Math 215: Introduction to Advanced Mathematics
Solution to trigonometry problem from Problem Set 1

Assignment: Write a careful complete solution of the following typical problem from a trigonometry text. Be sure you are clear about what $K$ refers to.

Show \( \sin A = \sin B \) if and only if \( A = B + 360K \) or \( A + B = 180 + 360K \).

Correct statement of problem (replacing the ambiguity of a typical high school text).

Show \( \sin A = \sin B \) if and only if for some integer \( K \), \( A = B + 360K \) or \( A + B = 180 + 360K \).

Logic fact: We will use the observation made in class that for any propositions \( P, Q \): \( (\exists k)(P(k) \text{ or } Q(k)) \) is equivalent to \( (\exists k)P(k) \) or \( (\exists k)Q(k) \).

Answer: Note first that for every integer \( K \) and any angle \( A \), \( \sin A = \sin(A + 360K) \). (We will discuss a more rigorous proof later in the semester but consider that fact as given now.)

Clearly if for some \( K \), \( A = B + 360K \), \( \sin A = \sin B \), since for all \( K \), \( \sin B = \sin(B + 360K) \). Moreover, if for some \( K \), \( A = 180 - B + 360K \), \( \sin A = \sin B \) since \( \sin A = \sin(180 - A) \) and for all \( K \), \( \sin(180 - A) = \sin(180 - A + 360K) \).

We have proved: if for some integer \( K \), \( A = B + 360K \) or \( A + B = 180 + 360K \) then \( \sin A = \sin B \). (The 'logic fact' allows us to distribute the quantifier.) Now we must show the converse.

We first consider the case \( 0 \leq A, B < 360 \).

Suppose first both \( A \) and \( B \) are angles with between \( 0 \leq A \leq 180 \) and \( 0 \leq B \leq 180 \). Then, looking at the unit circle, \( \sin A = \sin B \) implies either \( A = B \) or \( A = 180 - B \).

There is an additional possibility. If \( 0 \leq A, B < 360 \), by examining the unit circle we see either the pair \( A \) and \( B \) satisfy the case in the last paragraph or both satisfy \( 180 < A, B < 360 \). And then there are two possibilities, \( A = B \) or \( A + B = 540 \). The second possibility can be written \( A + B = 180 + 360 \).

We now reduce to the case \( 0 \leq A, B < 360 \).

Suppose \( \sin A = \sin B \).

For any angle \( A \) there is an integer \( K \) and an \( A' \) with \( 0 \leq A' < 360 \) so that \( A = A' + 360K \). So we can choose \( A', B' \), with \( 0 < A', B' < 360 \) and integers \( K_1, K_2 \) so that \( A = A' + 360K_1 \) and \( B = B' + 360K_2 \).
Now sin \( A = \sin B \) implies sin \( A' = \sin B' \) and so by our treatment of angles between 0 and 360, we know there are three possibilities. \( A' = B', A' + B' = 180, A' + B' = 540. \)

Now by substituting we will see that in the first case for some \( K \), \( A = B + 360K \) and in the other two cases, for some \( K \), \( A + B = 180 + 360K \). Again, the logic fact allows us to do find \( K \) separately for each case.

We carry out the detail of the substitution only for the third case. We have assumed \( A = A' + 360K_1 \) and \( B = B' + 360K_2 \) and by choice of case \( A' + B' = 540 \).

We have \( A = A' + 360K_1 \). That is, \( A = (540 - B') + 360K_1 \) and therefore \( A = (540 - (B - 360K_2)) + 360K_1 \). So \( A = 180 - B + 360(K_1 - K_2 + 1) \). We have the result with \( K = K_1 - K_2 + 1 \).