Lecture 10: Covers of the multiplicative group of $\mathbb{C}$

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The first approximation to a quasiminimal axiomatization of complex exponentiation considers short exact sequences of the following form.

$$0 \to Z \to H \to F^* \to 0. \tag{1}$$

H is a torsion-free divisible abelian group (written additively), F is an algebraically closed field, and exp is the homomorphism from $(H, +)$ to $(F^*, \cdot)$, the multiplicative group of F. We can code this sequence as a structure for a language L:

$$(H, +, E, S),$$

where $E(h_1, h_2)$ iff $\exp(h_1) = \exp(h_2)$ and we pull back sum by the defining $H \models S(h_1, h_2, h_3)$ iff $F \models \exp(h_1) + \exp(h_2) = \exp(h_3)$. Thus H now represents both the multiplicative and additive structure of F.

To guarantee Assumption **??**.3.) we expand the language further. Let $\exp : H \mapsto F^*$. For each affine variety over $\mathbb{Q}$, $\hat{V}(x_1, \ldots x_n)$, we add a relation symbol V interpreted by $H \models V(h_1, \ldots, h_n)$ iff $F \models V(\exp(h_1), \ldots, \exp(h_n))$. This includes the defintion of S mentioned above; we have some fuss to handle the pullback of relations which have 0 in their range.

Lemma 1 *There is an $L_{\omega_1,\omega}$-sentence Σ such that there is a* 1-1 *correspondence between models of Σ and sequences (1).*

The sentence asserts first that the quotient of H by E with $+$ corresponding to $\times$ and S to $+$ is an algebraically closed field. We use $L_{\omega_1,\omega}$ to guarantee the kernel is 1-generated. This same proviso insures that the relevant closure condition has countable closures.

Definition 2 *For $X \subset H \models \Sigma$,*

$$\text{cl}(X) = \exp^{-1}(\text{acl}(\exp(X))$$

where acl *is the field algebraic closure in F.*

Using this definition of closure the key result of [**?**] asserts:

Theorem 3 *Σ is quasiminimal excellent with the countable closure condition and categorical in all uncountable powers.*

Our goal is this section is to prove this result modulo one major algebraic lemma. We will frequently work directly with the sequence (1) rather than the coded model of Σ. Note that (1) includes the field structure on F. That is, two sequences are isomorphic if there are commuting maps H to H' etc. where the first two are group isomorphims but the third is a field isomorphism.

It is easy to check that Conditions I and IV and countable closures are satisfied: cl gives a combinatorial geometry such that the countable closure of countable sets is countable. We need more notation about the divisible closure (in the multiplicative group of the field to understand the remaining conditions.

Definition 4 *By a* divisibly closed multiplicative subgroup *associated with $a \in \mathbb{C}^*$, $a^{\mathbb{Q}}$, we mean a* **choice** *of a multiplicative subgroup containing a and isomorphic to the additive group $\mathbb{Q}$.*

Definition 5 *We say $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$,* determine the isomorphism type *of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$ over the subfield k of $\mathbb{C}$ if given subgroups of the form $c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$ and ϕ_m such that*

$$\phi_m : k(b_1^{\frac{1}{m}} \ldots b_\ell^{\frac{1}{m}}) \to k(c_1^{\frac{1}{m}} \ldots c_\ell^{\frac{1}{m}})$$

is a field isomorphism it extends to

$$\phi_\infty : k(b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}) \to k(c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}}).$$

To see the difficulty consider the following example.

Example 6 *Let a_1 and a_2 be linearly independent over $\mathbb{Q}$ complex numbers such that $(a_1 - 1)2 = a_2$. Suppose ϕ, which maps $\mathbb{Q}(a_1, a_2)$ to $\mathbb{Q}(c_1, c_2)$, is a field isomorphism. ϕ not extend to an isomorphism of their divisible hulls, we might have $a_1 - 1 = \sqrt{a_2}$ but $c_1 - 1 = -\sqrt{c_2}$.*

As in Lecture 3, for G a subgroup of H, H' and $H, H' \models \Sigma$, a partial function ϕ on H is called a G-monomorphism if it preserves L-quantifier free formulas with parameters from G.

Fact 7 *Suppose $b_1, \ldots b_\ell \in H$ and $c_1, \ldots c_\ell \in H'$ are each linearly independent sequences (from G) over $\mathbb{Q}$. Let $\hat{G}$ be the subfield generated by $\exp(G)$. If $\hat{G}(\exp(b_1)^{\mathbb{Q}}, \ldots \exp(b_\ell)^{\mathbb{Q}}) \approx \hat{G}(\exp(c_1)^{\mathbb{Q}}, \ldots \exp(c_\ell)^{\mathbb{Q}})$ as fields, then mapping b_i to c_i is a G-monomomorphism preserving each variety V.*

Proof. Let $G \subset H$ and suppose rational q_i, r_i are rational numbers, $h_i \in H - G, g_i \in G$. Then $H \models V(q_1b_1, \ldots q_\ell b_\ell, r_1g_1, \ldots r_m g_m)$ iff $\hat{H} \models \hat{V}(\exp(q_1b_1), \ldots \exp(q_\ell b_\ell), \exp(r_1g_1), \ldots \exp(r_m g_m))$ iff $\hat{G}(\exp(b_1)^{\mathbb{Q}}, \ldots \exp(b_\ell)^{\mathbb{Q}}, \exp(g_1)^{\mathbb{Q}}, \ldots \exp(g_m)^{\mathbb{Q}}) \models \hat{V}(\exp(q_1b_1), \ldots \exp(q_\ell b_\ell), \exp(r_1g_1), \ldots \exp(r_m g_m))$.

From this fact, it is straightforward to see that Condition II in the definition of quasiminimal excellence holds. For II.i) we need that there is only one type of a closure-independent sequence. But Fact 7 implies that for $\mathbf{b} \in H$ to be closure independent, the associated $\exp(\mathbf{b})$ must be algebraically independent and of course there is a unique type of an algebraically independent sequence. For II.ii) holds since added to language of Σ predicates for the pull-back of all quantifier-free relations on the field F. (Zilber doesn't do this.)

For Condition III and the excellence condition we need an algebraic result. In the following, $\sqrt{1}$ denotes the subgroup of roots of unity. We call this result the thumbtack lemma based on the following visualization of Kitty Holland. The various nth roots of $b_1, \ldots b_m$ hang on threads from the b_i. These threads can get tangled; but the theorem asserts that by sticking in a finite number of thumbtacks one can ensure that the rest of strings fall freely. The proof involves the theory of fractional ideals of number fields, Weil divisors, and the normalization theorem. For $a_1, \ldots a_r$ in $\mathbb{C}$, we write $\mathrm{gp}(a_1, \ldots a_r)$ for the multiplicative subgroup generated by $a_1, \ldots a_r$. The following general version of the theorem is applied for various sets of parameters to prove quasiminimal excellence.

In the following Lemma we write $\sqrt{1}$ for the group of roots of unity. If any of the L_i are defined, the reference to $\sqrt{1}$ is redundant. We write $\mathrm{gp}(\boldsymbol{a})$ for the multiplicative subgroup generated by $\boldsymbol{a}$.

Remark 8 *Let k be an algebraically closed subfield in $\mathbb{C}$ and let $\boldsymbol{a} \in \mathbb{C} - k$. A field theoretic description of the relation of $\boldsymbol{a}$ to k arises by taking the irreducible variety over k realized by $\boldsymbol{a}$. $\boldsymbol{a}$ is a generic realization of variety given by a finite conjunction $\phi(\mathbf{x}, \mathbf{b})$ of polynomials generating the ideal in $k[\mathbf{x}]$ of those polynomials which annihilate $\boldsymbol{a}$. From a model theoretic standpoint we can say, choose $\mathbf{b}$ so that the type of $\boldsymbol{a}/k$ is the unique nonforking extension of $\mathrm{tp}(\boldsymbol{a}/\mathbf{b})$. We use the model theoretic formulation below. See [**?**], page 39.*

Theorem 9 (thumbtack lemma) *[**?**]*

Let $P \subset \mathbb{C}$ be a finitely generated extension of $\mathbb{Q}$ and $L_1, \ldots L_n$ algebraically closed subfields of the algebraic closure $\hat{P}$ of P. Fix multiplicatively divisible subgroups $a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}}$ with $a_1, \ldots a_r \in \hat{P}$ and $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$. If $b_1 \ldots b_\ell$ are multiplicatively independent over $\mathrm{gp}(a_1, \ldots a_r) \cdot \sqrt{1} \cdot L_1^* \cdot \ldots L_n^*$ then for some m $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$, determine the isomorphism type of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}$ over $P(L_1, \ldots L_n, \sqrt{1}, a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}})$.

Lemma 10 *Condition III of quasiminimal excellence holds.*

Proof. We must show: If $G \models \Sigma$ and f is a partial G-monomorphism from H to H' with finite domain $X = \{x_1, \ldots, x_r\}$ then for any $y \in H$ there is y' in some H'' with $H' \prec_{\boldsymbol{K}} H''$ such that $f \cup \{\langle y, y' \rangle\}$ extends f to a partial G-monomorphism. Since $G \models \Sigma$, $\exp(G)$ is an algebraically closed field. For each i, let a_i denote $\exp(x_i)$ and similarly for x_i', a_i'. Choose a finite sequence $\mathbf{d} \in \exp(G)$ such that the sequence $(a_1, \ldots a_r)$ is independent (in the forking sense) from $\exp(G)$ over $\mathbf{d}$ and $\mathrm{tp}(a_1, \ldots a_r)/\mathbf{d})$ is stationary. Now we apply the thumbtack lemma. Let P_0 be $\mathbb{Q}(\mathbf{d})$. Let $n = 1$ and L_1 be the algebraic closure of P_0. We set $P_0(\mathbf{d}, a_1, \ldots a_r)$ as P. Take b_1 as $\exp(y)$ and set $\ell = 1$.

Now apply Lemma 9 to find m so that $b_1^{\frac{1}{m}}$ determines the algebraic type of $(b_1)^{\mathbb{Q}}$ over $L_1(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}}) = P_0(L_1, a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}})$. Let $\hat{f}$ denote the map f induces from $\hat{H}$ to $\hat{H}'$ over $\hat{G}$. Choose ${b_1'}^{\frac{1}{m}}$ to satisfy the quantifier free field type of $\hat{f}(\mathrm{tp}(b_1^{\frac{1}{m}}/L_1(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}})$. Now by Lemma 9, $\hat{f}$ extends to field isomorphism between $L_1(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}}, b_1^{\mathbb{Q}}))$ and $L_1((a_1')^{\mathbb{Q}}, \ldots (a_r')^{\mathbb{Q}}, (b_1')^{\mathbb{Q}})$. Since the sequence $a_1, \ldots a_r)$ is independent (in the forking sense) from $\exp(G)$ over L_1, we can extend this map to take $\exp(G)(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}}, b_1^{\mathbb{Q}})$ to $\exp(G)((a_1')^{\mathbb{Q}}, \ldots (a_r')^{\mathbb{Q}}, (b_1')^{\mathbb{Q}})$ and pull back to find y'; this suffices by Fact 7.

$\square_{10}$

Note there is no claim that $y' \in H'$ and there can't be.

One of the key ideas discovered by Shelah in the investigation of non-elementary classes is that in order for types to be well-behaved one may have to make restrictions on the domain. (E.g., we may be able to amalgamate types over models but not arbitrary types.) This principle is illustrated by the following definition and result of Zilber.

Definition 11 *$C \subseteq F$ is finitary if C is the union of the divisible closure (in $\mathbb{C}^*$) of a finite set and finitely many algebraically closed fields.*

Now we establish Condition IV, excellence. Note that this is a stronger condition than excellence since there is no independence requirement on the G_i.

Lemma 12 *Let $G_1, \ldots G_n \subset H$ all be models of Σ and suppose each has finite cl-dimension. If $h_1, \ldots h_\ell \in G^- = \mathrm{cl}(G_1 \cup \ldots G_n)$ then there is finite set $A \subset G^-$ such that any ϕ taking $h_1, \ldots, h_\ell$ into H which is an A-monomorphism is also a G^--monomorphism.*

Let $L_i = \exp(G_i)$ for $i = 1, \ldots, n$; $b_j^q = \exp(qh_j)$ for $j = 1, \ldots, \ell$ and $q \in \mathbb{Q}$. We may assume the h_i are linearly independent over the vector space generated by the G_i; this implies the b_i are muliplicatively independent over $L_1^* \cdot L_2^* \cdot \ldots L_n^*$. Now apply the thumbtack lemma with $r = 0$. This gives an m such that the field theoretic type of $b_1^{\frac{1}{m}}, \ldots, b_\ell^{\frac{1}{m}}$ determines the quantifier free type of $(h_1, \ldots, h_\ell)$ over G^-. So we need only finitely many parameters from G^- and we finish.

To prove the following result, apply the thumbtack lemma with the L_i as the fields and the $\boldsymbol{a}_i$ as the finite set.

Corollary 13 *Any almost finite n-type over a finitary set is a finite n-type.*

Since we have established all the conditions for quasiminimal excellence, we have proved Theorem 3.

Keisler[**?**] proved Morley's categoricity theorem for sentences in $L_{\omega_1,\omega}$, assuming that the categoricity model was $\aleph_1$-saturated. We give two examples showing this hypothesis is necessary. Marcus [**?**] showed:

Fact 14 *There is a first order theory T with a prime model M such that*

1. *M has no proper elementary submodel.*
2. *M contains an infinite set of indiscernibles.*

Exercise 15 *Show that the $L_{\omega_1,\omega}$-sentence satisfied only by atomic models of the theory T in Fact 14 has a unique model.*

Example 16 *Now construct an $L_{\omega_1,\omega}$-sentence ψ whose models are partitioned into two sets; on one side is an atomic model of T, on the other is an infinite set. Then ψ is categorical in all infinite cardinalities but no model is $\aleph_1$-homogeneous because there is a countably infinite maximal indiscernible set.*

Now we see that the example of this chapter has the same inhomogeneity property.

Consider the basic diagram:

$$0 \to Z \to H \to F^* \to 0. \tag{2}$$

Let a be a trancendental number in F^*. Fix h with $\exp(h) = a$ and define $a_n = \exp \frac{h}{n} + 1$ for each n. Now choose h_n so that $\exp(h_n) = a_n$. Let $X_r = \{h_i : i \leq r\}$. Note that $a_m = a^{\frac{1}{m}} + 1$ where we have chosen a specific mth root.

Claim 17 *$p_r = \mathrm{tp}(h/X_r)$ is a principal type.*

Proof. We make another application of the thumbtack Lemma 9 with $\mathbb{Q}(\exp(\mathrm{span}(X_r))$ as P , $a_1, \ldots a_r$ as themselves, all L_i are empty, and a as b_1. By the lemma there is an m such that $a^{\frac{1}{m}}$ determines the isomorphism type of $a^{\mathbb{Q}}$ over $P(a_1^{\mathbb{Q}}, \ldots a_r^{\mathbb{Q}})$. That is if ϕ_m is the minimal polynomial of $a^{\frac{1}{m}}$ over P, $(\exists y)\phi_m(y) \wedge y^m = x$ generates $\mathrm{tp}(a/\exp(\mathrm{span}(X_r)))$. Pulling back by Lemma 7, we see $\mathrm{tp}(h/X_r)$ is principal and even complete for $L_{\omega_1,\omega}$. In particular, for any $m' \geq m$, any two m'th roots of a have the same type over $\exp(X_r)$. But for sufficiently large r, one of these m'th roots is actually in X_r so $\mathrm{tp}(a/X_r)$ does not imply $p = \mathrm{tp}(a/X)$ for any X. That is, $\mathrm{tp}(a/X)$ is not implied by its restriction to any finite set. And by Lemma 7 this implies $\mathrm{tp}_{\omega_1,\omega}(h/X)$ is not implied by its restriction to any finite set.

Now specifically to answer the question of Keisler [**?**], page 123, we need to show there is a sentence ψ in a countable fragment L^* of $L_{\omega_1,\omega}$ such that ψ is $\aleph_1$-categorical but has a model with is not $(\aleph_1, L^*)$-homogeneous. Fix L^* as a countable fragment containing the categoricity sentence for 'covers'. We have shown no formula of $L_{\omega_1,\omega}$ (let alone L^*) with finitely many parameters from X implies p. By the omitting types theorem for L^*, there is a countable model H_0 of ψ which contains an L^*-equivalent copy X' of X and omits the associated p'. By categoricity, H_0 imbeds into H. But H also omits p'. As, if $h' \in H$, realizes p', then $\exp(h') \in \mathrm{acl}(X') \subseteq H_0$ so since the kernel of exp is standard, $h' \in H_0$, contradiction. Thus the type p' cannot be realized so H is not homogeneous.