In this section we consider \((K, \prec_K)\) to be the class of atomic models of a first order theory which is L-excellent. Our goal is to show that if \(K\) is \(\omega\)-stable then it is stable in all cardinalities.

The crucial point is that if \(M = \bigcup_{i<\alpha} M_i, p \neq q \in S_{\text{at}}(M)\) then for some \(i < \alpha, p \vdash M_i \neq q \vdash M_i\), because the types are syntactic.

Acknowledgement: This proof is virtually a copy of the first order case; the actual write-up is pirated from a more general version being developed by Baldwin-Kueker-Vandieren.

**Theorem 1** If \(K\) is excellent and \(\aleph_0\)-stable then \(K\) is stable in all \(\kappa\).

Proof. We show that if \(K\) in every cardinality less than \(\kappa\), then it is \(\kappa\)-stable.

Take any \(M\) of cardinality \(\kappa\). We may write \(M\) as the union of a continuous chain \(\langle M_i | i < \kappa\rangle\) under \(\prec_K\) of models of cardinality \(< \kappa\) in \(K\).

We say that a type over \(M_i\) has many extensions to mean that it has \(> \kappa\) distinct extensions to a type over \(M\).

**Claim 2** For every \(i\), there is some type over \(M_i\) with many extensions.

Proof. Each type over \(M^*\) is the extension of some type over \(M_i\) and, by our assumption, there are less than \(\kappa\) many types over \(M_i\), so at least one of them must have many extensions.

**Claim 3** For every \(i\), if the type \(p\) over \(M_i\) has many extensions, then for every \(j > i\), \(p\) has an extension to a type \(p'\) over \(M_j\) with many extensions.

Proof. Every extension of \(p\) to a type over \(M^*\) is the extension of some extension of \(p\) to a type over \(M_j\). By our assumption there are less than \(\kappa\) many such extensions to a type over \(M_j\), so at least one of them must have many extensions.

**Claim 4** For every \(i\), if the type \(p\) over \(M_i\) has many extensions, then for all sufficiently large \(j > i\), \(p\) can be extended to two types over \(M_j\) each having many extensions.
Proof. By Claim 3 it suffices to establish the result for some $j > i$. So assume that there is no $j > i$ such that $p$ has two extensions to types over $M_j$ each having many extensions. Then, by Claim 3 again, for every $j > i$, $p$ has a unique extension to a type $p_j$ over $M_j$ with many extensions. Let $S^*$ be the set of all extensions of $p$ to a type over $M^* - \{S^*\} \geq \kappa^+$. Then $S^*$ is the union of $S_0$ and $S_1$, where $S_0$ is the set of all $q$ in $S^*$ such that $p_j \not< q$ for all $j > i$, and $S_1$ is the set of all $q$ in $S^*$ such that $q$ does not extend $p_j$ for some $j > i$. Now if $q_1$ and $q_2$ are different types in $S^*$ then, since types are syntactic their restrictions to some $M_j$ must differ. Hence their restrictions to all sufficiently large $M_j$ must differ. Therefore, $S_0$ contains at most one type. On the other hand, if $q$ is in $S_1$ then, for some $j > i$, $q \mid M_j$ is an extension of $p$ to a type over $M_j$ which is different from $p_j$, hence has at most $\kappa$ extensions to a type over $M^*$. Since there are $< \kappa$ types over each $M_j$ (by assumption) and just $\kappa$ models $M_j$ there can be at most $\kappa$ types in $S_1$. Thus $S^*$ contains at most $\kappa^+$ types, a contradiction.

Claim 5 There is a countable $M \preceq_K M^*$ such that there are $2^{\aleph_0}$ types over $M$.

Proof. Let $p$ be a type over $M_0$ with many extensions. By Claim 4 there is a $j_1 > 0$ such that $p$ has two extensions $p_0, p_1$ to types over $M_{j_1}$ with many extensions. Iterating this construction we find a sequence of countable models $M_{j_n}$ and a tree $p_s$ of types for $s \in 2^\omega$ with the $2^n$ types $p_s$ (where $s$ has length $n$) all over $M_{j_n}$ and each $p_s$ has many extensions. Let $\hat{M}$ be the union of the $M_{j_n}$. Now for each $\sigma \in 2^\omega$, $p_\sigma = \bigcup_{s \subseteq \sigma} p_s$ is, by type realizability, in $S_{\text{at}}(\hat{M})$ contradicting $\omega$-stability.

This concludes the proof of Theorem 1.