

## The Mathematics for Essay 2

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The purpose of these notes are to explain some of the mathematics behind Essay 2. Your own essay should not just repeat these arguments but have a more geometric flavor. Write about how you can physically place the blocks. You may assume basic facts about geometric sums and series. Let  $r$  be any real number and let  $n$  be a non-negative integer. The sum

$$1 + r + r^2 + \cdots + r^n \tag{1}$$

is a *geometric sum* and the infinite series

$$1 + r + r^2 + \cdots + r^n + \cdots \tag{2}$$

is a *geometric series*.

Suppose further that  $r \neq 1$ . Then the geometric sum (1) can be computed by the formula

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

This fact, which you may assume, is easily proved by mathematical induction.

Now suppose that  $|r| < 1$ . Then  $\lim_{n \rightarrow \infty} r^n = 0$  which means the geometric series (2) converges to  $\frac{1}{1 - r}$  by the preceding equation. We write

$$1 + r + r^2 + \cdots + r^n + \cdots = \frac{1}{1 - r} \tag{3}$$

to indicate that the series converges and to designate the limit of the sequence of partial sums.

Your essay will involve the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots \tag{4}$$

Since  $|\frac{1}{2}| < 1$ , it follows by (3) that (4) converges and  $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2$ . The

Deluxe blocks are cubes with side lengths  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \dots$ . Your essay involves analyzing the sum of their side lengths

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \cdots.$$

The preceding series is called the *harmonic series*. Think of the terms of the geometric series (4) as markers for grouping terms of the harmonic series as follows:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \quad (5)$$

We will find an overestimate and an underestimate for the sum of the terms in each of the parenthesized groups. You will see a pattern emerging in our calculations:

$$\begin{aligned} 1 &= \frac{1}{2} + \frac{1}{2} > \frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \\ 1 &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} > \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}, \\ 1 &= \frac{1}{8} + \cdots + \frac{1}{8} = 8\left(\frac{1}{8}\right) > \frac{1}{9} + \cdots + \frac{1}{16} > \frac{1}{16} + \cdots + \frac{1}{16} = 8\left(\frac{1}{16}\right) = \frac{1}{2}, \\ &\vdots \end{aligned}$$

Using (5) and our underestimates, we see that

$$\begin{aligned} &1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \cdots \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \\ &> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \end{aligned}$$

Thus the partial sums of the harmonic series grow without bound which is expressed by

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$$

Below is a formal proof of the the fact that the sums of terms in parenthesized groupings lie between  $\frac{1}{2}$  and 1. You should *not* include the proof in your essay; the mathematics of your essay is to be treated informally. Observe that the terms of a parenthesized group are given by  $\frac{1}{2^n + 1}, \dots, \frac{1}{2^{n+1}}$  for some  $n \geq 1$ .

**Lemma 1** *Let  $n$  be a positive integer. Then  $\frac{1}{2} < \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} < 1$ .*

PROOF: Since  $2^{n+1} = 2^n + 2^n$  the sum in the statement of the lemma has  $2^n$  terms. Each term has the form  $\frac{1}{2^n + \ell}$  for some  $1 \leq \ell \leq 2^n$  and thus satisfies

$$\frac{1}{2^{n+1}} \leq \frac{1}{2^n + \ell} \leq \frac{1}{2^n}.$$

At least one of the terms is larger than  $\frac{1}{2}$  and one at least one is smaller than 1. Therefore

$$\frac{1}{2} = 2^n \left( \frac{1}{2^{n+1}} \right) < \frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}} < 2^n \left( \frac{1}{2^n} \right) = 1.$$

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