INFINITISTIC METHODS AND FINITE MODELS

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STUDY THEORIES NOT LOGICS

Shelah: Classify theories; then classify their models.

I. Shelah's dictum in infinite model theory

1. first order case
2. infinitary logic and amalgamation

II. 0-1 Laws

III. Defining classes of structures in other structures

IV. Stability in L^n

THE SPECTRUM PROBLEM

The *spectrum function* of T, $n(T, \kappa)$, is the number of models of T with cardinality κ.

Morley's conjecture: With the obvious exception of $\aleph_1$ but not $\aleph_0$ categorical theories, $n(T, \kappa)$ is an increasing function.

DECISIVE STRATEGY: Solve this my dividing theories into classes and computing a spectrum function for each class.

THE MONSTER MODEL

Every model we consider of T can be embedded in the monster model: $\mathcal{M}$.

In this sense $\mathcal{M}$ is universal. It is also homogenous

If $M \prec N$ then the embedding of M into $\mathcal{M}$ can be extended to N.

Since we are working in this monster model, the following two notions are the same.

$$S(A) = \{p : p \text{ is consistent type over A}\}.$$

$$S(A) = \{p : p \text{ is realized in } \mathcal{M}\}.$$

THE AMALGAMATION PROPERTY

$A \longrightarrow B$ means there is a $\mathbf{K}$-embedding f from A to B.

Definition. The class $(\mathbf{K}, \longrightarrow)$ satisfies the *amalgamation property* ($\mathbf{AP}$) if for any situation:

$$\begin{tikzcd}[row sep=tiny] & B \\ A \arrow[ur] \arrow[dr] & \\ & C \end{tikzcd}$$ there exist a D and strong embeddings,

such that $$\begin{tikzcd}[row sep=tiny] & B \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & D. \\ & C \arrow[ur] & \end{tikzcd}$$

A, B, C, D are in $\mathbf{K}$; A may be empty,

0-1 **LAWS**

A. STANDARD METHODS:

- back and forth games
- 'elimination' of quantifiers,
- categoricity

Deals with 'all' models of certain similarity type.

B. GENERIC MODELS:

METHOD: Fix a class $\mathbf{K}$ and a notion $\leq$ of strong submodel. Identify the generic (or semi-generic) for that class; show the associated theory is the almost sure theory.

ADVANTAGES: Relativizes, separates model theoretic and probabilistic aspects, model theoretic information clearer.

2ND ORDER LOGICS

Fix a first order language L, a new n-ary predicate symbol R and an $L(R)$ sentence $\theta(R)$.

We get different semantics by varying the set $\mathcal{R}$ of allowed interpretations of R.

1. $\mathcal{R}$ is all subsets of M^n: 2nd order logic;
2. $\mathcal{R}$ is all finite subsets of M^n: weak 2nd order logic:
   embedded finite model theory;
3. $\mathcal{R}$ is all L-definable subsets of M^n:
   constraint data bases;
4. $\mathcal{R}$ is all subsets of M^n satisfying some formula (only in R) $\psi(R)$: Shelah's
   'definable' second order quantifiers.

Progress in the last three areas have been made by making 'stability' assumptions on the ambient model.

INFINITARY LOGIC
ABSTRACT ELEMENTARY CLASSES

Goal: extend classification theory to infinitary logic.

Obstructions: compactness and amalgamation fail in general.

Paradigm: first order logic with a collection of types omitted

Palliatives:

1. Variants on amalgamation: set amalagamation, model amalgamation.
2. Corresponding variants on homegeneity.
3. Work inside a single model.
4. Substitute combinatorics for compactness.

FINITE VARIABLE LOGIC

Definition.

- An L^n-formula contains only n variables, free or bound
- An L^n-theory T is a consistent set of L^n-sentences, closed under logical consequence (in L^n).

We consider both infinite and finite models so we have compactness.

Topics:

1. basic properties of L^n.
2. types
3. stability and omega-stability
4. L^n-saturation: T^c
5. Djordjević's theorem; trivial forking
6. unstable L^n-theories

EXAMPLES

- Vector spaces over a finite field (L^3-complete)
- For any complete theory T, let T_n denote the L^n-sentences of T.
  - T_n is complete for L^n.
  - If T is totally categorical then for some n, T_n has arbitrarily large finite models.
- The n-extension axioms
- Dense linear order

L^n-THEORIES WITH FEW MODELS

Poizat characterized the complete L^n-theories which have models of size n or $n+1$ and larger finite models. He asked whether any complete L^n-theory with more than one finite model had infinitely many finite models.

Fact (Cherlin, Thomas)

- There are complete theories in L^n with models in finitely many finite cardinalities and *no* infinite model.
- There are complete theories in L^n with models in finitely many finite cardinalities and *with* infinite models.

L^n-TYPES OVER $\emptyset$

1. For $m \leq n$, an m-type in L^n, p, is the collection of L^n formulas satisfied by some $\mathbf{a} \in M \models T$.
2. $S_n^n(T)$ is the collection of such types.
3. The n-size of T is $|S_n^n(T)|$.

SCOTT SENTENCES

Lemma. For any finite structure M and any finite n there is a sentence of L^n which characterizes M up to L^n-equivalence. Moreover, the quantifier rank of this Scott sentence is less than $|S_n^n(T)|$ where $T = \mathrm{Th_{L^n}(M)}$.

Observe that the construction of the Scott sentence of a finite model, depends only on $S_n^n(T)$

Fact

1. If an L^n-theory T has a finite model, $S_n^n(T)$ is finite.
2. If $S_n^n(T)$ is finite, then
   - The L^n-theory T is axiomatized by a single sentence of L^n;
   - All models of T realize the same (all) members of $S_n^n(T)$.

L^n-STABILITY

A formula $\phi(\mathbf{x}, \mathbf{y})$ has the order property in a model M if there are sequences $\mathbf{a}_i, \mathbf{b}_j$ for $i, j < \omega$ such that

$$M \models \phi(\mathbf{a}_i, \mathbf{b}_j) \text{ iff } i < j.$$

A model M is *L^n-stable* if there is no L^n-formula $\phi(x, y)$ with the order property in M.

An L^n-theory T is stable if each model of T is.

The notions of stability and ω-stability in L^n were first worked out by Hyttinen and Djordjević.

SET AMALGAMATION PROPERTY

Definition. The L^n-theory T satisfies the *set amalgamation property* (**SAP**) if for any situation with $N, M \models T$:

$$\begin{tikzcd}[row sep=small] & N \\ A \arrow[ur] \arrow[dr] & \\ & M \end{tikzcd}$$

with $\mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{M}) = \mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{N})$ there exists an N_1

such that $\begin{tikzcd}[row sep=small] & N \arrow[dr] & \\ A \arrow[ur] \arrow[dr] & & N_1 \\ & M \arrow[ur] & \end{tikzcd}$

and $\mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{N_1}) = \mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{M}) = \mathrm{Th}_{\mathrm{L^k}}(\mathrm{A}, \mathrm{N})$.

FAILURE OF SAP

Let T be the L^3-theory of Abelian groups of exponent 2 in a language with one ternary relation symbol.

Let M be the Abelian group generated freely (in exponent 2) by b_1, b_2, b_3, b_4.

Let N be the Abelian group generated freely (in exponent 2) by c_1, c_2, c_3 and let

$$c_4 = c_1 + c_2 + c_3.$$

Now let f, g map $A = \{a_1, \ldots a_4\}$ into M and N by mapping a_i to b_i and c_i respectively.

- f and g are L^3 elementary.
- M and N can't be amalgamated over A.

WHAT IS AN L^n TYPE?

Two problems:

I. Formulas over parameters sneak in variables.

Solution: Each $\phi(\mathbf{x}, \mathbf{a})$ must be a substitution instance of an L^n-formula.

Drawback: Types are NOT closed under conjunction.

II. In the absence of amalgamation:

A. Types may be separately but not jointly consistent.

Hedman and Kueker built such an example where the countable model is ω-categorical.

B. $S(A)$ is ambiguous because we have no monster model.

TYPES

Let $M \models T$, $B \subseteq M, \mathbf{a} \in M$.

$$\mathrm{tp_{L^k}(\mathbf{a}/B, M) = \{\phi(\mathbf{x}, \mathbf{b}) : M \models \phi(\mathbf{a}, \mathbf{b})\}}$$

where:

1. $\mathbf{a}, \mathbf{b}, \mathbf{x}$ etc. are finite sequences, $\mathbf{b} \in B$
2. $\phi(\mathbf{x}, \mathbf{y}) \in L^k$

NOTA BENE:

1. Even if B is finite the conjunction of the type need not be in L^k.
2. We need the parameter M.
3. We write $\mathrm{tp_{L^k}(A, M)}$ or $\mathrm{Th_{L^k}(A, M)}$ for $\mathrm{tp_{L^k}(\mathbf{a}/\emptyset, M)}$ where $\mathbf{a}$ enumerates A.

$S(A)$?

Definition: T is ω-stable if for every $M \models T$ and every countable $A \subseteq M$, $S(A, M)$ is countable.

But, what is $S(A, M)$?

1. $S(A, M) = \{p : p \text{ is consistent}\}$.
2. $S(A, M) = \{p : p \text{ is realized in an } L^n\text{-extension of } M\}$.
3. $S(A, M) = \{p : p \text{ is realized in } M\}$.

Clearly ω-stability i implies ω-stability $i + 1$; but not conversely.

Amalgamation over models makes the last two equivalent; amalgamation over sets makes them all equivalent.

Fact:[Shelah] Let Δ be a finite set of formulas, closed under permutation of variables and negation. If M is a model which does not have the order property witnessed by a formula in Δ, then for each infinite cardinal λ, the model M realizes at most λ many Δ-types over each of its subsets of cardinality λ.

Corollary. If T is a complete stable L^n-theory and $S_n^n(T)$ is finite, then T is ω-stable in sense 3.

L^n-SATURATED MODELS

A structure M is L^n-κ-saturated if every $A \subset M$ with $|A| < \kappa$, every consistent type over A is realized in M.

Independently, Djordjević and Hyttinen showed:

Theorem. T has the L^n-set amalgamation property if and only if for any κ, every model of T can be L^n-embedded in an L^n-κ-saturated model.

Note that if T is the L^3 theory of $(Z, +, 1)$, there is no countable countably L^3-saturated model.

EXAMPLE

Consider the structure (M, E) consisting of an equivalence relation with one class of each finite cardinality and infinitely many infinite classes.

M is L-saturated but not L^n-saturated for any n.

Fix $m > n$. Let $a_0, \ldots a_{m-1}$ enumerate one equivalence class.

The L^n-type $p = \{xEa_1, x \neq a_i : i < m\}$ is omitted.

Note that p is inconsistent with the first order theory of $(M, \mathbf{a})$ but consistent with the L^n-theory of M.

T^c

Suppose the L^n theory T has the x-amalgamation property. (x = amalgamation over all sets, sufficiently closed subsets, or models).

Then T has L^n x-homogeneous models; all such models are elementarily equivalent. T^c is the first order theory of the L^n x-homogeneous models.

Theorem. Suppose T satisfies SAP. The following are equivalent.

1. Let M be a model of T. If $p(\overline{x}) = \mathrm{tp_L(\overline{a}/\emptyset, M)}$ and $q(\overline{x}) = \mathrm{tp_{L^n}(\overline{a}/\emptyset, M)}$ then
$$T^c \cup q(\overline{x}) \vdash p(\overline{x})$$
2. Every L^n-elementary extension of a model of T^c is an L-elementary extension.
3. Every model of T^c has an elementary extension which is x-homogenous.

GETTING FINITE MODELS

Theorem. (Djordjević) Suppose T satisfies SAP and $S_n^n(T)$ is finite. Let M be a model of T. If $p(\overline{x}) = \mathrm{tp_L(\overline{a}/\emptyset, M)}$ and $q(\overline{x}) = \mathrm{tp_{L^n}(\overline{a}/\emptyset, M)}$ then $T^c \cup q(\overline{x}) \vdash p(\overline{x})$.

Theorem. (Djordjević) If a stable L^k-theory T with the *set* amalgamation property has only finitely many L^k-types then T^c is ω-categorical and ω-stable. It follows that T has arbitrarily large finite models.

This requires Zil'ber/Cherlin-Harrington-Lachlan technology.

Theorem. (Baldwin-Lessmann) If a stable L^k-theory T with the *closure* amalgamation property has only finitely many L^k-types then T^c is ω-categorical and ω-stable. It follows that T has arbitrarily large finite models.

This extends Djordjević's result to theories with non-trivial forking.

UNSTABLE-L^n THEORIES

A model M has the L^n-independence property if it admits a parametrically definable family of independent sets.

Definition A. M has the L^n-independence property if there is an L^n-formula $\phi(\mathbf{x}, \mathbf{v})$ such that $A = \langle \mathbf{a}_n : n \in \omega \rangle \subset M$ and for each $X \subset \omega$, the type:

$$q_X = \{\phi(\mathbf{x}, \mathbf{a}_i) : i \in X\} \cup \{\neg\phi(\mathbf{x}, \mathbf{a}_i) : i \notin X\}$$

is consistent.

Definition B. M has the L^n-independence property if there is an L^n-formula $\phi(\mathbf{x}, \mathbf{v})$ such that $A = \langle \mathbf{a}_n : n \in \omega \rangle \subset M$ and for each $X \subset \omega$, the type:

$$q_X = \{\phi(\mathbf{x}, \mathbf{a}_i) : i \in X\} \cup \{\neg\phi(\mathbf{x}, \mathbf{a}_i) : i \notin X\}$$

is realized M

T has the independence property if some $M \models T$ does.

FURTHER DIRECTIONS

Lemma. (Shelah) If the n-variable formula ϕ does not have the have the independence property then there is a n'-formula ψ with the strict order property.

Lemma. (Baldwin-Dawar) If a complete theory T in L^n has arbitrarily large finite models then no L^n-formula has the strict order property.

Must every unstable L^n-theory with arbitrarily large finite models have the independence property?

Lessmann has shown the forking machinery can be developed in simple L^n-theories.

H1

Hypothesis (H1). For any countable models M_ℓ for $\ell = 0, 1, 2$ with $M_0 \prec_{L^n} M_\ell$ for $\ell = 1, 2$, and $\overline{a}_\ell \in M_\ell$ for $\ell = 1, 2$ if

$$\mathrm{tp_{L^n}(cl(\overline{a}_1)/M_0, M_1) = tp_{L^n}(cl(\overline{a}_2)/M_0, M_2)},$$

then there exist N and L^n-elementary embeddings $f_\ell : M_\ell \to N$ such that $f_1(\overline{a}_1) = f_2(\overline{a}_2)$ and f_ℓ is the identity over M_0 for $\ell = 1, 2$.

Let C denote $\mathrm{cl}(\emptyset)$.

Theorem. If T has (CAP) and satisfies $H1$ then $T^c(C)$ is $\aleph_0$-stable.