Two problem solutions

John T. Baldwin

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Page 184, number 9. Prove (11.1.4) that if there is an injection \( f : X \mapsto N_n \) then \( X \) is finite and the cardinality of \( X \) is at most \( n \).

Proof. We work by induction on \( n \). If \( n = 1 \), then an injection into \( N_1 \) must be onto. So \( f \) is invertible and \( X \) is a finite set with cardinality \( n \).

**Induction Hypothesis:** Suppose that for any \( X \) if there is an injection \( f \) from \( X \) into \( N_k \) then \( X \) is finite and the cardinality of \( X \) is at most \( k \).

**Induction step:** We must prove for any \( X \) if there is an injection \( f \) from \( X \) into \( N_{k+1} \) then \( X \) is finite and the cardinality of \( X \) is at most \( k + 1 \).

Case 1: \( k + 1 \) is not in the range of \( f \). Then \( f \) is an injection into \( N_k \) and the result is immediate from the induction hypothesis.

Case 2: \( k + 1 \) is in the range of \( f \). Say \( f(a) = k + 1 \). Now let \( g \) be the restriction of \( f \) to \( X - \{a\} \). Then \( g \) is an injection of \( X - \{a\} \) into \( N_k \). So again by induction, \( X - \{a\} \) is finite and \( |X - \{a\}| \) is some \( m \leq k \). Then by 10.2.1 (the addition principle), \( X = X \cup \{a\} \) is a disjoint union of finite sets, so \( X \) is finite and \( |X| = m + 1 \leq k + 1 \).

Page 184 number 10. Prove (11.1.6) that if \( X \) and \( Y \) are non-empty finite sets with \( |X| < |Y| \), there is no surjection from \( X \) onto \( Y \).

Proof. Suppose for contradiction that such \( f \) exists. By the definition of finite there exists an \( m < n \) and functions \( g_1, g_2 \) such that \( g_1 \) is a bijection from \( N_m \) onto \( X \) and \( g_2 \) is a bijection from \( N_m \) onto \( Y \). But then \( h = g_2^{-1} \circ f \circ g_1 \) is a surjection from \( N_m \) onto \( N_n \). Now we can find an injection \( h' : N_n \mapsto N_m \); \( h'(y) \) is defined to be the least \( k < m \) such that \( h(k) = y \). Now by 11.1.1 the existence of \( h' \) implies \( m \leq n \). This contradiction completes the proof.