

AXIOMS for ORDERED FIELDS

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Axioms about addition and multiplication

Commutativity $a + b = b + a$, $a \cdot b = b \cdot a$.

Associativity $(a + b) + c = a + (b + c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Distributivity $a(b + c) = a \cdot b + a \cdot c$

additive identity $a + 0 = a$

multiplicative identity $a \cdot 1 = a$

additive inverse There are two equivalent forms:

1. For every a there is a b such that $a + b = 0$.
2. $a + (-a) = 0$.

multiplicative inverse There are two equivalent forms:

1. For every a there is a b such that $a \cdot b = 1$.
2. $a \cdot \frac{1}{a} = 1$.

Order Axioms

trichotomy Either $a < b$ or $a = b$ or $b < a$.

transitivity If $a < b$ and $b < c$ then $a < c$.

Order and Addition $a < b$ if and only if $a + c < b + c$.

Order and Multiplication – If $c > 0$ then $a > b$ if and only if $c \cdot a > c \cdot b$.

– If $c < 0$ then $a > b$ if and only if $c \cdot a < c \cdot b$.

All of these axioms are true for the rational numbers and the reals. All but the existence of multiplicative inverses are true for the integers. Another essential axiom to prove results about the integers is that they are discretely ordered:

$$\forall x \forall y \forall z [x < y \rightarrow \neg(x < z \wedge z < y)].$$