Math 215: Introduction to Advanced Mathematics

Solution to trigonometry problem from Problem Set 1

Assignment: Write a careful complete solution of the following typical problem from a trigonometry text. Be sure you are clear about what K refers to.

Show $\sin A = \sin B$ if and only if $A = B + 360K$ or $A + B = 180 + 360K$.

Correct statement of problem (replacing the ambiguity of a typical high school text).

Show $\sin A = \sin B$ if and only if for some integer K, $A = B + 360K$ or $A + B = 180 + 360K$.

Logic fact: We will use the observation made in class that for any propositions P, Q: $(\exists k)[P(k) \text{ or } Q(k)]$ is equivalent to $(\exists k)P(k)$ or $(\exists k)Q(k)$.

Answer: Note first that *for every* integer K and any angle A, $\sin A = \sin(A + 360K)$. (We will discuss a more rigorous proof later in the semester but consider that fact as given now.)

Clearly if for some K, $A = B + 360K$, $\sin A = \sin B$, since for all K, $\sin B = sin(B + 360K)$. Moreover, if for some K, $A = 180 - B + 360K$, $\sin A = \sin B$ since $\sin A = \sin(180 - A)$ and for all K, $\sin(180 - A) = \sin(180 - A + 360K)$.

We have proved: if for some integer K, $A = B + 360K$ or $A + B = 180 + 360K$ then $\sin A = \sin B$. (The 'logic fact' allows us to distribute the quantifier.) Now we must show the converse.

We first consider the case $0 \leq A, B < 360$.

Suppose first both A and B are angles with between $0 \leq A \leq 180$ and $0 \leq B \leq 180$. Then, looking at the unit circle, $\sin A = \sin B$ implies either $A = B$ or $A = 180 - B$.

There is an additional possibility. If $0 \leq A, B < 360$, by examining the unit circle we see either the pair A and B satisfy the case in the last paragraph or both satisfy $180 < A, B < 360$. And then there are two possibilities, $A = B$ or $A + B = 540$. The second possibility can be written $A + B = 180 + 360$.

We now reduce to the case $0 \leq A, B < 360$.

Suppose $\sin A = \sin B$.

For any angle A there is an integer K and an A' with $0 \leq A' < 360$ so that $A = A' + 360K$. So we can choose A', B', with $0 < A', B' < 360$ and integers K_1, K_2 so that $A = A' + 360K_1$ and $B = B' + 360K_2$.

Now $\sin A = \sin B$ implies $\sin A' = \sin B'$ and so by our treatment of angles between 0 and 360, we know there are three possibilities. $A' = B'$, $A' + B' = 180$, $A' + B' = 540$.

Now by substituting we will see that in the first case for some K, $A = B + 360K$ and in the other two cases, for some K, $A + B = 180 + 360K$. Again, the logic fact allows us to do find K separately for each case.

We carry out the detail of the substitution only for the third case. We have assumed $A = A' + 360K_1$ and $B = B' + 360K_2$ and by choice of case $A' + B' = 540$.

We have $A = A' + 360K_1$. That is, $A = (540 - B') + 360K_1$ and therefore

$A = (540 - (B - 360K_2)) + 360K_1$. So $A = 180 - B + 360(K_1 - K_2 + 1)$. We have the result with $K = K_1 - K_2 + 1$.