Complexity and Absoluteness in $L_{\omega_1,\omega}$

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Sacks Dicta

"... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable."

Gerald Sacks, 1972

explained in Vaananen article in Model Theoretic Logic volume

Our question

Does Sacks dicta extend from $L_{\omega,\omega}$ to $L_{\omega_1,\omega}$?

FOL, $L_{\omega_1,\omega}$ and set theory

1970 - Close connection between model theory and set theory:

1. logics vrs theories
2. combinatorics vrs axiomatics
3. first order vrs infinitary

We study here absoluteness for theories, connecting $L_{\omega_1,\omega}$ with 'first order'.

Acknowledgements/References

We are indebted for discussions with Alf Dolich, Paul Larson, Chris Laskowski, Sy Friedman, Martin Koerwien, Christian Rosendal and Dave Marker for clarifying the issues.

This lecture is based on my paper for the Asia Logic Conference 2009 and Dave Marker's appendix to that paper. That paper is on my website: `www.math.uic.edu~jbaldwin` in my exposition 'Categoricity' of (primarily) Shelah's work which is an introduction to infinitary model theory.

Shelah also has a new book on categoricity in Abstract Elementary Classes. \$28 on Amazon

Shoenfield Absoluteness Lemma

Theorem (Shoenfield)

If

1. $V \subset V'$ are models of ZF with the same ordinals and
2. ϕ is a lightface Π^1_2 predicate of a set of natural numbers

then for any $A \subset N$, $V \models \phi(A)$ iff $V' \models \phi(A)$.

Note that this trivially gives the same absoluteness results for Σ^1_2-predicates.

Easy remark

The class of first order -sentences (formulas) is arithmetic, in fact recursive.

The class of satisfiable first order sentences is Π^0_1.

Sentences in $L_{\omega_1,\omega}$

Fix a vocabulary τ and let $\mathbb{X}_\tau$ be the Polish space of countable τ-structures with universe ω.

Fact The class of $L_{\omega_1,\omega}$-sentences (formulas) is complete-Π^1_1.

We sketch this argument.

What is a sentence?

Definition

1. A *labeled tree* is a non-empty tree $T \subseteq \omega^{<\omega}$ with functions l and v with domain T such that for any $\sigma \in T$ one of the following holds:
   - σ is a terminal node of T then $l(\sigma) = \psi$ where ψ is an atomic τ-formula and $v(\sigma)$ is the set of free variables in ψ;
   - $l(\sigma) = \neg$, $\sigma\hat{}0$ is the unique successor of σ in T and $v(\sigma) = v(\sigma\hat{}0)$;
   - $l(\sigma) = \exists v_i$, $\sigma\hat{}0$ is the unique successor of σ in T and $v(\sigma) = v(\sigma\hat{}0) \setminus \{i\}$;
   - $l(\sigma) = \bigwedge$ and $v(\sigma) = \bigcup_{\sigma\hat{}i \in T} v(\sigma\hat{}i)$ is finite.
2. A *formula* ϕ is a well founded labeled tree (T, l, v). A *sentence* is a formula where $v(\emptyset) = \emptyset$.

Truth Definition

Define in the natural inductive fashion a predicate 'f is a *truth definition* for the labeled tree (T, l, v) in M'.

The domain of f is pairs (σ, μ) where $\sigma \in T$ and $\mu : v(\sigma) \to M$ is an assignment of the free variables at node σ and $f(\sigma, \mu) \in \{0, 1\}$.

This predicate is arithmetic.

If ϕ is a sentence, there is a unique truth definition f for ϕ in M.

Satisfiability

Proposition

There is $R(x,y) \in \Pi^1_1$ and $S(x,y) \in \Sigma^1_1$ such that if ϕ is a sentence and $M \in \mathbb{X}_\tau$, then

1. $M \models \phi \Leftrightarrow R(M,\phi) \Leftrightarrow S(M,\phi)$.
2. $\{(M,\phi) : \phi$ is a sentence and $M \models \phi\}$ is Π^1_1.
3. For any fixed ϕ, $\mathrm{Mod}(\phi) = \{M \in \mathbb{X}_\tau : M \models \phi\}$ is Borel.

Define: $R(x,y) \Leftrightarrow x \in \mathbb{X}_\tau$ and y is a labeled tree and $f(\emptyset,\emptyset) = 1$ for all truth definitions f for y in x
and
$S(x,y) \Leftrightarrow y$ is a labeled tree and there is a truth definition f for y in x such that $f(\emptyset,\emptyset) = 1$.
Now Borel is obvious.

Complexity of first order concepts

For example a first order theory T is unstable just if there is a formula $\phi(\mathbf{x},\mathbf{y})$ such for every n

$$T \models (\exists \mathbf{x}_1, \ldots \mathbf{x}_n \exists \mathbf{y}_1, \ldots \mathbf{y}_n) \bigwedge_{i<j} \phi(\mathbf{x}_i, \mathbf{y}_j) \wedge \bigwedge_{i \geq j} \neg\phi(\mathbf{x}_i, \mathbf{y}_j)$$

This is an arithmetic statement and so is absolute by basic properties of absoluteness e.g. Kunen, Jech.
ω-stability, superstability and $\aleph_1$-categoricity are Π^1_1.

From $L_{\omega_1,\omega}$ to 'first order'

1. $\phi \in L_{\omega_1,\omega} \rightarrow (T, \Gamma)$
2. complete $\phi \in L_{\omega_1,\omega} \rightarrow (T, Atomic)$

Why is this not just a technical remark?

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Why is this not just a technical remark?

The first transformation is arithmetic.

The second is not even Borel.

The translation

Theorem

[Chang/Lopez-Escobar] Let ψ be a sentence in $L_{\omega_1,\omega}$ in a countable vocabulary τ. Then there is a countable vocabulary τ' extending τ, a first order τ'-theory T, and a countable collection of τ'-types Γ such that reduct is a 1-1 map from the models of T which omit Γ onto the models of ψ.

The proof is straightforward. E.g., for any formula ψ of the form $\bigwedge_{i<\omega} \phi_i$, add to the language a new predicate symbol $R_\psi(\mathbf{x})$. Add to T the axioms

$$(\forall \mathbf{x})[R_\psi(\mathbf{x}) \to \phi_i(\mathbf{x})]$$

for $i < \omega$ and omit the type $p = \{\neg R_\psi(\mathbf{x})\} \cup \{\phi_i : i < \omega\}$. How effective is the translation?

Reducing $L_{\omega_1,\omega}$ to 'first order'

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.
ϕ is Δ-complete if for every $\psi \in \Delta$
$\phi \models \psi$ or $\phi \models \neg\psi$.
(If Δ is omitted we mean complete for $L_{\omega_1,\omega}$.)

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
K is the class of atomic models (realize only principal types) of a first order theory (in an expanded language).

Completeness???

Łos-Vaught test

Let T be a set of first order sentences with no finite models, in a countable first order language.

If T is κ-categorical for some $\kappa \geq \aleph_0$, then T is complete.

Needs upward and downward Lowenheim-Skolem theorem for theories.

We search for a substitute in $L_{\omega_1,\omega}$.
There are models with no countable $L_{\omega_1,\omega}$-elementary submodel.

Small

Let Δ be a fragment of $L_{\omega_1,\omega}$ that contains ϕ.

Definition

A τ-structure M is Δ-small if M realizes only countably many Δ-types (over the empty set).

'small' means $\Delta = L_{\omega_1,\omega}$

Small implies complet(able)

If M is small then M satisfies a complete sentence.

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If ϕ is small then there is a complete sentence ψ_ϕ such that:

$\phi \wedge \psi_\phi$ have a countable model.

So ψ_ϕ implies ϕ.

But ψ_ϕ is not in general unique (real examples).

Shelah's lemma

Definition

An uncountable model M that is Δ-small for every countable Δ is called scattered.

Lemma

If ϕ has a scattered model M of cardinality $\aleph_1$, then ϕ has small model of cardinality $\aleph_1$.

The $L_{\omega_1,\omega}$-Vaught test

Shelah If ϕ has an uncountable model M that is Δ-small for every countable Δ and ϕ is $\aleph_1$-categorical then ϕ is implied by a complete sentence ψ with a model of cardinality $\aleph_1$.

Keisler If ϕ has $< 2^{\aleph_1}$ models of cardinality $\aleph_1$, each model is Δ-small for every countable Δ.

Do either of these hold for arbitrary κ?

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Do either of these hold for arbitrary κ?

Thus the model of ϕ in $\aleph_1$ is small.

So we effectively have Vaught's test.
But only in $\aleph_1$! And only for completability!

ω-stabilty

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
$\mathbf{K}$ is the class of atomic models (realize only principal types) of a first order theory (in an expanded language).

$(\mathbf{K}, \prec_{\mathbf{K}})$ is the class of atomic models of a first order theory under elementary submodel.

Definitions

$p \in S_{at}(A)$ if $a \models p$ implies Aa is atomic.

$\mathbf{K}$ is ω-stable if for every countable model M, $S_{at}(M)$ is countable.

ABSTRACT ELEMENTARY CLASSES

Generalizing Bjarni Jónsson:

A class of L-structures, $(\mathbf{K}, \prec_{\mathbf{K}})$, is said to be an *abstract elementary class: AEC* if both $\mathbf{K}$ and the binary relation $\prec_{\mathbf{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$ then $A \prec_{\mathbf{K}} B$;

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

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2. Closure under direct limits of $\prec_{\mathbf{K}}$-chains;
3. Downward Löwenheim-Skolem.

Examples

First order and $L_{\omega_1,\omega}$-classes
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One Completely General Result

WGCH: $2^\lambda < 2^{\lambda^+}$
Let **K** be an abstract elementary class (AEC).

Theorem

[WGCH] Suppose $\lambda \geq \mathrm{LS}(\mathbf{K})$ and **K** *is λ-categorical. If amalgamation fails in λ there are 2^{λ^+} models in* **K** *of cardinality $\kappa = \lambda^+$.*

Uses $[\hat{\Theta}_{\lambda^+}(S)]$ for many S.

λ-categoricity plays a fundamental role.

Definitely not provable in ZFC for AEC (but maybe for $L_{\omega_1,\omega}$).

Getting ω-stabilty

Theorem

[Keisler/Shelah]
$(2^{\aleph_0} < 2^{\aleph_1})$ If **K** has $< 2^{\aleph_1}$ models of cardinality $\aleph_1$, then **K** is ω-stable.

Two uses of WCH

1. WCH implies AP in $\aleph_0$. AP in $\aleph_0$ implies that if **K** is not ω-stable there are uncountably many types over a single countable model that are realized in uncountable models.

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Two uses of WCH

1. WCH implies AP in $\aleph_0$. AP in $\aleph_0$ implies that if **K** is not ω-stable there are uncountably many types over a single countable model that are realized in uncountable models.
2. WCH implies that if there are uncountably many types over a countable model there is another theory with uncountably many types over the empty set.

Contradicting the $L_{\omega_1,\omega}$-Vaught test.

Is WCH needed?

1. Yes for AEC, even $L_{\omega_1,\omega}(Q)$?

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1. Yes for AEC, even $L_{\omega_1,\omega}(Q)$?
2. $L_{\omega_1,\omega}$: open - equivalent to absoluteness by results below.

Absoluteness of atomicity

Lemma (Atomic models)

1. *'T has an atomic model' is an arithmetic property of T.*
2. *'M is an atomic model of T' is an arithmetic property of M and T.*
3. *For any vocabulary τ, the class of countable atomic τ-structures, M, is Borel.*

Some recursion theory

Definition

$x \in \omega^\omega$ is *hyperarithmetic* if $x \in \Delta^1_1$. x is *hyperarithmetic* in y, written $x \leq_{\mathrm{hyp}} y$, if $x \in \Delta^1_1(y)$.

Fact (Harrison's Lemma)

1. *The predicate* $\{(x, y) : x \leq_{hyp} y\}$ *is* Π^1_1.
2. *If* $K \subset \omega^\omega$ *is* Σ^1_1*, then for any* y*,* K *contains an element which is not hyperarithmetic in* y *if and only if* K *contains a perfect set.*

Marker realized that Harrison's lemma could reduce a number of Σ^1_2-definitions to Π^1_1.

Countable Stone space is Π^1_1

Lemma (Marker)

Let **K** *be an atomic class with a countable complete first order theory T. Let A be a countable atomic set.*

1. *The predicate of p and A, 'p is in $S_{\mathrm{at}}(A)$', is arithmetic.*
2. *'$S_{\mathrm{at}}(A)$ is countable' is a Π^1_1-predicate of A.*

Proof.

ii) By i), the set of p such that 'p is in $S_{\mathrm{at}}(A)$' is Σ^1_1) in A. By Harrison, each such p is hyperarithmetic in A. Since the continuum hypothesis holds for Σ^1_1-sets, '$S_{\mathrm{at}}(A)$ is countable' is formalized by:

$$(\forall p)[p \in S_{\mathrm{at}}(A) \rightarrow (p \leq_{\mathrm{hyp}} A)].$$

Definition of Excellence

Definition

Let **K** be an atomic class. **K** is *excellent* if **K** is ω-stable and any of the following equivalent conditions hold.
For any finite independent system of countable models with union C:

1. $S_{at}(C)$ is countable.
2. There is a unique primary model over C.
3. The isolated types are dense in $S_{at}(C)$.

Absoluteness of ω-stability and excellence: Atomic models

Lemma

Let T be a complete countable first order theory. The properties that the class of atomic models of T is

1. ω-stable
2. excellent

are each given by a Π^1_1-formula of set theory and so are absolute.

Proof. 1) The class of atomic models of T is ω-stable if and only if for every atomic model M, '$S_{\mathrm{at}}(M)$ is countable'. This property is Π^1_1 by the previous argument.
Excellence is slightly more complicated.

Complexity of model theoretic notions for $L_{\omega_1,\omega}$

Theorem

Each of the properties that a complete sentence of $L_{\omega_1,\omega}$ is ω-stable, excellent, or has no two-cardinal models is Σ^1_2.

Proof. Let $Q(T)$ denote any of the conditions above as a property of the first order theory T in a vocabulary τ^*. Now write the following properties of the complete sentence ϕ in vocabulary τ.

1. ϕ is a complete sentence.
2. There exists a $\tau^* \supseteq \tau$ and τ^* theory T satisfying the following.
    1. T is a complete theory that has an atomic model.
    2. The reduct to τ of any atomic model of T satisfies ϕ.
    3. There is a model M of ϕ and there exists an expansion of M to an atomic model of T.
    4. $Q(T)$.

CLI groups

A group is CLI if it admits a complete, compatible, left-invariant metric.

We prove the following claim. This result was developed in conversation with Martin Koerwien and Sy Friedman at the CRM Barcelona and benefitted from further discussion with Dave Marker and Christian Rosendal.

Some Model Theory

A countable model is *minimal* (equivalently *non-extendible*) if it has no proper $L_{\omega_1,\omega}$-elementary submodel.

Claim

If M is atomic, τ-elementary submodel is the same as $L_{\omega_1,\omega}(\tau)$-elementary submodel.

Thus, for atomic models: minimal iff first order minimal. Note that the class of minimal models is obviously Π^1_1. Now if the class of minimal models were Borel, it would follow that the class of minimal atomic (equal first order minimal prime) models is also Borel.

Lemma (Deissler)

There is a countable vocabulary τ such that the class of minimal first order prime models for τ is not Σ^1_1.

Back to Descriptive Set Theory

Lemma (Gao)

The following are equivalent:

1. *Aut(M) admits a compatible left-invariant complete metric.*
2. *There is no $L_{\omega_1,\omega}$-elementary embedding from M into itself which is not onto.*

Claim

The class of countable models whose automorphism groups admit a complete left invariant metric is Π^1_1 but not Σ^1_1.

Crossing Fields

Theorem

There is a Borel isomorphism between

1. The class of minimal atomic models for the vocabulary with infinitely many relations symbols of each arity.
2. Polish groups which admit a complete left invariant metric.

This result was worked out by myself and Christian Rosendal after noting that Malicki had proved the Π^1_1 but not Σ^1_1 definability result for the second class.

Absoluteness of existence of a model in $\aleph_1$

1. complete sentence - last slide
2. ϕ has countably many models - easy from previous
3. ϕ has less than $2^{\aleph_0}$ models: By Harnik-Makkai, there is model in $\aleph_1$. By Morley, ϕ is scattered. By Shelah there is a small model in $\aleph_1$.
4. But there exist ϕ such that every completion characterizes $\aleph_0$.

Does there exists a sentence ϕ such that categoricity in $\aleph_1$ of ϕ is not absolute?

Problems

1. Is $\aleph_1$-categoricity for $L_{\omega_1,\omega}$ absolute?
2. We have proved the important model theoretic properties of atomic classes are Π^1_1 or Σ^1_1.
   We have proved the important model theoretic properties of $L_{\omega_1,\omega}$ are Σ^1_2.

   Is this distinction real?

More Problems

3. (Sacks) Is the class of algebraically prime models absolute?
4. Is the class of minimal atomic models complete-Π^1_1?
5. What is the complexity of the class of first order theories with finite Morley rank?