The first approximation to a quasiminimal axiomatization of complex exponentiation considers short exact sequences of the following form.

\[ 0 \rightarrow \mathbb{Z} \rightarrow H \rightarrow F^* \rightarrow 0. \]  

(1)

\( H \) is a torsion-free divisible abelian group (written additively), \( F \) is an algebraically closed field, and \( \exp \) is the homomorphism from \((H, +)\) to \((F^*, \cdot)\), the multiplicative group of \( F \). We can code this sequence as a structure for a language \( L \):

\((H, +, E, S)\),

where \( E(h_1, h_2) \) iff \( \exp(h_1) = \exp(h_2) \) and we pull back sum by the defining \( H \models S(h_1, h_2, h_3) \) iff \( F \models \exp(h_1) + \exp(h_2) = \exp(h_3) \). Thus \( H \) now represents both the multiplicative and additive structure of \( F \).

To guarantee Assumption ??3.) we expand the language further. Let \( \exp : H \rightarrow F^* \). For each affine variety over \( \mathbb{Q} \), \( \hat{V}(x_1, \ldots, x_n) \), we add a relation symbol \( V \) interpreted by \( H \models V(h_1, \ldots, h_n) \) iff \( F \models V(\exp(h_1), \ldots, \exp(h_n)) \). This includes the definition of \( S \) mentioned above; we have some fuss to handle the pullback of relations which have 0 in their range.

**Lemma 1** There is an \( L_{\omega_1, \omega} \)-sentence \( \Sigma \) such that there is a 1-1 correspondence between models of \( \Sigma \) and sequences (1).

The sentence asserts first that the quotient of \( H \) by \( E \) with + corresponding to \( \times \) and \( S \) to + is an algebraically closed field. We use \( L_{\omega_1, \omega} \) to guarantee the kernel is 1-generated. This same proviso insures that the relevant closure condition has countable closures.

**Definition 2** For \( X \subset H \models \Sigma \),

\[ \text{cl}(X) = \exp^{-1}(\text{acl}(\exp(X))) \]

where \( \text{acl} \) is the field algebraic closure in \( F \).

Using this definition of closure the key result of [?] asserts:
**Theorem 3** $\Sigma$ is quasiminimal excellent with the countable closure condition and categorical in all uncountable powers.

Our goal is this section is to prove this result modulo one major algebraic lemma. We will frequently work directly with the sequence (1) rather than the coded model of $\Sigma$. Note that (1) includes the field structure on $F$. That is, two sequences are isomorphic if there are commuting maps $H$ to $H'$ etc. where the first two are group isomorphisms but the third is a field isomorphism.

It is easy to check that Conditions I and IV and countable closures are satisfied: $\text{cl}$ gives a combinatorial geometry such that the countable closure of countable sets is countable. We need more notation about the divisible closure (in the multiplicative group of the field to understand the remaining conditions.

**Definition 4** By a divisibly closed multiplicative subgroup $a \in \mathbb{C}^*$, $aQ$, we mean a choice of a multiplicative subgroup containing $a$ and isomorphic to the additive group $Q$.

**Definition 5** We say $b_1^\frac{1}{m} \in b_1^Q \ldots b_t^\frac{1}{m} \subset \mathbb{C}^*$, determine the isomorphism type of $b_1^Q \ldots b_t^Q \subset \mathbb{C}^*$ over the subfield $k$ of $\mathbb{C}$ if given subgroups of the form $c_1^Q \ldots c_l^Q \subset \mathbb{C}^*$ and $\phi_m$ such that

$$\phi_m : k(b_1^\frac{1}{m} \ldots b_t^\frac{1}{m}) \rightarrow k(c_1^\frac{1}{m} \ldots c_l^\frac{1}{m})$$

is a field isomorphism it extends to

$$\phi_\infty : k(b_1^Q \ldots b_t^Q) \rightarrow k(c_1^Q \ldots c_l^Q).$$

To see the difficulty consider the following example.

**Example 6** Let $a_1$ and $a_2$ be linearly independent over $Q$ complex numbers such that $(a_1 - 1)2 = a_2$. Suppose $\phi$, which maps $Q(a_1, a_2)$ to $Q(c_1, c_2)$, is a field isomorphism. $\phi$ not extend to an isomorphism of their divisible hulls, we might have $a_1 - 1 = \sqrt{a_2}$ but $c_1 - 1 = -\sqrt{c_2}$.

As in Lecture 3, for $G$ a subgroup of $H$, $H'$ and $H', H' \models \Sigma$, a partial function $\phi$ on $H$ is called a $G$-monomorphism if it preserves $L$-quantifier free formulas with parameters from $G$.

**Fact 7** Suppose $b_1, \ldots, b_t \in H$ and $c_1, \ldots, c_t \in H'$ are each linearly independent sequences (from $G$) over $Q$. Let $G$ be the subfield generated by $\exp(G)$. If $\hat{G}(\exp(b_1)^Q, \ldots, \exp(b_t)^Q) \approx \hat{G}(\exp(c_1)^Q, \ldots, \exp(c_t)^Q)$ as fields, then mapping $b_i$ to $c_i$ is a $G$-monomorphism preserving each variety $V$.

Proof. Let $G \subset H$ and suppose rational $q_i, r_i$ are rational numbers, $h_i \in H - G, g_i \in G$. Then $H \models V(q_1 b_1, \ldots, q t b_t, r_1 g_1, \ldots, r_m g_m)$ iff $H \models \hat{V}(\exp(q_1 b_1), \ldots, \exp(q t b_t), \exp(r_1 g_1), \ldots, \exp(r_m g_m))$ iff $\hat{G}(\exp(b_1)^Q, \ldots, \exp(b_t)^Q, \exp(g_1)^Q, \ldots, \exp(g_m)^Q) \models \hat{V}(\exp(q_1 b_1), \ldots, \exp(q t b_t), \exp(r_1 g_1), \ldots, \exp(r_m g_m)).$

From this fact, it is straightforward to see that Condition II in the definition of quasiminimal excellence holds. For II.i) we need that there is only one type of a closure-independent sequence. But Fact 7 implies that for $b \in H$ to be closure independent, the associated $\exp(b)$ must be algebraically independent and of course there is a unique type of an algebraically independent sequence. For II.ii) holds since added to language of $\Sigma$ predicates for the pull-back of all quantifier-free relations on the field $F$. (Zilber doesn’t do this.)
For Condition III and the excellence condition we need an algebraic result. In the following, $\sqrt{1}$ denotes the subgroup of roots of unity. We call this result the thumbstack lemma based on the following visualization of Kitty Holland. The various $nth$ roots of $b_1 \ldots b_m$ hang on threads from the $b_i$. These threads can get tangled, but the theorem asserts that by sticking in a finite number of thumbstacs one can ensure that the rest of strings fall freely. The proof involves the theory of fractional ideals of number fields, Weil divisors, and the normalization theorem. For $a_1, \ldots a_r$ in $\mathbb{C}$, we write $gp(a_1, \ldots a_r)$ for the multiplicative subgroup generated by $a_1, \ldots a_r$. The following general version of the theorem is applied for various sets of parameters to prove quasiminimal excellence.

In the following Lemma we write $\sqrt{1}$ for the group of roots of unity. If any of the $L_i$ are defined, the reference to $\sqrt{1}$ is redundant. We write $gp(a)$ for the multiplicative subgroup generated by $a$.

**Remark 8** Let $k$ be an algebraically closed subfield in $\mathbb{C}$ and let $a \in \mathbb{C} - k$. A field theoretic description of the relation of $a$ to $k$ arises by taking the irreducible variety over $k$ realized by $a$. $a$ is a generic realization of variety given by a finite conjunction $\phi(x, b)$ of polynomials generating the ideal in $k[x]$ of those polynomials which annihilate $a$. From a model theoretic standpoint we can say, choose $b$ so that the type of $a/k$ is the unique nonforking extension of $tp(a/b)$. We use the model theoretic formulation below. See [10], page 39.

**Theorem 9 (thumbstack lemma)** Let $P \subset \mathbb{C}$ be a finitely generated extension of $Q$ and $L_1, \ldots L_n$ algebraically closed subfields of the algebraic closure $\hat{P}$ of $P$. Fix multiplicatively divisible subgroups $a_1^Q, \ldots a_r^Q$ with $a_1, \ldots a_r \in \hat{P}$ and $b_1^Q, \ldots b_n^Q \subset \mathbb{C}^*$. If $b_1 \ldots b_r$ are multiplicatively independent over $gp(a_1, \ldots a_r) \cdot \sqrt{1} \cdot L_1^* \cdots L_n^*$ then for some $m b_1^{\ell m} \in b_1^Q, \ldots b_r^{\ell m} \in b_r^Q \subset \mathbb{C}^*$, determine the isomorphism type of $b_1^Q, \ldots b_r^Q$ over $P(L_1, \ldots L_n, \sqrt{1}, a_1^Q, \ldots a_r^Q)$.

**Lemma 10** Condition $III$ of quasiminimal excellence holds.

Proof. We must show: If $G \models \Sigma$ and $f$ is a partial $G$-monomorphism from $H$ to $H'$ with finite domain $X = \{x_1, \ldots x_r\}$ then for any $y \in H$ there is $y'$ in some $H''$ with $H' \prec K H''$ such that $f \cup \{(y, y')\}$ extends $f$ to a partial $G$-monomorphism. Since $G \models \Sigma$, exp($G$) is an algebraically closed field. For each $i$, let $a_i$ denote exp($x_i$) and similarly for $x_i, a_i'$. Choose a finite sequence $d \in$ exp($G$) such that the sequence $(a_1, \ldots a_r)$ is independent (in the forking sense) from exp($G$) over $d$ and $tp(a_1, \ldots a_r)/d$ is stationary. Now we apply the thumbstack lemma. Let $P_0$ be $Q(d)$. Let $n = 1$ and $L_1$ be the algebraic closure of $P_0$. We set $P_0(d, a_1, \ldots a_r)$ as $P$. Take $b_1$ as exp($y$) and set $\ell = 1$.

Now apply Lemma 9 to find $m$ so that $b_1^{\ell m}$ determines the algebraic type of $(b_1)^Q$ over $L_1(a_1^Q, \ldots a_r^Q) = P_0(L_1, a_1^Q, \ldots a_r^Q)$. Let $\bar{f}$ denote the map $f$ induces from $\hat{H}$ to $\hat{H}'$ over $\hat{G}$. Choose $b_1^{'\ell m}$ to satisfy the quantifier free field type of $\bar{f}(tp(b_1^{\ell m}/L_1(a_1^Q, \ldots a_r^Q))$. Now by Lemma 9, $\bar{f}$ extends to field isomorphism between $L_1(a_1^Q, \ldots a_r^Q, b_1^Q))$ and $L_1((a_1')^Q, \ldots (a_r')^Q, (b_1')^Q)$. Since the sequence $a_1, \ldots a_r$ is independent (in the forking sense) from exp($G$) over $L_1$, we can extend this map to take exp($G$)/$(a_1^Q, \ldots a_r^Q, b_1^Q)$ to exp($G$)/(a_1', a_r', b_1') and pull back to find $y'$; this suffices by Fact 7.

\[ \square \]

Note there is no claim that $y' \in H'$ and there can't be.

One of the key ideas discovered by Shelah in the investigation of non-elementary classes is that in order for types to be well-behaved one may have to make restrictions on the domain. (E.g., we may be able to amalgamate
types over models but not arbitrary types.) This principle is illustrated by the following definition and result of Zilber.

**Definition 11** \( C \subseteq F \) is finitary if \( C \) is the union of the divisible closure (in \( C^* \)) of a finite set and finitely many algebraically closed fields.

Now we establish Condition IV, excellence. Note that this is a stronger condition than excellence since there is no independence requirement on the \( G_i \).

**Lemma 12** Let \( G_1, \ldots, G_n \subseteq H \) all be models of \( \Sigma \) and suppose each has finite cl-dimension. If \( h_1, \ldots, h_\ell \in G^\sim = \text{cl}(G_1 \cup \ldots \cup G_n) \) then there is finite set \( A \subseteq G^\sim \) such that any \( \phi \) taking \( h_1, \ldots, h_\ell \) into \( H \) which is an \( A \)-monomorphism is also a \( G \)-monomorphism.

Let \( L_i = \exp(G_i) \) for \( i = 1, \ldots, n \); \( b_i^q = \exp(qh_i) \) for \( j = 1, \ldots, \ell \) and \( q \in \mathbb{Q} \). We may assume the \( h_i \) are linearly independent over the vector space generated by the \( G_i \); this implies the \( b_i \) are multiplicatively independent over \( L_1^* \cdot L_2^* \cdot \ldots \cdot L_n^* \). Now apply the thumbtack lemma with \( r = 0 \). This gives an \( m \) such that the field theoretic type of \( b_1^m, \ldots, b_\ell^m \) determines the quantifier free type of \( (h_1, \ldots, h_\ell) \) over \( G^\sim \). So we need only finitely many parameters from \( G^\sim \) and we finish.

To prove the following result, apply the thumbtack lemma with the \( L_i \) as the fields and the \( a_i \) as the finite set.

**Corollary 13** Any almost finite \( n \)-type over a finitary set is a finite \( n \)-type.

Since we have established all the conditions for quasiminimal excellence, we have proved Theorem 3.

Keisler[?] proved Morley’s categoricity theorem for sentences in \( L_{\omega_1, \omega} \), assuming that the categoricity model was \( \aleph_1 \)-saturated. We give two examples showing this hypothesis is necessary. Marcus[?] showed:

**Fact 14** There is a first order theory \( T \) with a prime model \( M \) such that

1. \( M \) has no proper elementary submodel.
2. \( M \) contains an infinite set of indiscernibles.

**Exercise 15** Show that the \( L_{\omega_1, \omega} \)-sentence satisfied only by atomic models of the theory \( T \) in Fact 14 has a unique model.

**Example 16** Now construct an \( L_{\omega_1, \omega} \)-sentence \( \psi \) whose models are partitioned into two sets; on one side is an atomic model of \( T \), on the other is an infinite set. Then \( \psi \) is categorical in all infinite cardinalities but no model is \( \aleph_1 \)-homogeneous because there is a countably infinite maximal indiscernible set.

Now we see that the example of this chapter has the same inhomogeneity property.

Consider the basic diagram:

\[
0 \rightarrow Z \rightarrow H \rightarrow F^* \rightarrow 0. \tag{2}
\]
Let $a$ be a transcendental number in $F^*$. Fix $h$ with $\exp(h) = a$ and define $a_n = \exp \frac{h}{n} + 1$ for each $n$. Now choose $h_n$ so that $\exp(h_n) = a_n$. Let $X_r = \{h_i : i \leq r\}$. Note that $a_m = a^\frac{1}{m} + 1$ where we have chosen a specific $m$th root.

**Claim 17** $p_r = \text{tp}(h/X_r)$ is a principal type.

Proof. We make another application of the thumbtack Lemma 9 with $Q(\exp(\text{span}(X_r)))$ as $P$, $a_1, \ldots, a_r$ as themselves, all $L_i$ are empty, and $a$ as $b_1$. By the lemma there is an $m$ such that $a^\frac{1}{m}$ determines the isomorphism type of $a^Q$ over $P(a_1^Q, \ldots, a_r^Q)$. That is if $\phi_m$ is the minimal polynomial of $a^\frac{1}{m}$ over $P$, $(\exists y) \phi_m(y) \land y^m = x$ generates $\text{tp}(a/\exp(\text{span}(X_r)))$. Pulling back by Lemma 7, we see $\text{tp}(h/X_r)$ is principal and even complete for $L_{\omega_1, \omega}$. In particular, for any $m' \geq m$, any two $m'$th roots of $a$ have the same type over $\exp(X_r)$. But for sufficiently large $r$, one of these $m'$th roots is actually in $X_r$ so $\text{tp}(a/X_r)$ does not imply $p = \text{tp}(a/X)$ for any $X$. That is, $\text{tp}(a/X)$ is not implied by its restriction to any finite set. And by Lemma 7 this implies $\text{tp}_{\omega_1, \omega}(h/X)$ is not implied by its restriction to any finite set.

Now specifically to answer the question of Keisler [?], page 123, we need to show there is a sentence $\psi$ in a countable fragment $L^*$ of $L_{\omega_1, \omega}$ such that $\psi$ is $\aleph_1$-categorical but has a model with is not $(\aleph_1, L^*)$-homogeneous. Fix $L^*$ as a countable fragment containing the categoricity sentence for ‘covers’. We have shown no formula of $L_{\omega_1, \omega}$ (let alone $L^*$) with finitely many parameters from $X$ implies $p$. By the omitting types theorem for $L^*$, there is a countable model $H_0$ of $\psi$ which contains an $L^*$-equivalent copy $X'$ of $X$ and omits the associated $p'$. By categoricity, $H_0$ imbeds into $H$. But $H$ also omits $p'$. As, if $h' \in H$, realizes $p'$, then $\exp(h') \in \text{acl}(X') \subseteq H_0$ so since the kernel of $\exp$ is standard, $h' \in H_0$, contradiction. Thus the type $p'$ cannot be realized so $H$ is not homogeneous.