Ehrenfreuht-Mostowski models in Abstract Elementary Classes

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Remark 0.1 (Preliminary version.) This is an extremely preliminary version of my paper for the Ann Arbor proceedings. It contains some material (e.g. guide to reading 394) which will almost certainly come out. But it does provide I think some insight into Shelah’s use of Ehrenfeucht-Mostowski models which I hadn’t understood. Even more than usual comments on both the general direction and specific mathematical problems are warmly appreciated.

We prove three results using EM models. 1) Morley’s omitting types theorem – for Galois types. 2) If an AEC (with amalgamation) is categorical in some uncountable power $\mu$ it is stable in (every) $\lambda < \mu$. 3) Categoricity implies tameness. The first two results easily lead the following theorem: If an AEC (with amalgamation) $K$ is categorical in a regular $\lambda$ greater than $\theta$, then it is categorical in all cardinals between $\theta$ and $\lambda$.

In [11], Shelah proclaims the aim of reconstructing model theory, ‘with no use of even traces compactness’. We analyze here one aspect of this program. Keisler organizes [6] around four kinds of constructions: the Henkin method, Ehrenfeucht-Mostowski models, unions of chains, and ultraproducts. The later history of model theory reveals a plethora of tools arising in stability theory. Fundamental is a notion of dependence which arises from Morley’s study of rank, and passes through various avatars of splitting, strong splitting, and dividing before being fully actualized in the first order setting as forking. We eschew this technique altogether in this paper—to isolate its role.

For the axioms of an AEC $(K, \preceq_K)$, see for example [12, 11, 14, 4]. In English, we write strong for $\preceq_K$.

Sections 1, 2, 3 define most of the terminology. The ‘result’ sections 4, 6, 5 are almost independent. One result from 4 is used in 6.

1 Assumptions

Begin with a countable vocabulary, $\tau$. We use variants on $\tau$ to denote vocabularies. In addition to this usage, Shelah uses $\tau$ as an operator: $\tau(\Phi)$ denotes the vocabulary of the set of sentences $\Phi$. We may write $\tau$-structure or $L$-structure; uniformity should appear in later versions.

Assumption 1.1 1. $K$ has arbitrarily large models.

2. $K$ satisfies the amalgamation property and the joint embedding property.

3. The Lowenheim-number of $K$, $\text{LN}(K)$, is $\aleph_0$.

We say $K$ has the amalgamation property if $M \leq N_1$ and $M \leq N_2 \in K$ with all three in $K$ implies there is a common strong extension $N_3$ completing the diagram. Joint embedding means any two members of $K$
have a common strong extension. Crucially, we amalgamate only over members of \( K \); this distinguishes this context from the context of homogeneous structures. Probably, amalgamation does not imply the existence of arbitrarily large models. Disjoint amalgamation (the images of \( N_1 \) and \( N_2 \) in \( N_3 \) intersect in the image of \( M \)) easily implies the existence of arbitrarily large models.

**Notation 1.2** \( \Theta = \Theta(K) \) denotes \( \beth_\omega(2^{\aleph_0}) \). A cardinal is called huge if it is bigger than \( \Theta \).

This cardinal is sometimes called the Hanf number of \( K \). This is somewhat misleading because a single class cannot have a Hanf number – a Hanf number is a maximum for all similarity types. It is in fact not the Hanf number of \( K \) but the Hanf number for all AEC with Löwenheim number \( \aleph_0 \). But as we’ll see there is a still wider basis for this name; we will consider other classes of models (which are not AEC) and it is crucial that all of them have the property: for any model \( M \) with \( |M| \geq \Theta \), there are models in the class of all cardinalities which omit all types omitted in \( M \).

Many of the results here will go through without change, replacing \( \aleph_0 \) by \( LS(K) \). We have chosen to err on the side of concreteness.

There is some vestige of compactness here. Both the existence of arbitrarily large models and amalgamation are proved in first order logic using compactness. But they have completely semantic statements and you have to start somewhere.

## 2 The presentation theorem and E-M models

We call the next result: the presentation theorem. It allows us to replace the entirely semantic description of an abstract elementary class by a syntactic one. I find it extraordinary that the notion of an AEC which is designed to give a version of the Fraïssé construction and thus saturated models, also turns out to allow the use of the second great model theoretic technique of the 50’s: Ehrenfeucht-Mostowski models.

**Theorem 2.1** If \( K \) is an AEC with Löwenheim number \( \aleph_0 \) (in a countable vocabulary \( L \)), there is a countable language \( L' \), a first order \( L' \)-theory \( T' \) and a set of \( 2^{\aleph_0} \) types \( \Gamma \) such that:

\[
K = \{ M' \models L : M' \models T' \text{ and } M' \text{ omits } \Gamma \}. 
\]

Moreover, if \( M' \) is an \( L' \)-substructure of \( N' \) where \( M' \), \( N' \) satisfy \( T' \) and omit \( \Gamma \) then \( M' \models L \preceq \bigcirc K \ N' \models L \).

Proof. Let \( L' \) contain \( n \)-ary function symbols \( F^n_i \) for \( n < \omega \) and \( i < \omega \). We take as \( T' \) the theory which asserts only that nonempty models exist. For any \( a \in M \), let \( M'_a \) denote the \( L' \)-structure generated by \( a \). Let \( \Gamma \) be the set of quantifier free \( L' \)-types of finite tuples \( a \) such that \( M'_a \models L \notin K \) or for some \( b \subseteq a \), \( M'_b \models L \cong \bigcirc K \ M'_a \models L \).

We claim \( T' \) and \( \Gamma \) suffice. That is, if \( K' = \{ M' \models L : M' \models T' \text{ and } M' \text{ omits } \Gamma \} \) then \( K = K' \). If \( M' \models L \in K' \), write \( M' \) as a direct limit of finitely generated \( L' \)-structures \( M'_a \). By the choice of \( \Gamma \), each \( M'_a \models L \in K \) if \( a \subseteq a' \), \( M'_a \models L \preceq \bigcirc K \ M'_a \models L \), and so by the unions of chains axioms \( M' \models L \in K \). Conversely, if \( M \in K \), write \( M \) as a \( \preceq \bigcirc K \)-direct limit of countable \( L \)-structures. Expand each countable \( \preceq \bigcirc K \)-substructure of \( M \) to an \( L' \)-structure by letting \( \{ F^n_i(a) : i < \omega \} \) enumerate the universe of \( M \). By proceeding inductively, we can guarantee that these expansions cohere and verify that \( M \in K' \).

The moreover holds for countable structures directly by the choice of \( \Gamma \) and extends to arbitrary structures by the union of chain axioms on an AEC. In more detail, we have \( M' \) a direct limit of \( M_a \) and \( N' \) is a \( \preceq \bigcirc K \)-direct
limit of \( N_a \) where \( M_a = N_a \) for \( a \in M \). Each \( M_a \models L \preceq_K N \) so the direct limit \( M \) is a strong submodel of \( N \).

We say \( K \) is a \( \text{PC}(\aleph_0, \mathbb{N}) \) class if it satisfies the conclusion of Theorem 2.1.

**Remark 2.2**

1. There is no use of amalgamation in this theorem.
2. The only penalty for increasing the size of the language or the L"owenheim number is that the size of \( \mathcal{L}' \) and the number of types omitted; thus \( \theta \) must be chosen larger.
3. We can (and Shelah does) observe that the class of pairs \((M, N)\) with \( M \preceq_K N \) forms a \( \text{PC}(\aleph_0, \mathbb{N}) \) class if it satisfies the conclusion of Theorem 2.1. A more useful version. See Theorem 2.5 and its applications. This clause appears in Grossberg’s account: [2]

**Notation 2.3**

1. For any linearly ordered set \( X \subseteq M \) where \( M \) is a \( \tau \)-structure we write \( D_\tau(X) \) (diagram) for the set of \( \tau \)-types of finite sequences (in the given order) from \( X \). We will omit \( \tau \) if it is clear from context.
2. Such a \( D_\tau(X) = \Phi \) is called ‘proper for linear orders’ by Shelah.
3. If \( X \) is a sequence of \( \tau \)-indiscernibles with diagram \( \Phi = D_\tau(X) \), for any linear ordering \( I \), \( EM(I, \Phi) \) denotes the \( \tau \)-Skolem hull of a sequence of order indiscernibles realizing \( \Phi \).
4. If \( \tau_0 \subseteq \tau \), the reduct of \( EM(I, \Phi) \) to \( \tau \) is denoted \( EM_{\tau_0}(I, \Phi) \).

‘Morley’s method’ (Section 7.2 of [2]) is a fundamental technique in first order model theory. It is essential for the foundations of simplicity theory and for the construction of indiscernibles in infinitary logic. We quote the first order version here; in Lemma 5.1, we prove the analog for abstract elementary classes.

**Lemma 2.4** If \( (X, <) \) is a sufficiently long linearly ordered subset of a \( \tau \)-structure \( M \), for any \( \tau' \) extending \( \tau \) (the length needed for \( X \) depends on \(|\tau'||\) there is a countable set \( Y \) of \( \tau' \)-indiscernibles (and hence one of arbitrary order type) such that \( D_\tau(Y) \subseteq D_\tau(X) \). This implies that the only (first order) \( \tau \)-types realized in \( EM(X, D_\tau(X)) \) are realized in \( M \).

Further, we find Skolem models over indiscernibles in an AEC.

**Theorem 2.5** If \( K \) is an abstract elementary class which is represented as a \( \text{PC}(\aleph_0, \mathbb{N}) \) class witnessed by \( \tau', T', \Gamma \) that has arbitrarily large models, then for every linear order \((I, <)\) there is an \( \tau' \)-structure \( M = EM(I, \Phi) \) such that:

1. \( M \models T' \).
2. The \( \tau' \)-structure \( M = EM(I, \Phi) \) is the Skolem hull of \( I \).
3. \( I \) is a set of \( \tau' \)-indiscernibles in \( M \).
4. \( M \models L \) is in \( K \).
5. If \( I' \subset I \) then \( EM_\tau(I', \Phi) \preceq_K EM_\tau(I, \Phi) \).
Proof. The first four clauses are a direct application of Lemma 2.4, Morley’s theorem on omitting types. See also problem 7.2.5 of Chang-Keisler [7]. It is automatic that $EM(I', \Phi)$ is an $L'$ substructure of $EM(I, \Phi)$. The moreover clause allows us to extend this to $EM_I(I', \Phi) \preceq_K EM_I(I, \Phi)$.

Note that we have simplified our presentation of many members of $K$. Inside the class $K$, which is the set of reducts of models which omit $\Gamma$, sits a class $K'$, which is the class of reducts of Skolem hulls of order indiscernibles. In general, $K'$ is a proper subclass of $K$. It may not be an AEC because we don’t know closure under unions of chains. In [10], under strong hypotheses the closure is proved.

Silver (Chapter 18 of [7]) gives a simple example of a pseudoelementary class where the categoricity spectrum and its complement are both cofinal in the class of cardinals. The example is the class of models $(M, X)$ where $|2^{|X|} \geq |M|$. This class is not an AEC because it is not closed under unions of chains.

## 3 Galois types and saturation

In this section we take advantage of joint embedding and amalgamation to find a monster model. We then define types in terms of orbits of stabilizers of submodels. This allows an identification of ‘model-homogeneous’ with ‘saturated’. That is, we give an abstract account of Morley-Vaught [8].

**Definition 3.1** $M$ is $\mu$-model homogenous if for every $N \preceq_K M$ and every $N' \in K$ with $|N'| < \mu$ and $N \preceq_K N'$ there is a $K$ embedding of $N'$ into $M$ over $N$.

To emphasize, this differs from the homogenous context because the $N$ must be in $K$. It is easy to show:

**Lemma 3.2** If $M_1$ and $M_2$ are $\mu$-model homogenous of cardinality $\mu > \text{LS}(K)$ then $M_1 \sim M_2$.

Proof. If $M_1$ and $M_2$ have a common submodel $N$ of cardinality $< \mu$, this is an easy back and forth. Now suppose $N_1$, $(N_2)$ is a small model of $M_1$, $(M_2)$ respectively. By the joint embedding property there is a small common extension $N$ of $N_1, N_2$ and by model homogeneity $N$ is embedded in both $M_1$ and $M_2$. □

Note that in the absence of joint embedding to get uniqueness, we would (as in [12]) have to add to the definition of ‘$M$ is model homogenous’ that all models of cardinality $< \mu$ are embedded in $M$.

**Theorem 3.3** If $K$ has the amalgamation property and $\mu^* < \mu^* = \mu^* > \text{LS}(K)$ then there is a model $\mathcal{M}$ of cardinality $\mu^*$ which is model homogenous.

We call the model constructed in Theorem 3.3, the monster model. From now on all, structures considered are substructures of $\mathcal{M}$ with cardinality $< \mu^*$. The standard arguments for the use of a monster model in first order model theory ([5, 1] apply here.

**Definition 3.4** Let $M \in K$, $M \preceq_K \mathcal{M}$ and $a \in \mathcal{M}$. The Galois type of $a$ over $M$ is the orbit of $a$ under the automorphisms of $\mathcal{M}$ which fix $M$.

When amalgamation is not assumed, the Galois type is an equivalence class of an equivalence relation on triples $(M, a, N)$. Since we have amalgamation and have fixed $\mathcal{M}$, we don’t need the extra notation.

**Definition 3.5** The set of Galois types over $M$ is denoted $ga - S(M)$. 
We say a Galois type $p$ over $M$ is realized in $N$ with $M \preceq_K N \preceq_K M$ if $p \cap N \neq \emptyset$.

**Definition 3.6** The model $M$ is $\mu$-Galois saturated if for every $N \preceq_K M$ with $|N| < \mu$ and every Galois type $p$ over $N$, $p$ is realized in $M$.

The following model-homogeneity=saturativity theorem is the only result beyond Morley we are taking without proof in this article. The result is announced with an incomplete proof in [13]. Full proofs are given in Theorem 6.7 of [4] and .26 of [11].

**Theorem 3.7** For $\lambda > \text{LS}(K)$, The model $M$ is $\lambda$-Galois saturated if and only if it is $\lambda$-model homogeneous.

In the remainder of this section we discuss some important ways in which Galois types behave differently from ‘syntactic types’.

Note that if $M \subset N \preceq_K M$, then $p \in \text{ga} - S(N)$ extends $p' \in \text{ga} - S(N)$ if for some (any) $a$ realizing $p$ and some (any) $b$ realizing $p'$ there is an automorphism $\alpha$ fixing $N$ and taking $a$ to $b$.

**Lemma 3.8** If $M = \bigcup_{i<\omega} M_i$ is an increasing chain of members of $K$ and $\{p_i : i < \omega\}$ satisfies $p_{i+1} \mid M_i = p_i$, there is a $p_\omega \in \text{ga} - S(M)$ with $p_\omega \mid M_i = p_i$ for each $i$.

Proof. Let $a_i$ realize $p_i$. By hypothesis, for each $i < \omega$, there exists $f_i$ which fixes $M_{i-1}$ and maps $a_i$ to $a_{i-1}$. Let $g_i$ be the composition $f_0 \circ f_1 \circ \ldots \circ f_i$. Then $g_i$ maps $a_i$ to $a_0$, fixes $M_0$ and $g_i \mid M_{i-1} = g_{i-1} \mid M_{i-1}$. Let $M'_i$ denote $g_i(M_i)$ and $M'$ their union. Then $\bigcup_{i<\omega} g_i$ is an isomorphism between $M$ and $M'$. So by model-homogeneity there exists an automorphism $h$ of $M$ with $h \mid M_i = g_i \mid M_i$ for each $i$. Now $g_i^{-1} \circ h$ fixes $M_i$ and maps $a_\omega$ to $a_i$ for each $i$. This completes the proof. □

Now suppose we wanted to prove Lemma 3.8 for chains of length $\delta > \omega$. The difficulty can be seen at stage $\omega$. In addition to the assumptions of Lemma 3.8, we are given $\{a_i : i \leq \omega\}$ and $f_{\omega,i}$ which fixes $M_i$ and maps $a_\omega$ to $a_i$. We can construct $g_i$ as in the original proof. The difficulty is to find $g_\omega$ which extends all the $g_i$ and maps $a_\omega$ to $a_0$. In the argument for Lemma 3.8, we found a map $h$ and an element (which we will now call $a'_\omega$) such that $h$ takes $a'_\omega$ to $a_0$ while $h$ extends all the $g_i$. We would be done if $a_\omega$ and $a'_\omega$ realized the same galois type over $M = M_\omega$. In fact, $a_\omega$ and $a'_\omega$ realized the same galois type over each $M_i$. So the following locality condition (for chains of length $\omega$) would suffice for this special case. Moreover, by a further induction locality would give Lemma 3.8 for chains of arbitrary length. Unfortunately, locality probably does not hold for all AEC with amalgamation.

This is a key distinction between the general AEC case and homogenous structures. In homogeneous structures, types are syntactic objects and locality is trivial. Thus, as pointed out by Shelah, Hyttinen, and Buechler-Lessmann, Lemma 3.8 is true for arbitrary length of chains in the homogeneous context.

**Definition 3.9** $K$ has local galois types if for every $M = \bigcup_{i<\kappa} M_i$ in a continuous increasing chain of members of $K$ and for any $p, q \in \text{ga} - S(M)$: if $p \mid M_i = q \mid M_i$ for every $i$ then $p = q$.

**4 Getting stability**

The key idea is that for a linear order $I$ and model $EM(I, \Phi)$ automorphisms of $I$ induce automorphisms of $EM(I, \Phi)$. And, automorphisms of $EM(I, \Phi)$ preserve types in any reasonable logic; in particular, automorphisms of $EM(I, \Phi)$ preserve Galois types. Note that a model $N$ is (defined to be) stable if few types are realized in $N$. So if $N$ is a brimful model (Definition 4.2) then the model $N$ is $\sigma$-stable for every $\sigma < |N|$. 

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Since we deal with reducts, we will consider several structures with the same universe; it is crucial to keep the vocabulary of the structure in mind. The AEC under consideration has vocabulary \( \tau \); it is presented as reducts of models of theory \( T' \) (which omit certain types) in a vocabulary \( \tau' \). In addition, we have the class of linear orderings (LO) in the background.

We really have three classes: \(( \text{LO}, \subset), \mathcal{K}' \) which is \( \text{Mod}(T') \) with submodel as \( \tau' \)-closed subset, and \(( \mathcal{K}, \preceq_{\mathcal{K}}) \). The first and third are AEC's; I haven't thought carefully about whether the second is. We are describing the properties of the EM-functor between \(( \text{LO}, \subset) \) and \( \mathcal{K}' \) or \( \mathcal{K} \). \( \mathcal{K}' \) is only a tool that we are singling out to see the steps in the argument. The following definitions hold for any of the three classes and I write \( \preceq \) for the notion of substructure. In this section of the paper I am careful to use \( \preceq \) when discussing all three cases versus \( \preceq_{\mathcal{K}} \) for the AEC.

**Definition 4.1** \( M_2 \) is \( \sigma \)-universal over \( M_1 \) in \( N \) if for every \( M_1 \leq M_2 \leq N \) and whenever \( M_1 \leq M_2' \leq N \), with \( |M_1| \leq |M_2'| \leq \sigma \), there is a \(( \text{partial isomorphism}) \) fixing \( M_1 \) and taking \( M_2' \) into \( M_2 \).

I introduce one term for shorthand.

**Definition 4.2** \( N \) is brimful if for every \( \sigma < |N| \), and every \( M_1 \leq N \) with \( |M_1| = \sigma \), there is an \( M_2 \) that is \( \sigma \)-universal over \( M_1 \) in \( N \).

The next notion just makes it easier to write the proof of the following Lemma.

**Notation 4.3** Let \( I \subset J \) be linear orders. We say \( a \) and \( b \) in \( J \) realize the same cut over \( I \) and write \( a \sim_I b \) if for every \( j \in J, a < j \) if and only if \( b < j \).

**Claim 4.4** The linear order \( I = \lambda^{< \omega} \) is brimful. (Lemma 3.7 of 362).

Proof. Let \( J \subset I \) have cardinality \( \theta < \lambda \). Without loss of generality we can assume \( J = A^{< \omega} \) for some \( A \subset \lambda \). Note that \( \sigma \sim_J \tau \) if and only if for the least \( n \) such that \( \sigma \restriction n = \tau \restriction n \in J, \sigma(n) \sim_A \tau(n) \). Thus there are only \( \theta \) cuts over \( J \) realized in \( I \). For each cut \( C_\alpha, \alpha < \theta \), we choose a representative \( \sigma_\alpha \in I - J \) of length \( n \) such that \( \sigma \restriction n = \tau \restriction n, \sigma(n) \sim_C \tau(n) \). For any \( J^* \) extending \( J \), we can assume \( J^* = B^{< \omega} \) for some \( B \subset \lambda \), say with \( \text{otp}(B) = \gamma \). Thus, the intersection of \( J^* \) with a cut in \( J \) is isomorphic to a subset of \( \gamma^{< \omega} \). We finish by noting for any ordinal \( |\gamma| = \theta \), \( \gamma^{< \omega} \) can be embedded in \( \theta^{< \omega} \).

Qing Zhang has provided the following elegant argument for the last claim. First show by induction on \( \gamma \) there is a map \( g \) embedding \( \gamma \) in \( \theta^{< \omega} \). (E.g. if \( \gamma = \lim_{i<\omega} \gamma_i \), and \( g_i \) maps \( \gamma_i \) into \( \theta^{< \omega} \), let for \( \beta < \gamma \), \( g(\beta) = \hat{\gamma}_i(\beta) \) where \( \gamma_i \leq \beta < \gamma_{i+.} \) Then let \( h \) map \( \gamma^{< \omega} \) into \( \theta^{< \omega} \) by, for \( \sigma \in \gamma^{< \omega} \) of length \( n \), setting \( h(\sigma) = (g(\sigma(0)), \ldots, g(\sigma(n-1))) \). \[ \Box_{4.4} \]

Exercise: For an ordinal \( \gamma \), let \( \gamma^{< *} \) denote the functions from \( \omega \) to \( \gamma \) with only finitely many non-zero values. Show \( \gamma^{< *} \) is a dense linear order and so is not isomorphic to \( \gamma^{< \omega} \). Vary the proof above to show \( \gamma^{< *} \) is brimful.

Since every \( L' \)-substructure of \( \text{EM}(I, \Phi) \) has the form \( \text{EM}(I_0, \Phi) \) for some subset \( I_0 \) of \( I \), we have immediately:

**Claim 4.5** If \( I \) is brimful as linear order, \( \text{EM}(I, \Phi) \) is brimful as an \( L' \)-structure.

Now using amalgamation and categoricity, we move to the AEC \( \mathcal{K} \). There are some subtle uses here of the 'coherence axiom': \( M \subseteq N \preceq_{\mathcal{K}} N_1 \) and \( M \preceq_{\mathcal{K}} N_1 \) implies \( M \preceq_{\mathcal{K}} N \).
Claim 4.6 If $I$ is brimful as linear order, $EM_r(I, \Phi)$ is brimful as a member of $K$.

Proof. Let $M = EM(I, \Phi)$; we must show $M \upharpoonright \tau$ is brimful as a member of $K$. Suppose $M_1 \preceq_K M \upharpoonright \tau$ with $|M_1| = \sigma < |M|$. Then there is $N_1 = EM(I', \Phi)$ with $|I'| = \sigma$ and $M_1 \leq N_1$. By Lemma 2.5.5, $N_1 \upharpoonright \tau \preceq_K M \upharpoonright \tau$. So $M_1 \preceq_K N_1 \upharpoonright \tau$ by the coherence axiom. Let $M_2$ have cardinality $\sigma$ and $M_1 \preceq_K M_2 \preceq_K M \upharpoonright \tau$. Choose a $\tau'$-structure $N_2$ of $M$ containing $N_1$ and $M_2$. Now, $M_2$ can be embedded by a map $f$ into the $\sigma$-universal $\tau'$-structure $N_3$ containing $N_1$ which is guaranteed by Claim 4.5. But $f(N_2) \upharpoonright \tau \preceq_K N_3 \upharpoonright \tau$ by the coherence axiom so $N_3 \upharpoonright \tau$ is the required $K$-universal extension of $M_1$. \hfill \Box_{4.6}

Definition 4.7

1. Let $N \subset M$. $N$ is $\lambda$-galois-stable if for every $M \subset N$ with cardinality $\lambda$, only $\lambda$ Galois types over $M$ are realized in $N$.

2. $K$ is $\lambda$ stable if $M$ is. That is aut$_M(M)$ has only $\lambda$ orbits for every $M \subset M$ with cardinality $\lambda$.

Since each galois type over $M_0$ realized in $M$ is represented by an $M_1$ with $M_0 \preceq_K M_1 \preceq_K M$ and $|M_1| = |M_0|$, Claim 4.6 implies immediately:

Claim 4.8 If $K$ is $\lambda$-categorical, the model $M$ with $|M| = \lambda$ is $\sigma$-Galois stable for every $\sigma < \lambda$.

Theorem 4.9 If $K$ is categorical in $\lambda$, then $K$ is $\sigma$-galois-stable for every $\sigma < \lambda$.

Proof. Suppose $K$ is not $\sigma$-stable for some $\sigma < \lambda$. Then by Löwenheim-Skolem, there is a model $N$ of cardinality $\sigma^+$ which is not $\sigma$-stable. Let $M$ be the $\sigma$ stable model with cardinality $\lambda$ constructed in Claim 4.8. Categoricity implies $N$ can be embedded in $M$. The resulting contradiction proves the result. \hfill \Box_{4.9}

Corollary 4.10 If $\lambda$ is regular, the model of power $\lambda$ is saturated and so model homogeneous.

Proof. Now choose in $M$ using $< \lambda$-stability and Löwenheim-Skolem, $M_i$ for $i < \lambda$ so that each $M_i$ has cardinality $< \lambda$ and $M_{i+1}$ realizes all types over $M_i$. By regularity, it is easy to check that $M_\lambda$ is saturated. \hfill \Box_{4.10}

More strongly we have.

Corollary 4.11 Suppose $K$ is categorical in $\lambda$ and $\lambda$ is regular. Then for every regular $\mu$, LS($K$) $< \mu < \lambda$ there is a model $M_\mu = EM(I_\mu, \Phi)$ which is saturated. In particular, it is $\mu$-model homogeneous.

Proof. Let $N = EM(J, \phi)$ be the model of cardinality $\lambda$. We construct an alternating chain of submodels of length $\mu$. $M_0 \leq M$ is arbitrary with cardinality $\mu$. $M_{2^\alpha+1}$ has cardinality $\mu$ and realizes all types over $M_{2^\alpha}$ (possible by Corollary 4.10). $M_{2^\alpha+2}$ has cardinality $\mu$, $M_{2^\alpha+2} \leq M_{2^\alpha+2}$ and $M_{2^\alpha+2}$ is $EM(I_{\alpha+1}, \Phi)$ where $I_\alpha \subset I_{\alpha+1} \subset J$ and all $I_\alpha$ have cardinality $\mu$. Then $EM(I_\mu, \Phi) = \bigcup_{\alpha < \mu} EM(I_\alpha, \Phi)$ is saturated by regularity. \hfill \Box_{4.11}

Now using stability we can get a still stronger result, eliminating the hypothesis that $\mu$ is regular. We show the proofs of both Corollary 4.11 and Corollary 4.12 since in the first case we constructed a saturated model directly and in the second a model homogeneous structure.

Claim 4.12 Suppose $K$ is categorical in $\lambda$ and $\lambda$ is regular. Then for every $\mu$, LS($K$) $< \mu < \lambda$ there is a model $M_\mu = EM(I_\mu, \Phi)$ which is $\mu$-model homogeneous.
Proof. Fix \( I \) with \(|I| < \lambda \) that is brimful as a linear order. We show \( EM_\tau(I, \Phi) \) is model homogenous. We must show \( M \upharpoonright \tau \) is brimful as a member of \( K \). Suppose \( M_1 \preceq_K M \upharpoonright \tau \) with \(|M_1| = \sigma < |M| \). Then there is \( N_1 = EM(I', \Phi) \) with \(|I'| = \sigma \) and \( M_1 \subset N_1 \). By Lemma 2.5, \( N_1 \upharpoonright \tau \preceq_K M \upharpoonright \tau \). So \( M_1 \preceq_K N_1 \upharpoonright \tau \) by the coherence axiom. Let \( M_2 \) have cardinality \( \sigma \) and \( M_1 \preceq_K M_2 \). By amalgamation, choose \( N_2 \subset K \) which is an amalgam of \( N_1 \) and \( M_2 \) over \( M_1 \). Now by categoricity, \( N_2 \) can be strongly embedded into \( M \). And this map can be taken over \( N \) by the model homogeneity of \( M \) (Corollary 4.10). Now extend the image of \( N_2 \) to a \( \tau' \)-closed substructure of \( M \) and then map that structure into the \( \sigma \)-universal \( L' \)-structure \( N_3 \) containing \( N_1 \) which is guaranteed by Claim 4.5. Another use of the coherence axiom shows \( N_3 \upharpoonright \tau \) is the required \( K \)-universal extension of \( M_1 \).

**Remark 4.13**

1. Note that for each \( \sigma \) less than the categoricity cardinal \( \lambda \), the \( \sigma \)-universal model that is constructed has the form \( EM_\tau(I', \Phi) \) for some \( I' \).

2. Compare Claim 4.12 to I.3.1 in [15], which has the same conclusion but weakening the amalgamation property to: there are no maximal models. There are two uses of the amalgamation property in the argument for Claim 4.12. The first requires only that \( M_1 \) be an amalgamation base for models in \( K \) of size \( \mu \) and so extends easily to prove the analogous result where \( K \) has amalgamation is replaced by \( K \) has no maximal models. The second is that \( M \) is \(< \lambda \) model homogenous. This step is done in quite a different way in the proof of I.3.1 in [15]; stability is not used but GCH is.

## 5 Morley’s method for Galois Types: Downward categoricity

Now we prove ‘Morley’s method’ for Galois types.

**Lemma 5.1** [II.1.5 of 394] If \( M_0 \leq M \) and \( M \) is huge we can find an \( EM \)-set \( \Phi \) such that the following hold.

1. The \( \tau \)-reduct of the Skolem closure of the empty set is \( M_0 \).
2. For every \( I, M_0 \leq EM(I, \Phi) \).
3. If \( I \) is finite, \( EM_\tau(I, \Phi) \) can be embedded in \( M \).
4. \( EM_\tau(I, \Phi) \) omits every galois type over \( N \) which is omitted in \( M \).

Proof. Let \( \tau_1 \) be the Skolem language given by the presentation theorem and consider \( M \) as the reduct of \( \tau_1 \) structure \( M^1 \). Add constants for \( M_0 \) to form \( \tau_1^1 \)-structure \( EM(I, \Phi) \). We need to show that if \( a \in N \), \( p = tp(a/M_0) \) is realized in \( M \). For some \( e \in I \), \( a \) is in the \( \tau_1 \)-Skolem hull \( N_e \) of \( e \). (Recall the notation from the presentation theorem.) By 3) there is an embedding \( \alpha \) of \( N_e \) into \( M^1 \) over \( M_0 \). \( \alpha \) is also an isomorphism of \( N_e \upharpoonright \tau_1 \) into \( M \). Now, by the model homogeneity, \( \alpha \) extends to an automorphism of \( M \) fixing \( M_0 \) and \( \alpha(a) \in M \) realizes \( p \).

This has immediate applications in the direction of transferring categoricity.

**Theorem 5.2** If \( K \) is categorical in \( \lambda \) and the model of power \( \lambda \) is saturated, the every model of \( K \) with \( \theta \leq |M| \leq \lambda \) is saturated. So saturation in \( \lambda \) transfer down as far as \( \theta \).
Proof: If some $M$ of cardinality $\mu$ with $\theta \leq \mu \leq \lambda$ is not saturated then by Theorem 5.1, the model of power $\lambda$ is not saturated.

It is easy (Lemma 4.10) to show the model of power $\lambda$ is saturated if $\lambda$ is regular. Judging from results stated in [14], it is not easy to eliminate the hypothesis that $\lambda$ is regular. Of course, it is not a problem if $K$ is categorical above $\lambda$.

6 Tameness

I don’t see how to make this work at the moment. I believe Lemma 6.4 is correct as written. Shelah says the hypothesis $I$ is $\aleph_0$-homogeneous in $J$ can be eliminated. If so, the argument for 6.7 would work roughly as indicated here. But I don’t see this. Some use of the notions I dismiss in Remark 6.1 are needed. I thank Tapani Hyttinen for pointing out my oversights here.

In this section we prove: categoricity implies tame – i.e. it is an exposition of 9.3 of [14].

Remark 6.1 Shelah introduces two relations $\leq^\otimes$ and $\leq^\oplus$ between EM-sets (in his language between $\Phi$ proper for linear order). These are more subtle notions that mere containment but right now I don’t seem to need the subtlety.

Definition 6.2 The linear ordering $I$ is $\aleph_0$-homogeneous in the linear ordering $J$ if $I$ is contained in $J$ and every order preserving finite partial function from $I$ to $I$ can be extended to an automorphism of $J$ which fixes $I$ setwise.

If $J$ is 2-transitive, then $J$ is $\aleph_0$-homogeneous. There are $\aleph_0$-homogeneous linear orderings of all cardinalities. (The order type of any ordered field; see [9].) We can guarantee every suborder $I$ is $\aleph_0$ homogeneous in the linear ordering $J$.

The following notations are established for stating Lemma 6.4 precisely.

Notation 6.3 Let $I$ be a linear order and $u = u_1 u_2 \subseteq I$. We write $I_u$ for $I - u_2$ and $M_u$ for $EM_\tau(I_u, \Phi)$.

This is 8.7 simplified.

Lemma 6.4 Let $\Phi$ be an EM-diagram in vocabulary $\tau_1$. Fix $n$ and $\tau_1$-terms $\sigma_1, \sigma_2$ in $n$-variables. Suppose there are huge linear orders $I \subset J$ with $I$ $\aleph_0$-homogeneous in $J$, such that for some increasing sequence $u = u_1 u_2$ from $I$ there is an automorphism $f$ of $M = EM_\tau(I, \Phi)$ which fixes $M_u = EM_\tau(I_u, \Phi)$ and

$$f(\sigma_1(u)) = \sigma_2(u).$$

Then there is an EM-diagram $\Phi'$ containing $\Phi$ with vocabulary $\tau'_1$, $|\tau'_1| = |\tau_1|$, such that for every linear order $I_1$ and increasing sequence $u = u_1 u_2$ from $I_1$, there is an automorphism $f$ of $M = EM_\tau(I_1, \Phi')$ which fixes $M_u = EM_\tau(I'_u, \Phi')$ such that

$$f(\sigma_1(u_1 u_2)) = \sigma_2(u_1 u_2).$$
Proof. For any increasing $n$-tuple $c = c_1c_2 \in I$, there is an automorphism of $M$ mapping $\sigma_1(c)$ to $\sigma_2(c)$ and fixing $I_2$. To see this, let $\alpha$ be an automorphism of $J$ which fixes $I$ setwise and takes $u$ to $c$. The required automorphism, $f_c$, is $\alpha^{-1}$. Add to $\tau_1$ a new $|u| + 1$-ary function symbol $F$ and interpret $F$ in $M$ by $F(b, c) = f_c(b)$. Also add a predicate $Q$ satisfied by the elements of $I$. Applying Lemma 2.4, we get the required $\Phi'$ where we choose the set of indiscernibles to realize types from $Q$. Note that $\Phi'$ asserts that in $EM(I, \Phi')$ each function $F(x, a)$ (with $a = a_1a_2 \in I$) is an automorphism, which fixes the $\tau_1$-Skolem hull of the indiscernibles satisfying $Q$ except those in $a_2$.

\[ \square_{6.4} \]

**Definition 6.5** We say $K$ is $(\chi, \mu)$-tame if for any saturated $N \in K$ with $|N| = \mu < \lambda$ if $p, q, \in ga - S(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \not\in N_0 = q \not\in N_0$ then $q = p$.

**Remark 6.6** The previous definition is weaker in a number of ways than Grossberg- Van Dieren [3] where the term ‘tame’ is coined. But it seems to be what Shelah establishes in 394.

The use of $\chi$-character in the next argument is a bit subtle. We apply Lemma 6.4 for only one $\chi$-subset $I'$. But we had to have every tuple $u$ from $I$ contained in some $I'$.

**Theorem 6.7** Suppose $K$ is $\lambda$-categorical for a huge $\lambda$ and $\lambda$ is regular. Then $K$ is $(\chi, \mu)$-tame for regular $\mu$ with $\Theta \leq \chi < \mu \leq \lambda$.

Proof. Let $M$ be the model of power $\lambda$ so $M$ is saturated. Let $N \leq M$ be saturated of cardinality $\mu$. We want to show every type over $N$ realized in $M$ is $\chi$-determined for some (indeed any) $\chi < \mu$ and $\chi > \Theta$. By Lemma 4.11, we may assume $N = EM_{\tau}(I, \Phi)$ and we can take $M = EM_{\tau}(J, \Phi)$. Let $a, b$ realize $p \not= q$, distinct types in $S(N)$. So for some $u \in J$, and a pair of $\tau_1$-terms, $\sigma_1, \sigma_2$, $a = \sigma_1(u)$ and $b = \sigma_2(u)$. Let $u_1 = u \cap I$ and $u_2 = u-I$. Now let $I''$ be a subset of $I$ with cardinality $\chi$ containing $u \cap I$ and let $I''$ be $I'\cup I$.

Now, by the assumption on $\chi$, $a$ and $b$ realize the same galois type over $N_0 = EM_{\tau}(I'', \Phi)$, so the hypothesis of Lemma 6.4 holds (the $I_u$ is $I''$). Now, in a Gödelian touch, apply Lemma 6.4 and use its conclusion on $EM_{\tau}(Iu_2, \Phi')$. So there is an automorphism of $EM_{\tau}(Iu, \Phi')$ which fixes the $\tau_1$ Skolem hull of $I$ and sends $a$ to $b$. In particular, the automorphism fixes the $\tau_1$-Skolem Hull of $I$, $N$. By model homogeneity, this automorphism extends to an automorphism of $M$. That is, $a$ and $b$ realize the same Galois type over $M$.

\[ \square \]

Maybe we need the homogeneity for a last step here.

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Replace 1.1 by our standing assumptions. I don’t think a implies a’ but not to worry now.

$\tau$ is the similarity type i.e. the vocabulary of $K$. $\tau(\Phi)$ is the vocabulary of a set of formulas $\Phi$.

As far as I can see at the moment II.1.6 (8.6), 8.8, and 8.9 are not needed for tameness. Here is the beginning of deciphering 8.6 but it should be skipped until you are actually interested.

Lemma 1.6 is still being deciphered. Some obvious typos are that in $\gamma$, the second $N_0'$ should be $N_1'$ and in remark 2) $\zeta$ should be $\theta$. I write $p_i$ for Shelah’s $p_i^1$. I don’t bother to enumerate the $p_i^0$. 

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Lemma 7.1 (II.1.6 of 394) If $N_0 \leq N_1 \leq M$ and $M$ is huge we can find an EM-set $\Phi$ such that the following hold.

1. $EM_r(I, \Phi)$ omits every galois type over $N_0$ which is omitted in $M$.

2. Let $\Gamma_1 = \{ p_i : i < i_1 \}$ enumerate the galois types over $N_1$ omitted in $M$. We can find $N'_j$ for $j \leq \omega$ and $\{ q_i : i < i_1 \} \in S(N'_0)$ such that
   (a) $N'_j$ for $j \leq \omega$ is a continuous chain of small models, beginning with $N_0$ and all strong in $N_1$.
   (b) The $\tau$-reduct of the Skolem closure of the empty set is $N'_\omega$.
   (c) For every $I$, $EM_r(I, \Phi)$ omits each $q_i$ in the strong sense that if $a \in EM_r(I, \Phi)$, then for some $j < \omega$, $q_i \upharpoonright N'_j \neq \text{tp}(a, N'_j, EM_r(I, \Phi))$.

We still need to decipher clause $\eta$. But possibly only in the proof. No, it must be part of the statement as it establishes the connection between the $p_i$ and $q_i$.

References