Lecture 7: Galois stability

John T. Baldwin
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

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In this section we show that a countable λ-categorical AEC is μ-stable for μ above the Löwenheim number and below λ. The key idea is that for a linear order I and model $EM(I, \Phi)$ automorphisms of I induce automorphisms of $EM(I, \Phi)$. And, automorphisms of $EM(I, \Phi)$ preserve types in *any* reasonable logic; in particular, automorphisms of $EM(I, \Phi)$ preserve Galois types. Note that a model N is (defined to be) stable if few types are realized *in* N. So if N is a brimful model (Definition 2) then the model N is σ-stable for every $\sigma < |N|$.

Since we deal with reducts, we will consider several structures with the same universe; it is crucial to keep the vocabulary of the structure in mind. The AEC under consideration has vocabulary τ; it is presented as reducts of models of theory T' (which omit certain types) in a vocabulary τ'. In addition, we have the class of linear orderings (LO) in the background.

We really have three AEC's: $(LO, \subset)$, $\boldsymbol{K}'$ which is $Mod(T')$ with submodel as τ'-closed subset, and $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$. We are describing the properties of the EM-functor between $(LO, \subset)$ and $\boldsymbol{K}'$ or $\boldsymbol{K}$. $\boldsymbol{K}'$ is only a tool that we are singling out to see the steps in the argument. The following definitions hold for any of the three classes and I write $\leq$ for the notion of substructure. In this section of the paper I am careful to use $\leq$ when discussing all three cases versus $\prec_{\boldsymbol{K}}$ for the AEC.

Definition 1 *M_2 is σ-universal over M_1 in N if for every $M_1 \leq M_2 \leq N$ and whenever $M_1 \leq M_2' \leq N$, with $|M_1| \leq |M_2'| \leq \sigma$, there is a (partial isomorphism) fixing M_1 and taking M_2' into M_2.*

I introduce one term for shorthand. It is related to Shelah's notion of *brimmed* in [**?**].

Definition 2 *N is* brimful *if for every $\sigma < |N|$, and every $M_1 \leq N$ with $|M_1| = \sigma$, there is an M_2 that is σ-universal over M_1 in N.*

The next notion just makes it easier to write the proof of the following Lemma.

Notation 3 *Let $I \subset J$ be linear orders. We say a and b in J realize the same cut over I and write $a \sim_I b$ if for every $j \in J$, $a < j$ if and only if $b < j$.*

Claim 4 (Lemma 3.7 of [?]) *The linear order $I = \lambda^{<\omega}$ is brimful.*

Proof. Let $J \subset I$ have cardinality $\theta < \lambda$. Since we can increase J without harm, we can assume $J = A^{<\omega}$ for some $A \subset \lambda$. Note that $\sigma \sim_J \tau$ if and only if for the least n such that $\sigma \restriction n = \tau \restriction n \in J$, $\sigma(n) \sim_A \tau(n)$. Thus there are only θ cuts over J realized in I. For each cut C_α, $\alpha < \theta$, we choose a representative $\sigma_\alpha \in I - J$ of length n such that $\sigma_\alpha \restriction n-1 \in J$, so a cut has the form $\{\sigma_\alpha\hat{}\tau : \tau \in \lambda^{<\omega}, \alpha < \theta\}$. We can assume any J^* extending J has the form $J^* = B^{<\omega}$ for some $B \subset \lambda$, say with $\mathrm{otp}(B) = \gamma$. Thus, the intersection of J^* with a cut in J is isomorphic to a subset of $\gamma^{<\omega}$. We finish by noting for any ordinal $|\gamma| = \theta$, $\gamma^{<\omega}$ can be embedded in $\theta^{<\omega}$. Thus, the required θ-universal set over J is $J \cup \{\sigma_\alpha\hat{}\tau : \tau \in \theta^{<\omega}, \alpha < \theta\}$.

Qing Zhang has provided the following elegant argument for the last claim. First show by induction on γ there is a map g embedding γ in $\theta^{<\omega}$. (E.g. if $\gamma = \lim_{i<\theta} \gamma_i$, and g_i maps γ_i into $\theta^{<\omega}$, let for $\beta < \gamma$, $g(\beta) = i\hat{}g_i(\beta)$ where $\gamma_i \leq \beta < \gamma_{i+}$.) Then let h map $\gamma^{<\omega}$ into $\theta^{<\omega}$ by, for $\sigma \in \gamma^{<\omega}$ of length n, setting $h(\sigma) = \langle g(\sigma(0)), \ldots, g(\sigma(n-1))\rangle$. $\square_4$

The argument for Claim 4 yields:

Corollary 5 *Suppose $\mu < \lambda$ are cardinals. Then for any $X \subset \mu^{<\omega}$ and any Y with$X \subseteq Y \subset \lambda^{<\omega}$ and $|X| = |Y| < \mu$, there is an order embedding of Y into $\mu^{<\omega}$ over X.*

Exercise: For an ordinal γ, let $\gamma^{\omega *}$ denote the functions from ω to γ with only finitely many non-zero values. Show $\gamma^{\omega *}$ is a dense linear order and so is not isomorphic to $\gamma^{<\omega}$. Vary the proof above to show $\gamma^{\omega *}$ is brimful.

Since every L'-substructure of $EM(I, \Phi)$ has the form $EM(I_0, \Phi)$ for some subset I_0 of I, we have immediately:

Claim 6 *If I is brimful as a linear order, $EM(I, \Phi)$ is brimful as an L'-structure.*

Recall, Morley's omitting types theorem.

Lemma 7 *If $(X, <)$ is a sufficiently long linearly ordered subset of a τ-structure M, for any τ' extending τ (the length needed for X depends on $|\tau'|$) there is a countable set Y of τ'-indiscernibles (and hence one of arbitrary order type) such that $\mathbf{D}_\tau(Y) \subseteq \mathbf{D}_\tau(X)$. This implies that the only (first order) τ-types realized in $EM(X, \mathbf{D}_{\tau'}(Y))$ were realized in M.*

Using this result, we can find Skolem models over indiscernibles in an AEC.

Theorem 8 *If $\boldsymbol{K}$ is an abstract elementary class in the vocabulary τ, which is represented as a $PC\Gamma$ class witnessed by τ', T', Γ that has arbitrarily large models, there is a τ'-diagram Φ such that for every linear order $(I, <)$ there is a τ'-structure $M = EM(I, \Phi)$ such that:*

1. *$M \models T'$.*
2. *The τ'-structure $M = EM(I, \Phi)$ is the Skolem hull of I.*
3. *I is a set of τ'-indiscernibles in M.*
4. *$M \restriction \tau$ is in $\boldsymbol{K}$.*
5. *If $I' \subset I$ then $EM_\tau(I', \Phi) \prec_{\boldsymbol{K}} EM_\tau(I, \Phi)$.*

Proof. The first four clauses are a direct application of Lemma 7, Morley's theorem on omitting types. See also problem 7.2.5 of Chang-Keisler [**?**]. It is automatic that $EM(I', \Phi)$ is an L' substructure of $EM(I, \Phi)$. The moreover clause allows us to extend this to $EM_\tau(I', \Phi) \prec_{\boldsymbol{K}} EM_\tau(I, \Phi)$. $\square_8$

Now using amalgamation and categoricity, we move to the AEC $\boldsymbol{K}$. There are some subtle uses here of the 'coherence axiom': $M \subseteq N \prec_{\boldsymbol{K}} N_1$ and $M \prec_{\boldsymbol{K}} N_1$ implies $M \prec_{\boldsymbol{K}} N$.

Claim 9 *If I is brimful as linear order, $EM_\tau(I,\Phi)$ is brimful as a member of $\boldsymbol{K}$.*

Proof. Let $M = EM(I,\Phi)$; we must show $M \restriction \tau$ is brimful as a member of $\boldsymbol{K}$. Suppose $M_1 \prec_{\boldsymbol{K}} M \restriction \tau$ with $|M_1| = \sigma < |M|$. Then there is $N_1 = EM(I',\Phi)$ with $|I'| = \sigma$ and $M_1 \subseteq N_1 \leq M$. By Lemma 8.5, $N_1 \restriction \tau \prec_{\boldsymbol{K}} M \restriction \tau$. So $M_1 \prec_{\boldsymbol{K}} N_1 \restriction \tau$ by the coherence axiom. Let M_2 have cardinality σ and $M_1 \prec_{\boldsymbol{K}} M_2 \prec_{\boldsymbol{K}} M \restriction \tau$. Choose a τ'-substructure N_2 of M with cardinality σ containing N_1 and M_2. Now, N_2 can be embedded by a map f into the σ-universal τ'-structure N_3 containing N_1 which is guaranteed by Claim 6. But $f(N_2) \restriction \tau \prec_{\boldsymbol{K}} N_3 \restriction \tau$ by the coherence axiom so $N_3 \restriction \tau$ is the required $\boldsymbol{K}$-universal extension of M_1. $\square_9$

Definition 10 1. *Let $N \subset \mathbb{M}$. N is λ-Galois-stable if for every $M \subset N$ with cardinality λ, only λ Galois types over M are realized in N.*

2. *$\boldsymbol{K}$ is λ-Galois-stable if $\mathbb{M}$ is. That is $\mathrm{aut}_M(\mathbb{M})$ has only λ orbits for every $M \subset \mathbb{M}$ with cardinality λ.*

Since we are usually working in an AEC, we will frequently abuse notation and write stable rather than Galois-stable.

Since each Galois type over M_0 realized in M is represented by an M_1 with $M_0 \prec_{\boldsymbol{K}} M_1 \prec_{\boldsymbol{K}} M$, $M = EM(I,\phi)$ brimful, and $|M_1| = |M_0|$, Claim 9 implies immediately:

Claim 11 *If $\boldsymbol{K}$ is λ-categorical, the model M with $|M| = \lambda$ is σ-Galois stable for every $\sigma < \lambda$.*

Theorem 12 *If $\boldsymbol{K}$ is categorical in λ, then $\boldsymbol{K}$ is σ-Galois-stable for every $\sigma < \lambda$.*

Proof. Suppose $\boldsymbol{K}$ is not σ-stable for some $\sigma < \lambda$. Then by Löwenheim-Skolem, there is a model N of cardinality σ^+ which is not σ-stable. Let M be the σ-stable model with cardinality λ constructed in Claim 11. Categoricity and joint embedding imply N can be embedded in M. The resulting contradiction proves the result. $\square_{12}$

Remark 13 *Again, the assumption that $\boldsymbol{K}$ has amalgamation isn't needed here; instead of using Löwenheim-Skolem from the monster, one can use amalgamation on $\boldsymbol{K}_{<\lambda}$ and get joint embedding by restricting to the equivalence class of the categoricity model.*

Corollary 14 *Suppose $\boldsymbol{K}$ is categorical in λ and λ is regular. The model of power λ is saturated and so model homogeneous.*

Proof. Choose in $M_i \prec_{\boldsymbol{K}} \mathbb{M}$ using $< \lambda$-stability and Löwenheim-Skolem, for $i < \lambda$ so that each M_i has cardinality $< \lambda$ and M_{i+1} realizes all types over M_i. By regularity, it is easy to check that M_λ is saturated. $\square_{14}$

The same argument gives saturated models in smaller regular cardinals; more strongly we can demand that the saturated model be an Ehrenfreuht-Mostowski model.

Corollary 15 *Suppose $\boldsymbol{K}$ is an AEC with vocabulary τ that is categorical in λ and λ is regular. Then for every regular μ, $\mathrm{LS}(\boldsymbol{K}) < \mu < \lambda$ there is a model $M_\mu = EM_\tau(I_\mu,\Phi)$ which is saturated. In particular, it is μ-model homogeneous.*

Proof. For any ordered set J of cardinality λ, let $N = EM_\tau(J,\phi)$ be the model of cardinality λ. We construct an alternating chain of $\boldsymbol{K}$-submodels of length μ. $M_0 \prec_{\boldsymbol{K}} M$ is arbitrary with cardinality μ. $M_{2\alpha+1}$ has cardinality

μ and realizes all types over $M_{2\alpha}$ (possible by Corollary 14). $M_{2\alpha+2}$ has cardinality μ, $M_{2\alpha+1} \prec_{\boldsymbol{K}} M_{2\alpha+2}$ and $M_{2\alpha+2}$ is $EM_\tau(I_{\alpha+1}, \Phi)$ where $I_\alpha \subset I_{\alpha+1} \subset J$ and all I_α have cardinality μ. Then $EM_\tau(I_\mu, \Phi) = \bigcup_{\alpha<\mu} EM_\tau(I_\alpha, \Phi)$ is saturated by regularity. $\square_{15}$

Now using stability we can get a still stronger result, eliminating the hypothesis that μ is regular. We show the proofs of both Corollary 15 and Corollary 16 since in the first case we constructed a saturated model directly and in the second a model homogeneous structure.

Corollary 16 *Suppose $\boldsymbol{K}$ is categorical in λ and λ is regular. Then for every μ, $\mathrm{LS}(\boldsymbol{K}) < \mu < \lambda$ there is a model $M_\mu = EM(\mu^{<\omega}, \Phi)$ which is μ-model homogeneous.*

Proof. Represent the categoricity model as $M^* = EM_\tau(\lambda^{<\omega}, \Phi)$. We show $M_\mu = EM_\tau(\mu^{<\omega}, \Phi)$ is model homogenous. Suppose $M_1 \prec_{\boldsymbol{K}} M \restriction \tau$ with $|M_1| = \sigma < |M|$. Then there is $N_1 = EM_\tau({}_1, \Phi)$ with $|I_1| = \sigma$, $M_1 \subset N_1$ and $I_1 \subset \mu^{<\omega}$. Let M_2 have cardinality σ and $M_1 \prec_{\boldsymbol{K}} M_2$. By amalgamation, choose $N_2 \in \boldsymbol{K}$ which is an amalgam of N_1 and M_2 over M_1. By the λ-model homogeneity of M^*, there is an embedding of N_2 into M^* over N_1 say with image N_2'. Then $M_2' \subset EM(J, \Phi)$ for some J with $I_1 \subset J \subset (\lambda^{<\omega}$ and $|J| = \sigma$. Now by Corollary 5 and an argument like that in Claim 9, there is an embedding of $EM_\tau(J, \Phi)$ into $M = EM_\tau(\mu^{<\omega}, \Phi)$ over N_1, and *a fortiori* over M_1 and we finish. $\square_{16}$

Exercise 17 *Show Corollary 16 can be marginally strengthened by dropping the hypothesis that λ is regular but requiring that μ be less than the cofinality of λ.*