Lecture 5: Abstract Elementary Classes

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When Jónsson generalized the Fraïsse construction to uncountable cardinalities [**?**, **?**], he did so by describing a collection of axioms, which might be satisfied by a class of models, that guaranteed the existence of a homogeneous-universal model; the substructure relation was an integral part of this description. Morley and Vaught [**?**] replaced substructure by elementary submodel and developed the notion of saturated model. Shelah [**?**, **?**] generalized this approach in two ways. He moved the amalgamation property from a basic axiom to a constraint to be considered. (But this was a common practice in universal algebra as well.) He made the *substructure* notion a 'free variable' and introduced the notion of an *Abstract Elementary Class*: a class of structures and a 'strong' substructure relation which satisfied variants on Jonsson's axioms. To be precise

Definition 1 *A class of L-structures,* $(\boldsymbol{K}, \leq)$*, is said to be an* abstract elementary class: AEC *if both* $\boldsymbol{K}$ *and the binary relation* $\leq$ *are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If* $M \leq N$ *then* $M \subseteq N$.
- **A2**. $\leq$ *is a partial order on* $\boldsymbol{K}$.
- **A3**. *If* $\langle A_i : i < \delta \rangle$ *is* $\leq$*-increasing chain:*
  1. $\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;
  2. *for each* $j < \delta$, $A_j \leq \bigcup_{i<\delta} A_i$
  3. *if each* $A_i \leq M \in \boldsymbol{K}$ *then* $\bigcup_{i<\delta} A_i \leq M$.
- **A4**. *If* $A, B, C \in \boldsymbol{K}$, $A \leq C$, $B \leq C$ *and* $A \subseteq B$ *then* $A \leq B$.
- **A5**. *There is a Löwenheim-Skolem number* $\kappa(\boldsymbol{K})$ *such that if* $A \subseteq B \in K$ *there is a* $A' \in \boldsymbol{K}$ *with* $A \subseteq A' \leq B$ *and* $|A'| < \kappa(\boldsymbol{K})$.

Property **A5** is sometimes called the coherence property and sometimes 'the funny axiom'. Perhaps best is the Tarski-Vaught property since it easily seen to follow in the first order case as an application the Tarski-Vaught test for elementary submodel. However, Shelah sometimes uses 'Tarski-Vaught' for the union axioms.

Exercise 2 *Show the class of well-orderings with* $\leq$ *taken as end extension satisfies the first four properities of an AEC. Does it have a Löwenheim number?*

Exercise 3 *The models of a sentence of first order logic or any countable fragment of* $L_{\omega_1,\omega}$ *with the associated notion of elementary submodel as* $\leq$ *gives an AEC with Löwenheim number* $\aleph_0$.

Definition 4 *The logic $L(Q)$ adds to first order logic the expression $(Qx)\phi(x)$ which holds if there are uncountably many solutions of ϕ. The analogous expansion of $L_{\omega_1,\omega}$ is called $L_{\omega_1,\omega}(Q)$.*

Exercise 5 *The models of a sentence of $L(Q)$ with the associated notion of elementary submodel as $\leq$ does* not *give an AEC.*

It is easy to verify the following statement.

Lemma 6 *Let ψ be a sentence in $L_{\omega_1,\omega}(Q)$ and let L^* be the smallest countable fragment of $L_{\omega_1,\omega}(Q)$ containing ψ. Define a class $(\boldsymbol{K}, \leq)$ by letting $\boldsymbol{K}$ be the class of models of ψ in the standard interpretation and $M \leq N$ if*

1. *$M \prec_{L^*} N$ and*
2. *$M \models (Qx)\theta(x, \boldsymbol{a})$ iff $\{b \in N : N \models \theta(b, \boldsymbol{a})$ properly contains $\{b \in M : N \models \theta(b, \boldsymbol{a})$.*

Exercise 7 *What is the Löwenhheim number of the AEC defined in Lemma 6 ?*

Question 8 *Is there a way to translate an $L_{\omega_1,\omega}(Q)$sentence to an AEC with Löwenhheim number $\aleph_0$ and which has at least approximately the same number of models in each uncountable cardinality?*

We approach this question by passing through a more abstract treatment. We call the next result: the presentation theorem. It allows us to replace the entirely semantic description of an abstract elementary class by a syntactic one. I find it extraordinary that the notion of an AEC which is designed to give a version of the Fraïsse construction and thus saturated models, also turns out to allow the use of the second great model theoretic technique of the 50's: Ehrenfeucht-Mostowski models.

Theorem 9 *If K is an AEC with Lowenheim number $\aleph_0$ (in a countable vocabulary L), there is a countable language L', a first order L'-theory T' and a set of $2^{\aleph_0}$ types Γ such that:*

$$\boldsymbol{K} = \{M' \restriction L : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' is an L'-substructure of N' where M', N' satisfy T' and omit Γ then $M' \restriction L \leq N' \restriction L$.

Proof. Let L' contain n-ary function symbols F_i^n for $n < \omega$ and $i < \omega$. We take as T' the theory which asserts only that nonempty models exist. For any $\boldsymbol{a} \in M$, let $M'_{\boldsymbol{a}}$ denote the L' structure generated by $\boldsymbol{a}$. Let Γ be the set of quantifier free L'-types of finite tuples $\boldsymbol{a}$ such that $M'_{\boldsymbol{a}} \restriction L \notin \boldsymbol{K}$ or for some $\mathbf{b} \subset \boldsymbol{a}$, $M'_{\mathbf{b}} \restriction L \not\leq M'_{\boldsymbol{a}} \restriction L$.

We claim T' and Γ suffice. That is, if $\boldsymbol{K}' = \{M' \restriction L : M' \models T'$ and M' omits $\Gamma\}$ then $\boldsymbol{K} = \boldsymbol{K}'$. If $M' \restriction L \in \boldsymbol{K}'$, write M' as a direct limit of finitely generated L'-structures $M'_{\boldsymbol{a}}$. By the choice of Γ, each $M'_{\boldsymbol{a}} \restriction L \in \boldsymbol{K}$ and if $\boldsymbol{a} \subseteq \boldsymbol{a}'$, $M'_{\boldsymbol{a}} \restriction L \leq M'_{\boldsymbol{a}'} \restriction L$, and so by the unions of chains axioms $M' \restriction L \in \boldsymbol{K}$. Conversely, if $M \in \boldsymbol{K}$, write M as a $\leq$-direct limit of countable L-structures. Expand each countable $\leq$-substructure of M to an L'-structure by letting $\{F_i^n(\boldsymbol{a}) : i < \omega\}$ enumerate the universe of M. By proceeding inductively, we can guarantee that these expansions cohere and verify that $M \in \boldsymbol{K}'$.

The moreover holds for countable structures directly by the choice of Γ and extends to arbitrary structures by the union of chain axioms on an AEC. In more detail, we have M' a direct limit of $M_{\boldsymbol{a}}$ and N' is a $\leq$-direct limit of $N_{\boldsymbol{a}}$ where $M_{\boldsymbol{a}} = N_{\boldsymbol{a}}$ for $\boldsymbol{a} \in M$. Each $M_{\boldsymbol{a}} \restriction L \leq N$ so the direct limit M is a strong submodel of N. $\square_9$

We have shown that an arbitrary countable AEC $\boldsymbol{K}$ is a $PC(\aleph_0, 2^{\aleph_0})$ class. We will show that under further conditions on $\boldsymbol{K}$, we can get an ω-presentation.

Exercise 10 *Show that if $\boldsymbol{K}$ is an AEC in a similarity type of cardinality λ, $\boldsymbol{K}$ can be presented as a $PC(\lambda, 2^{\lambda})$-class.*

Remark 11

1. *There is no use of amalgamation in this theorem.*
2. *The only penalty for increasing the size of the language or the Löwenheim number is that the size of L' and the nunber of types omitted; thus θ must be chosen larger.*
3. *We can (and Shelah does) observe that the class of pairs (M, N) with $M \leq N$ forms a $PC(\aleph_0, 2^{\aleph_0})$ but it seems that the moreover clause of Theorem 9 is a more useful version. See Theorem* **??** *and its applications. This clause appears in Grossberg's account: [*?*] and in Makowsky's [*?*].*

We will see many problems can be reduced classes of structures of the following sort.

Definition 12

1. *A* finite diagram *or $EC(T, \Gamma)$* class *is the class of models of first order theory which omit all types from a specified collection Γ of types in finitely many variables over the empty set.*
2. *$EC(T, Atomic)$ denotes the class of* atomic *models of T.*

The last definition abuses the $EC(T, \Gamma)$ notation, since for consistency, we really should write nonatomic. But atomic is shorter and emphasizes that we are restricting to the atomic models of T.

Exercise 13 *The models of an $EC(T, \Gamma)$ with the ordinary first order notion of elementary submodel as $\leq$ gives an AEC with Löwenheim number $\aleph_0$.*

Definition 14 *A λ-abstract class $\boldsymbol{K}_\lambda$ is a collection of τ-structures of cardinality λ and a binary relation $\leq_\lambda$ refining substructure on these structures such that both $\boldsymbol{K}_\lambda$ and $\leq_\lambda$ are closed under isomorphism and the properties of an AEC hold with one modification. The union of chains axioms is revised to apply to chains of length $\delta < \lambda$.*

Exercise 15 *If If $(\boldsymbol{K}, \leq)$ is an abstract elementary class then the restriction of $\boldsymbol{K}$ and $\leq$) to models of cardinality λ gives a λ-abstract elementary class.*

Exercise 16 *If $\boldsymbol{K}_\lambda$ is an abstract elementary class show $(\boldsymbol{K}, \leq)$ is an AEC with Löwenheim number λ if $\boldsymbol{K}$ and $\leq$ are all direct limits of $\boldsymbol{K}_\lambda$ and $\leq_\lambda$ respectively.*

Exercise 17 *Show that if the AEC's $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ have Lówenheim number λ and the same restriction to models of size λ they are identical above λ.*