FOR THE THEORY OF ABSOLUTE CONTINUOUS MAGNITUDE*
BY EDWARD V. HUNTINGTON

"The following paper presents a complete set of postulates or primitive propositions from which the mathematical theory of absolute continuous magnitude can be deduced."

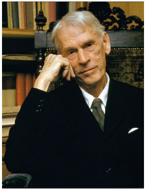
What does complete mean?

"The object of the work which follows is to show that these six postulates form a complete set; that is, they are
(I) consistent,
(II) sufficient,
(III) independent (or irreducible).

By these three terms we mean:

(I) there is at least one assemblage in which the chosen rule of combination satisfies all the six requirements;
(II) there is essentially only one such assemblage possible;
(III) none of the six postulates is a consequence of the other five."

1904: name changes, concept doesn't:

Oswald Veblen 

Following a suggestion of his University of Chicago Colleague, John Dewey,

Oswald Veblen renamed sufficiency as categoricity.

Both Huntington and Veblen have the modern notion of isomorphism.

Veblen proved the categoricity of a set of second order axioms for Euclidean Geometry.

Modern Terminology

A theory T is a collection of sentences in some logic $\mathcal{L}$.

E.G. first order, second order, $L_{\omega_1,\omega}$ and $L_{\omega_1,\omega}(Q)$.)

For simplicity, we will assume that T is consistent (has at least one model) and has only infinite models.

T is categorical if it has exactly one model (up to isomorphism).

Complete – the ultimate homonym

A logic $\mathcal{L}$ is deductively complete if there is deductive system such that for every ϕ

$$\vdash \phi \text{ if and only if } \models \phi.$$

A theory T in a logic $\mathcal{L}$ is semantically complete if for every sentence $\phi \in \mathcal{L}$

$$T \models \phi \text{ or } T \models \neg\phi.$$

Note that for any structure M any logic $\mathcal{L}$,

$$\mathrm{Th}_{\mathcal{L}}(M) = \{\phi \in \mathcal{L} : M \models \phi\}$$

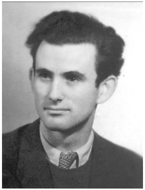
is a semantically complete theory.

3 intertwined notions

The distinction between

1. semantic completeness
2. categoricity
3. deductive completeness

was not really understood until the late 1920's.

Categoricity in Power 1954: Łoś 

T is categorical in power κ if it has exactly one model in cardinality κ.

T is totally categorical if it is categorical in every infinite power.

I assume that theories are closed under semantic consequence.

Detlefsen asked:

Detlefsen 

Question A

Which view is the more plausible—that theories are the better the more nearly they are categorical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?

Detlefsen also asked

Question B

Is there a single answer to the preceding question? Or is it rather the case that categoricity is a virtue in some theories but not in others? If so, how do we tell these apart, and how do we justify the claim that categoricity is or would be a virtue just in just former?

return

Goals Matter

Two motives of Axiomatization

1. Understand a single significant structure such as $(N, +, \cdot)$ or $(R, +, \cdot)$.
2. Find the common characteristics of a number of structures: theories of the second sort include groups, rings, fields etc.

CONCLUSION: There is not a single answer to question A.
But we will argue that usually the answer is that it is better to be closer to categorical in power.

Questions

What is virtue?

From the second standpoint it is better to take theories as closed under logical consequence.

What is virtue?

I take ‘ a virtuous property’ to be one which has significant mathematical consequences for a theory or its models.
Thus, a better property of theories has more mathematical consequences for the theory.

Changing the question

I will argue categoricity of a second order theory does not, by itself, shed any mathematical light on the categorical structure.

But categoricity in power for first order and infinitary logic yields significant structural information about models of theory.

This kind of structural analysis leads to a fruitful classification theory for complete first order theories.
Indeed, fewer models usually indicates a better structure theory for models of the theory.

Choice of Logic matters

No first order theory is categorical.

There are important categorical second order axiomatizations.

Second Order Categoricity - Examples

The second order axiom which imposes categoricity also explains the central property of the structure

1. Second order induction guarantees that arithmetic has order type ω.
2. Order completeness of the real numbers is the central point for developing analysis.

Second Order Categoricity- generalities I

Completeness does not imply categoricity

There are $2^{\aleph_0}$ theories and a proper class of structures.

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Categoricity in power does not imply Completeness

The second order sentence ‘I am a cardinal’ is categorical (in ZFC) in every power.
Some cardinals are regular; some aren’t.

Second Order Categoricity- generalities II

Sometimes Completeness implies categoricity

Marek-Magidor/Ajtai (V=L) The second order theory of a countable structure is categorical.

H. Friedman (V=L) The second order theory of a Borel structure is categorical.

Solovay (V=L) A recursively axiomatizable complete 2nd order theory is categorical.

Solovay/Ajtai It is consistent with ZFC that there is a complete finitely axiomatizable second order theory that is not categorical.

Ali Enayat has nicely orchestrated this discussion on FOM and Mathoverflow. `http://mathoverflow.net/questions/72635/categoricity-in-second-order-logic`

Second Order Categoricity- Summary

The specific axiomatization of central mathematical structures that are second order categorical can have important explanatory power.

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The general theory of categoricity of second order structures

1. doesn't show categoricity yields structural properties or indeed any similarities.
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The close connection of categoricity and completeness for second order logic partially explains the early 20th century difficulty in disentangling those two notions.

First Order Categoricity- generalities

Completeness does not imply categoricity

There are $2^{\aleph_0}$ theories and a proper class of structures.

Categoricity implies Completeness

Obvious

Categoricity in power implies Completeness

Use the upward and downward Löwenheim-Skolem theorems.

Our Argument

1. Categoricity in power implies strong structural properties of each categorical structure.
2. These structural properties can be generalized to all models of certain (syntactically described) complete first order theories.

GEOMETRIES

Definition. A pregeometry is a set G together with a 'dependence' relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \ X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.
A4. $cl(cl(X)) = cl(X)$
If points are closed the structure is called a geometry.

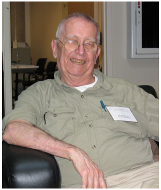
STRONGLY MINIMAL

$a \in \mathrm{acl}(B)$ if $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.
A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

These two conditions assign a unique dimension which determines each model of T.
The complex field is strongly minimal.

$\aleph_1$-categorical theories

Morley  Lachlan  Zilber 

Strongly minimal set are the building blocks of structures whose first order theories are categorical in uncountable power.

$\aleph_1$-categorical theories

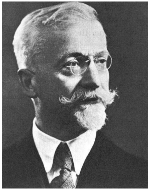
Theorem (Morley/ Baldwin-Lachlan/Zilber) TFAE

1. T is categorical in one uncountable cardinal.
2. T is categorical in all uncountable cardinals.
3. T is ω-stable and has no two cardinal models.
4. Each model of T is prime over a strongly minimal set.
5. Each model of T can be decomposed by finite 'ladders'. Classical groups are first order definable in non-trivial categorical theories.

Item 3) implies categoricity in power is absolute.

Any theory satisfying these properties has either one or $\aleph_0$ models of cardinality $\aleph_0$.

Bourbaki on Axiomatization:

Dieudonne  Bourbaki  Cartan 

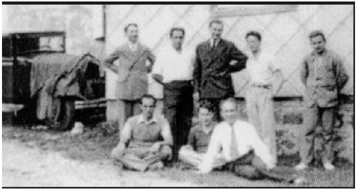
Bourbaki wrote:

> *Many of the latter (mathematicians) have been unwilling for a long time to see in axiomatics anything other else than a* futile logical hairsplitting not capable of fructifying any theory whatever.

More Bourbaki

This critical attitude can probably be accounted for by a purely historical accident.

The first axiomatic treatments and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories, i.e. *theories which are entirely determined by their complete systems of axioms; for this reason they could not be applied to any theory except the one from which they had been abstracted (quite contrary to what we have seen, for instance, for the theory of groups).*

More Bourbaki: Bourbaki 

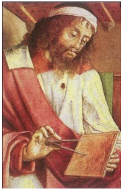
If the same had been true of all other structures, the reproach of sterility brought against the axiomatic method, would have been fully justified.

Bourbaki realizes but then forgets that the hypothesis of this last sentence is false.

They miss the distinctions between

1. axiomatization and theory
2. first and second order logic.

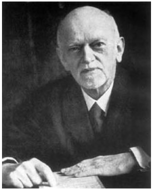
Euclid-Hilbert formalization 1900:

Euclid  Hilbert 

The Euclid-Hilbert (the Hilbert of the Grundlagen) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.

But the arguments and statements take place in natural language. For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski formalization 1956:

Hilbert  Gödel  Tarski 

In the Hilbert (the founder of proof theory)-Gödel-Tarski framework, logic is a mathematical subject.

There are explicit rules for defining a formal language and proof. Semantics is defined set-theoretically.

First order logic is complete. The theory of the real numbers is complete and easily axiomatized. The first order Peano axioms are not complete.

Formalization

Anachronistically, *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
   a Specify a class of well formed formulas.
   b Specify truth of a formula from this class in a structure.
   c Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

Item 2c) is the least important from our standpoint.

Formalization as a mathematical tool

The study of complete first order theories provides a tool for understanding and proving theorems in everyday mathematics.

This study is enhanced by using syntactic properties to classify theories and find underlying reasons for mathematical theorems.

Formalism freeness

This paper is a counterpoint to discussions of trends away from fully formalized theories in model theory.

Baldwin,J.T., *Formalization, Primitive Concepts, and Purity*, 2013, Review of Symbolic Logic.

Juliette Kennedy, *On formalism freeness*, 2013 , to appear Bulletin of Symbolic Logic

Bourbaki Again

Bourbaki distinguishes between ‘logical formalism’ and the ‘axiomatic method’.

‘We emphasize that it (logical formalism) is but one aspect of this (the axiomatic) method, indeed the least interesting one’.

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We reverse this aphorism:
The axiomatic method is but one aspect of logical formalism.

And the foundational aspect of the axiomatic method is the least important for mathematical practice.

Two roles of formalization

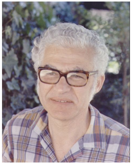
1. Building a piece or all of mathematics on a firm ground specifying the underlying assumptions
2. When mathematics is organized by studying first order (complete) theories, syntactic properties of the theory induce profound similarities in the structures of models. These are tools for mathematical investigation.

Theories are important

The breakthroughs of classification theory as a tool for organizing mathematics come in several steps.

1. (complete) first order theories are important.

Polynomial Maps:

Bailynicki-Birula  Rosenlicht 

Theorem: Bailynicki-Birula/Rosenlicht 1962

Every injective polynomial map on an affine algebraic variety over $\Re$ is surjective.

The proof uses basic real algebraic geometry and the homology theory of Borel and Haefliger.

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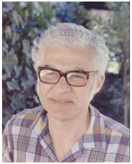
Bailynicki-Birula  Rosenlicht 

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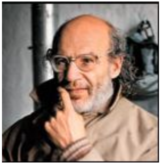
Theorem: Bailynicki-Birula/Rosenlicht 1962

Every injective polynomial map from k^n to k^n where k is algebraically closed is surjective.

The Ax-Grothendieck version:

Grothendieck  Ax 

Theorem: 1968, 1966

Every injective polynomial map on an affine algebraic variety over $\mathcal{C}$ is surjective.

The Ax model theoretic proof

The condition is axiomatized by a family of ‘for all -there exist’ first order sentences ϕ_i (one for each pair of a map and a variety).

Such sentences are preserved under direct limit and the ϕ_i are trivially true on all finite fields. So they hold for the algebraic closure of F_p for each p (as it is a direct limit of finite fields).

Ax proof continued

Note that $T = \mathrm{Th}(\mathcal{C})$, the theory of algebraically closed fields of characteristic 0, is axiomatized by a schema Σ asserting each polynomial has a root and stating for each p that the characteristic is not p.

Since each ϕ_i is consistent with every finite subset of Σ, it is consistent with Σ and so proved by Σ, since the consequences T of Σ form a complete theory.

$\in \mathfrak{K}$

$\in \mathcal{C}$

Ax proof works because $\mathcal{C}$ is an ultraproduct of locally finite fields.

But Not for Reals

No model theoretic proof for $\Re$

$\Re$ is not such an ultraproduct because $(\exists z)(x + z^2 = y)$ defines a linear order of the universe. So this must be true for a family of fields indexed by a member of the ultrafilter.
But, no locally finite field can be linearly ordered.

Even more, the natural variant for o-minimal theories that Nash functions (ie. semialgebraic + real analytic functions) satisfy the condition fails. $f(x) = \frac{x}{(x^2+1)^{\frac{1}{2}}}$ is a one-to-one map from R to $(-1, 1)$.
So there are too many definable sets for the mathematical purpose.

Mathematical Applications of Completeness

We gave in some detail a striking example, (see also
the web site of Terry Tao.)
of the role of complete theories and formalization in proving a theorem in algebraic geometry.

Many more examples are in the paper: classification of division algebras over Real closed fields, definition of schemes over fields, Lefschetz principle, foundations of algebraic geometry

Kashdan summarises some of the advantages and difficulties.

From a mathematician: Kazhdan 

On the other hand, the Model theory is concentrated on gap between an abstract definition and a concrete construction. Let T be a complete theory. On the first glance one should not distinguish between different models of T, since all the results which are true in one model of T are true in any other model. One of main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty.

Kazhdan continued

Different models of complete theories are of different flavors and support different intuitions. So an attack on a problem often starts [with] a choice of an appropriate model. Such an approach lead to many non-trivial techniques for constructions of models which all are based on the compactness theorem which is almost the same as the fundamental existence theorem.

On the other hand the novelty creates difficulties for an outsider who is trying to reformulate the concepts in familiar terms and to ignore the differences between models.

Classes of Theories are important

The breakthroughs of classification theory as a tool for organizing mathematics come in several steps.

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1. (complete) first order theories are important.
2. Classes of (complete) first order theories are important.

Shelah on Dividing Lines: Shelah 

I am grateful for this great honour. While it is great to find full understanding of that for which we have considerable knowledge, I have been attracted to trying to find some order in the darkness, more specifically, finding meaningful dividing lines among general families of structures. This means that there are meaningful things to be said on both sides of the divide: characteristically, understanding the tame ones and giving evidence of being complicated for the chaotic ones.

Shelah on Dividing Lines

It is expected that this will eventually help in understanding even specific classes and even specific structures. Some others see this as the aim of model theory, not so for me. Still I expect and welcome such applications and interactions. It is a happy day for me that this line of thought has received such honourable recognition. Thank you

on receiving the Steele prize for seminal contributions.

Mathematical Applications of the stability hierarchy

We quickly sketch the first order stability hierarchy and then

1. Show how it provides a new organization scheme for some mathematics.
2. List a few examples of mathematical applications of these tools.

Bourbaki again

Bourbaki has some beginning notions of combining the 'great mother-structures' (group, order, topology). They write:

'the organizing principle will be the concept of a hierarchy of structures, going from the simple to complex, from the general to the particular.'

But this is a vague vision. We now sketch a realization of a more sophisticated organization of parts of mathematics.
In particular, 'geometry' should have been one of the 'great mother structures'.

Properties of classes of theories

The Stability Hierarchy

Every complete first order theory falls into one of the following 4 classes.

1. ω-stable
2. superstable but not ω-stable
3. stable but not superstable
4. unstable

Stability is Syntactic

Definition

T is stable if no formula has the order property in any model of T.

ϕ is unstable in T just if for every n the sentence
$\exists x_1, \ldots x_n \exists y_1, \ldots y_n \bigwedge_{i<j} \phi(x_i, y_i) \wedge \bigwedge_{j \geq i} \neg\phi(x_i, y_i)$ is in T.

This formula changes from theory to theory.

1. dense linear order: $x < y$;
2. real closed field: $(\exists z)(x + z^2 = y)$,
3. $(\mathbb{Z}, +, 0, \times)$:$(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. infinite boolean algebras: $x \neq y$ & $(x \wedge y) = x$.

The stability hierarchy: examples

ω-stable

Algebraically closed fields (fixed characteristic), differentially closed fields (infinite rank), complex compact manifolds

strictly superstable

$(\mathbb{Z}, +)$, $(2^\omega, +) = (Z_2^\omega, H_i)_{i<\omega}$.

strictly stable

$(\mathbb{Z}, +)^\omega$, separably closed fields, the free group on 2 generators

Consequences: Main Gap

Shelah proved:

Main Gap

For every first order theory T, either

1. Every model of T is decomposed into a tree of countable models with uniform bound on the depth of the tree, or
2. The theory T has the maximal number of models in all uncountable cardinalities.

Groups of finite Morley rank: Cherlin Zilber

Definition

A group of finite Morley rank (FMR) is an infinite structure which admits a group operation and is ω-stable with finite rank.

This class properly contains algebraic groups over algebraically closed fields.

conjecture[Cherlin/Zilber]

A simple group of finite Morley rank is algebraic over an algebraically closed field.

The 25 year project to solve the conjecture has developed as an amalgam of basic stability theoretic tools with many different tools from finite and, recently, combinatorial group theory. It engages all 3 generations of the proof of the classification of finite simple groups.

Borovik-Nesin book:

Borovik  Nesin 

The notion of interpretation in model theory corresponds to a number of familiar phenomena in algebra which are often considered distinct: coordinatization, structure theory, and constructions like direct product and homomorphic image.

For example a Desarguesian projective plane is coordinatized by a division ring; Artinian semisimple rings are finite direct products of matrix rings over divisions rings; many theorems of finite group theory have as their conclusion that a certain abstract group belongs to a standard family of matrix groups over [. . .].

Borovik-Nesin book

All of these examples have a common feature: certain structures of one kind are somehow encoded in terms of structures of another kind. All of these examples have a further feature which plays no role in algebra but which is crucial for us: in each case the encoded structures can be recovered from the encoding structures definably.

Model theoretic hypotheses:

Theorem (Hrushovski)

Let T be a stable theory. Let $\tilde{p}$ and $\tilde{q}$ be nonorthogonal stationary, regular types and let n be maximal such that $\tilde{p}^n$ is almost orthogonal to $\tilde{q}^\omega$. Then there exist p almost bidominant to $\tilde{p}$ and q dominated by $\tilde{q}$ such that:

There is no group or field in sight.
Regular types support geometries, so geometry is in view.

Mathematical Conclusion

$n = 1$ q is the generic type of a (type) definable group that has the regular action on the realizations for p.

$n = 2$ q is the generic type of a (type) definable algebraically closed field that acts on the realizations for p as an affine line.

$n = 3$ q is the generic type of a (type) definable algebraically closed field that acts on the realizations for p as a projective line.

$n \geq 4$ is impossible.

Summation: Hrushovski 

Hrushovski ICM talk 1998

Instead of defining the abstract context for the [stability] theory, I will present a number of its results in a number of special and hopefully more familiar, guises: compact complex manifolds, ordinary differential equations, difference equations, highly homogeneous finite structures. Each of these has features of its own and the transcription of results is not routine; they are nonetheless readily recognizable as instances of a single theory.