Ehrenfeucht-Mostowski models in Abstract Elementary Classes

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We work in the context of an abstract elementary class (AEC) with the amalgamation and joint embedding properties and arbitrarily large models. We prove two results using Ehrenfeucht-Mostowski models: 1) Morley's omitting types theorem – for *Galois* types. 2) If an AEC (with amalgamation) is categorical in some uncountable power μ it is stable in (every) $\lambda < \mu$.

These results are lemmas towards Shelah's consideration [**12**] of the downward transfer of categoricity, which we discuss in Section 6. This paper expounds some of the main ideas of [**12**], filling in vague allusions to earlier work and trying to separate those results which depend only on the Ehrenfeucht-Mostowski method from those which require more sophisticated stability theoretic tools.

In [**15**], Shelah proclaims the aim of reconstructing model theory, 'with no use of even traces compactness'. We analyze here one aspect of this program. Keisler organizes [**8**] around four kinds of constructions: the Henkin method, Ehrenfeucht-Mostowski models, unions of chains, and ultraproducts. The later history of model theory reveals a plethora of tools arising in stability theory. Fundamental is a notion of dependence which arises from Morley's study of rank, and passes through various avatars of splitting, strong splitting, and dividing before being fully actualized in the first order setting as forking. We eschew this technique altogether in this paper– to isolate its role.

The axioms of an AEC $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$, were first set down in [**17**]. We repeat for convenience.

Definition 0.1. *A class of L-structures,* $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$, *is said to be an* abstract elementary class: AEC *if both* $\boldsymbol{K}$ *and the binary relation* $\preceq_{\boldsymbol{K}}$ *are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If* $M \preceq_{\boldsymbol{K}} N$ *then* $M \subseteq N$.
- **A2**. $\preceq_{\boldsymbol{K}}$ *is a partial order on* $\boldsymbol{K}$.
- **A3**. *If* $\langle A_i : i < \delta \rangle$ *is* $\preceq_{\boldsymbol{K}}$*-increasing chain:*

(1) $\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;
(2) *for each* $j < \delta$, $A_j \preceq_{\boldsymbol{K}} \bigcup_{i<\delta} A_i$
(3) *if each* $A_i \preceq_{\boldsymbol{K}} M \in \boldsymbol{K}$ *then* $\bigcup_{i<\delta} A_i \preceq_{\boldsymbol{K}} M$.

- **A4**.*[Coherence Axiom] If* $A, B, C \in \boldsymbol{K}$, $A \preceq_{\boldsymbol{K}} C$, $B \preceq_{\boldsymbol{K}} C$ *and* $A \subseteq B$ *then* $A \preceq_{\boldsymbol{K}} B$.
- **A5**. *There is a Löwenheim-Skolem number* $\kappa(\boldsymbol{K})$ *such that if* $A \subseteq B \in K$ *there is a* $A' \in \boldsymbol{K}$ *with* $A \subseteq A' \preceq_{\boldsymbol{K}} B$ *and* $|A'| < |A| + \kappa(\boldsymbol{K})$.

In English, we often write B is a strong extension of A for $A \preceq_{\boldsymbol{K}} B$.

Sections 1, 2, 3 define most of the terminology and lay out the basic results. In Sections 4, we show categoricity implies stability and establish the existence of saturated models. Section 5 lifts Morley's omitting types theorem to the AEC setting. Finally in Section 6, we survey the additional steps needed to prove Shelah's downward categoricity theorem. I thank Greg Cherlin for some trenchant observations, Tapani Hyttinen for pointing out an error in an earlier draft, and Alex Usvyatsov for a careful reading.

1. Assumptions

We work with classes of structures in a fixed vocabulary, τ. When results are uniform functions of such invariants as the cardinality of τ or LS($\boldsymbol{K}$) we may write them in terms of these numbers. We use variants on τ to denote vocabularies. In addition to this usage, Shelah uses τ as an operator: $\tau(\Phi)$ denotes the vocabulary of the set of sentences Φ. We may write τ-structure or L-structure.

Assumption 1.1. (1) $\boldsymbol{K}$ *has arbitrarily large models.*
(2) $\boldsymbol{K}$ *satisfies the amalgamation property and the joint embedding property.*

We say $\boldsymbol{K}$ has the amalgamation property if $M, N_1, N_2 \in \boldsymbol{K}$ and there are strong embeddings of M into N_1 and N_2 then there is an N_3 and strong embeddings of N_1 and N_2 into N_3 so that the composition maps agree on M. Joint embedding means that for any two members of $\boldsymbol{K}$ there is a third into which both can be strongly embedded. Crucially, we amalgamate only over members of $\boldsymbol{K}$; this distinguishes this context from the context of homogeneous structures. Amalgamation does not imply the existence of arbitrarily large models; the class of initial segments of $\aleph_1$ with end extension as strong extension is an AEC. An AEC with disjoint amalgamation (the images of N_1 and N_2 in N_3 intersect in the image of M) and at least two models can easily be seen to have arbitrarily large models.

We stress that amalgamation is a very strong assumption and we make full use of it. However, many of the results can be achieved under some weaker conditions with somewhat more effort; we allude to some of these. Much of the Shelah work involves two kinds of argument of a more local nature: failure of amalgamation in κ implies many models in, say, κ^+ (with various variants), and arguments which assume only amalgamation below (or in) a certain cardinality.

Notation 1.2. (1) *Let* $\mu(\lambda, \kappa)$ *be the least cardinal* μ *such that if a first order theory* T *with* $|T| = \lambda$ *has models of every cardinal less than* μ *which omit* *each of a set Γ of types, with $|\Gamma| = \kappa$, then there are arbitrarily large models of T which omit Γ.*

(2) *Write $\mu(\kappa)$ for $\mu(\kappa, \kappa)$.*
(3) *We say λ is* substantial *for κ if $\lambda \geq \mu(\kappa)$.*
(4) *For a similarity type τ, $\mu(\tau)$ means $\mu(|\tau|)$.*

Note that an old theorem of Morley [**11**], VII.5, [**3**] says $\mu(\kappa, \kappa) \leq \beth_{(2^\kappa)^+}$. For simplicity, we assume the Löwenheim number is at least $|\tau|$.

When $\mathrm{LS}(\boldsymbol{K}) = |\tau(\boldsymbol{K})| = \kappa$, $\mu(\kappa)$ is sometimes called the Hanf number of $\boldsymbol{K}$. This is somewhat misleading because a single class cannot have a Hanf number – a Hanf number is a maximum for all similarity types of a given cardinality. It is in fact not the Hanf number of $\boldsymbol{K}$ but the Hanf number for all AEC with the same Löwenheim number. But as we'll see there is a still wider basis for this name; we will consider other classes of models (which are not AEC) and it is crucial that all of them have the property: for any model M with $|M| \geq \mu(\tau)$, there are models in the class of all cardinalities that omit all types omitted in M.

There is some vestige of compactness here. Both the existence of arbitrarily large models and amalgamation are proved in first order logic using compactness. But they have completely semantic statements and you have to start somewhere.

2. The presentation theorem and E-M models

We call the next result: the presentation theorem. It allows us to replace the entirely semantic description of an abstract elementary class by a syntactic one. I find it extraordinary that the notion of an AEC which is designed to give a version of the Fraïsse construction and thus saturated models, also turns out to allow the use of the second great model theoretic technique of the 50's: Ehrenfeucht-Mostowski models.

Theorem 2.1. *If K is an AEC with Lowenheim number $\mathrm{LS}(\boldsymbol{K})$ (in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\boldsymbol{K})$), there is a vocabulary τ' with cardinality $|\mathrm{LS}(\boldsymbol{K})|$, a first order τ'-theory T' and a set of $2^{\mathrm{LS}(\boldsymbol{K})}$ types Γ such that:*

$$\boldsymbol{K} = \{M' \restriction \tau : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, if M' is a τ''-substructure of N' where M', N' satisfy T' and omit Γ then $M' \restriction \tau \preceq_{\boldsymbol{K}} N' \restriction \tau$.

Proof. Let τ' contain n-ary function symbols F_i^n for $n < \omega$ and $i < \mathrm{LS}(\boldsymbol{K})$. We take as T' the theory which asserts only that its models are nonempty. For any τ'-structure M' and any $\boldsymbol{a} \in M$, let $M'_{\boldsymbol{a}}$ denote the subset of M' enumerated as $\{F_i^n(\boldsymbol{a}) : i < \mathrm{LS}(\boldsymbol{K})\}$ where $n = \lg(\boldsymbol{a})$; the only requirement on this enumeration is that the first n-elements are $\boldsymbol{a}$. The isomorphism type of $'_{\boldsymbol{a}}$ is determined by the quantifier free τ'-type of $\boldsymbol{a}$. Note that $M'_{\boldsymbol{a}}$ may not be either a τ' or even a τ-structure. Let Γ be the set of quantifier free τ'-types of finite tuples $\boldsymbol{a}$ such that $M'_{\boldsymbol{a}} \restriction \tau \notin \boldsymbol{K}$ or for some $\mathbf{b} \subset \boldsymbol{a}$, $M'_{\mathbf{b}} \restriction \tau \not\preceq_{\boldsymbol{K}} M'_{\boldsymbol{a}} \restriction \tau$.

We claim T' and Γ suffice. That is, if $\boldsymbol{K}' = \{M' \restriction \tau : M' \models T' \text{ and } M' \text{ omits } \Gamma\}$ then $\boldsymbol{K} = \boldsymbol{K}'$. Let the τ'-structure M' omit Γ; in particular, each $M'_{\boldsymbol{a}}$ is a τ-structure. Write M' as a direct limit of the finitely generated τ-structures $M'_{\boldsymbol{a}}$. (These may not be closed under the operations of τ'.) By the choice of Γ, each $M'_{\boldsymbol{a}} \restriction \tau \in \boldsymbol{K}$ and if $\boldsymbol{a} \subseteq \boldsymbol{a}'$, $M'_{\boldsymbol{a}} \restriction \tau \preceq_{\boldsymbol{K}} M'_{\boldsymbol{a}'} \restriction \tau$, and so by the unions of chains axioms $M' \restriction \tau \in \boldsymbol{K}$. Conversely, if $M \in \boldsymbol{K}$ we define by induction on $|\boldsymbol{a}|$, structures $M_{\boldsymbol{a}}$ for each finite subset $\boldsymbol{a}$ of M. Let $M_\emptyset$ be any $\preceq_{\boldsymbol{K}}$-substructure of M with cardinality $\mathrm{LS}(\boldsymbol{K})$ and let the $\{F^0_i : i < \mathrm{LS}(\boldsymbol{K})\}$ be constants enumerating the universe of $M_\emptyset$. Given a sequence $\mathbf{b}$ of length $n+1$, choose $M_{\mathbf{b}} \preceq_{\boldsymbol{K}} M$ with cardinality $\mathrm{LS}(\boldsymbol{K})$ containing all the $M_{\boldsymbol{a}}$ for $\boldsymbol{a} \subset \mathbf{b}$ of smaller cardinality. Let $\{F^{n+1}_i(\mathbf{b}) : i < \mathrm{LS}(\boldsymbol{K})\}$ enumerate the universe of $M_{\mathbf{b}}$ (and give the function the same value on any ordering of the range of $\mathbf{b}$). Now each $M_{\boldsymbol{a}} \restriction \tau \in \boldsymbol{K}$ and if $\mathbf{b} \subset \mathbf{c}$, $M_{\mathbf{b}} \preceq_{\boldsymbol{K}} M_{\mathbf{c}}$ so M' omits Γ as required.

The moreover holds for the partial τ'-structures $M'_{\boldsymbol{a}}$ directly by the choice of Γ and extends to arbitrary structures by the union of chain axioms on an AEC. In more detail, we have M' is a direct limit of finite structures $M'_{\boldsymbol{a}}$ and N' is a $\preceq_{\boldsymbol{K}}$-direct limit of $N'_{\boldsymbol{a}}$ where $M'_{\boldsymbol{a}} = N'_{\boldsymbol{a}}$ for $\boldsymbol{a} \in M$ because $M' \restriction \tau$ is a τ-substructure of $N' \restriction \tau$. Each $M'_{\boldsymbol{a}} \restriction \tau \preceq_{\boldsymbol{K}} N' \restriction \tau$ so the direct limit $M' \restriction \tau$ is a strong submodel of $N' \restriction \tau$. $\square_{2.1}$

We have represented $\boldsymbol{K}$ as a $PC\Gamma$ class in the following sense.

Definition 2.2. *A $PC(T, \Gamma)$* class *is the class of reducts to $\tau \subset \tau'$ of models of a first order theory τ'-theory which omit all types from the specified collection Γ of types in finitely many variables over the empty set.*

We write $PC\Gamma$ to denote such a class without specifying either T or Γ. And we write $\boldsymbol{K}$ is $PC(\lambda, \mu)$ if $\boldsymbol{K}$ can be presented as $PC(T, \Gamma)$ with $|T| \leq \lambda$ and $|\Gamma| \leq \mu$. In the simplest case, we say $\boldsymbol{K}$ is λ-presented if $\boldsymbol{K}$ is $PC(\lambda, \lambda)$.

In this language we have shown any AEC $\boldsymbol{K}$ is $2^{\mathrm{LS(K)}}$-presented.

Remark 2.3.
(1) *There is no use of amalgamation in this theorem.*
(2) *The only penalty for increasing the size of the language or the Löwenheim number is that the size of τ' and the nunber of types omitted increases as well. This will meant that for the use of EM models below, the θ must be chosen larger.*
(3) *We can observe with Shelah* [**17**] *that the class of pairs (M, N) with $M \preceq_{\boldsymbol{K}} N$ also forms a $PC(\mathrm{LS}(\boldsymbol{K}), 2^{\mathrm{LS}(\boldsymbol{K})})$. This observation is important for some applications but will not be used here; see Theorem 2.7 and its applications. The moreover clause appears in Grossberg's account:* [**5**] *and in Makowsky's* [**9**].

We immediately conclude the required computation of Hanf numbers for abstract elementary classes; we will use in a significant way the fact that this is, in fact, the Hanf number for $PC\Gamma$ classes where $|\Gamma| \leq 2^{|\tau|}$.

Corollary 2.4. *Let $\boldsymbol{K}$ be an AEC with similarity type τ. If $\boldsymbol{K}$ has a model with cardinality at least $\mu(\tau)$ then $\boldsymbol{K}$ has arbitrarily large models.*

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Notation 2.5. (1) *For any linearly ordered set $X \subseteq M$ where M is a τ-structure we write $\mathbf{D}_\tau(X)$ (diagram) for the set of τ-types of finite sequences (in the given order) from X. We will omit τ if it is clear from context.*
(2) *Such a diagram of an order indiscernible set, $\mathbf{D}_\tau(X) = \Phi$, is called 'proper for linear orders'.*
(3) *If X is a sequence of τ-indiscernibles with diagram $\Phi = \mathbf{D}_\tau(X)$ and any τ model of Φ has built in Skolem functions, then for any linear ordering I, $EM(I, \Phi)$ denotes the τ-hull of a sequence of order indiscernibles realizing Φ.*
(4) *If $\tau_0 \subset \tau$, the reduct of $EM(I, \Phi)$ to τ_0 is denoted $EM_{\tau_0}(I, \Phi)$.*

'Morley's method' (Section 7.2 of [**4**]) is a fundamental technique in first order model theory. It is essential for the foundations of simplicity theory and for the construction of indiscernibles in infinitary logic. We quote the first order version here; in Lemma 5.1, we prove the analog for abstract elementary classes.

Lemma 2.6. *If $(X, <)$ is a sufficiently long linearly ordered subset of a τ-structure M, for any τ' extending τ (the length needed for X depends on $|\tau'|$) with $<$ in τ' there is a countable set Y of τ'-indiscernibles (and hence one of arbitrary order type) such that $\mathbf{D}_\tau(Y) \subseteq \mathbf{D}_\tau(X)$. This implies that the only (first order) τ-types realized in $EM(X, \mathbf{D}_{\tau'}(Y))$ were realized in M.*

Further, we find Skolem models over indiscernibles in an AEC.

Theorem 2.7. *If $\boldsymbol{K}$ is an abstract elementary class in the vocabulary τ, which is presented as a $PC\Gamma$ class witnessed by τ', T', Γ that has arbitrarily large models, there is a τ'-diagram Φ such that for every linear order $(I, <)$ there is a τ'-structure $M = EM(I, \Phi)$ such that:*

(1) *$M \models T'$.*
(2) *The τ'-structure $M = EM(I, \Phi)$ is the Skolem hull of I.*
(3) *I is a set of τ'-indiscernibles in M.*
(4) *$M \restriction \tau$ is in $\boldsymbol{K}$.*
(5) *If $I' \subset I$ then $EM_\tau(I', \Phi) \preceq_{\boldsymbol{K}} EM_\tau(I, \Phi)$.*

Proof. The first four clauses are a direct application of Lemma 2.6, Morley's theorem on omitting types. See also problem 7.2.5 of Chang-Keisler [**4**] or [**3**]. It is automatic that $EM(I', \Phi)$ is an L' substructure of $EM(I, \Phi)$. The moreover clause of Theorem 2.1 allows us to extend this to $EM_\tau(I', \Phi) \preceq_{\boldsymbol{K}} EM_\tau(I, \Phi)$. $\square_{2.7}$

Note that we have simplified our presentation of many members of $\boldsymbol{K}$. Inside the class $\boldsymbol{K}$, which is the set of reducts of models which omit Γ, sits a class $\boldsymbol{K}'$, which is the class of reducts of Skolem hulls of order indiscernibles. In general, $\boldsymbol{K}'$ is a proper subclass of $\boldsymbol{K}$. It may not be an AEC because we don't know closure under unions of chains. In [**14**], under strong hypotheses this closure is proved.

Remark 2.8. *Silver (Chapter 18 of [**8**]) gives a simple example of a psuedoelementary class where the categoricity spectrum and its complement are both cofinal in* *the class of cardinals. The example is the class of models (M, X) where $2^{|X|} \geq |M|$. This class is not an AEC because it is not closed under unions of chains.*

The arguments below depend on classes being both AEC and PCΓ.

3. Galois types and saturation

In this section we take advantage of joint embedding and amalgamation to find a monster model. We then define types in terms of orbits of stabilizers of submodels. This allows an identification of 'model-homogeneous' with 'saturated'. That is, we give an abstract account of Morley-Vaught [**10**].

DEFINITION 3.1. *M is μ-model homogenous if for every $N \preceq_{\boldsymbol{K}} M$ and every $N' \in \boldsymbol{K}$ with $|N'| < \mu$ and $N \preceq_{\boldsymbol{K}} N'$ there is a $\boldsymbol{K}$-embedding of N' into M over N.*

To emphasize, this differs from the homogenous context because the N must be *in* $\boldsymbol{K}$. It is easy to show:

LEMMA 3.2. *If M_1 and M_2 are μ-model homogenous of cardinality $\mu > \mathrm{LS}(\boldsymbol{K})$ then $M_1 \approx M_2$.*

Proof. If M_1 and M_2 have a common submodel N of cardinality $< \mu$, this is an easy back and forth. Now suppose N_1, (N_2) is a small model of M_1, (M_2) respectively. By the joint embedding property there is a small common extension N of N_1, N_2 and by model homogeneity N is embedded in both M_1 and M_2. $\Box_{3.2}$

Note that in the absence of joint embedding, to get uniqueness we would (as in [**17**]) have to add to the definition of 'M is model homogeneous' that all models of cardinality $< \mu$ are embedded in M.

THEOREM 3.3. *If $\boldsymbol{K}$ has the amalgamation property and $\mu*^{<\mu*} = \mu*$ and $\mu* \geq 2^{\mathrm{LS}(\boldsymbol{K})}$ then there is a model $\mathbb{M}$ of cardinality $\mu*$ which is $\mu*$-model homogeneous.*

We call the model constructed in Theorem 3.3, the monster model. From now on all, structures considered are substructures of $\mathbb{M}$ with cardinality $< \mu*$. The standard arguments for the use of a monster model in first order model theory ([**7, 2**] apply here.

DEFINITION 3.4. *Let $M \in \boldsymbol{K}$, $M \preceq_{\boldsymbol{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.*

We freely use the phrase, 'Galois type of a over M'. Note that *a priori* this notion depends on the embedding of Ma into an $N \in \boldsymbol{K}$ and the embedding of N into $\mathbb{M}$. Since we have assumed amalgamation, our usage is justified as long as the base is an $M \in \boldsymbol{K}$. In more general situations, the Galois type is an equivalence class of an equivalence relation on triples (M, a, N). This is an equivalence relation on the class of M that are amalgamation bases for extensions in the same cardinality. (See [**18, 19**].) Since we have amalgamation and have fixed $\mathbb{M}$, we don't need the extra notation. The following definition and exercise show the connection of the situation as described here with the more complicated description elsewhere. They are needed only to link with the literature.

DEFINITION 3.5. *For $M \preceq_{\boldsymbol{K}} N_1 \in \boldsymbol{K}$, $M \preceq_{\boldsymbol{K}} N_1 \in \boldsymbol{K}$ and $a \in N_1 - M$, $b \in N_2 - M$, write $(M, a, N_1) \sim (M, b, N_2)$ if there exist strong embeddings f_1, f_2 of N_1, N_2 into some N^* which agree on M and with $f_1(a) = f_2(b)$.*

EXERCISE 3.6. *If $\boldsymbol{K}$ has amalgamation, $\sim$ is an equivalence relation.*

EXERCISE 3.7. *Suppose $\boldsymbol{K}$ has amalgamation and joint embedding. Show $(M, a, N_1) \sim (M, b, N_2)$ if and only if there are embeddings g_1 and g_2 of N_1, N_2 into $\mathbb{M}$ that agree on M and such that $g_1(a)$ and $g_2(b)$ have the same Galois type over $g_1(M)$.*

DEFINITION 3.8. *The set of Galois types over M is denoted* $\mathrm{ga-S(M)}$.

We say a Galois type p over M is realized in N with $M \preceq_{\boldsymbol{K}} N \preceq_{\boldsymbol{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

DEFINITION 3.9. *The model M is μ-Galois saturated if for every $N \preceq_{\boldsymbol{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is realized in M.*

Again, *a priori* this notion depend on the embedding of M into $\mathbb{M}$; but with amalgamation it is well-defined.

THEOREM 3.10. *For $\lambda > \mathrm{LS}(\boldsymbol{K})$, The model M is λ-Galois saturated if and only if it is λ-model homogeneous.*

Proof. It is obvious that λ-model homogeneous implies λ-Galois saturated. Let $M \preceq_{\boldsymbol{K}} \mathbb{M}$ be λ-saturated. We want to show M is λ-model homogeneous. So fix $M_0 \preceq_{\boldsymbol{K}} M$ and N with $M_0 \preceq_{\boldsymbol{K}} N \preceq_{\boldsymbol{K}} \mathbb{M}$. Say, $|N| = \mu < \lambda$. We must construct an embedding of N into M. Enumerate $N - M$ as $\langle a_i : i < \mu \rangle$. We will define f_i for $i < \mu$ an increasing continuous sequence of maps with domain N_i and range M_i so that $M_0 \preceq_{\boldsymbol{K}} N_i \preceq_{\boldsymbol{K}} \mathbb{M}$, $M_0 \preceq_{\boldsymbol{K}} M_i \preceq_{\boldsymbol{K}} M$ and $a_i \in N_{i+1}$. The restriction of $\bigcup_{i<\mu} f_i$ to N is required embedding. Let $N_0 = M_0$ and f_0 the identity. Suppose f_i has been defined. Choose the least j such that $a_j \in N - N_i$. By the model homogeneity of $\mathbb{M}$, f_i extends to an automorphism $\hat{f}_i$ of $\mathbb{M}$. Using the saturation, let $b_j \in M$ realize the Galois type of $\hat{f}_i(a_j)$ over M_i. So there is an $\alpha \in \mathrm{aut}\,\mathbb{M}$ which fixes M_i and takes b_j to $\hat{f}_i(a_j)$. Choose $M_{i+1} \preceq_{\boldsymbol{K}} M$ with cardinality μ and containing $M_i b_j$. Now $\hat{f}_i^{-1} \circ \alpha$ maps M_i to N_i and b_j to a_j. Let $N_{i+1} = \hat{f}_i^{-1} \circ \alpha(M_{i+1})$ and define f_{i+1} as the restriction of $\alpha^{-1} \circ \hat{f}_i$ to N_{i+1}. Then f_{i+1} is as required. $\square_{3.10}$

The last argument makes full use of the amalgamation property. We discuss some generalizations in the last paragraph of this article. In the remainder of this section we discuss some important ways in which Galois types behave differently from 'syntactic types'.

Note that if $M \preceq_{\boldsymbol{K}} N \preceq_{\boldsymbol{K}} \mathbb{M}$, then $p \in \mathrm{ga-S(N)}$ extends $p' \in \mathrm{ga-S(N)}$ if for some (any) a realizing p and some (any) b realizing p' there is an automorphism α fixing M and taking a to b.

Lemma 3.11. *If $M = \bigcup_{i<\omega} M_i$ is an increasing chain of members of $\boldsymbol{K}$ and $\{p_i : i < \omega\}$ satisfies $p_{i+1} \restriction M_i = p_i$, there is a $p_\omega \in \mathrm{ga-S(M)}$ with $p_\omega \restriction M_i = p_i$ for each i.*

Proof. Let a_i realize p_i. By hypothesis, for each $i < \omega$, there exists f_i which fixes M_{i-1} and maps a_i to a_{i-1}. Let g_i be the composition $f_0 \circ f_1 \circ \ldots f_i$. Then g_i maps a_i to a_0, fixes M_0 and $g_i \restriction M_{i-1} = g_{i-1} \restriction M_{i-1}$. Let M'_i denote $g_i(M_i)$ and M' their union. Then $\bigcup_{i<\omega} g_i$ is an isomorphism between M and M'. So by model-homogeneity there exists an automorphism h of $\mathbb{M}$ with $h \restriction M_i = g_i \restriction M_i$ for each i. Let $a_\omega = h^{-1}(a_0)$. Now $g_i^{-1} \circ h$ fixes M_i and maps a_ω to a_i for each i. This completes the proof. $\square_{3.11}$

Now suppose we wanted to prove Lemma 3.11 for chains of length $\delta > \omega$. The difficulty can be seen at stage ω. In addition to the assumptions of Lemma 3.11, we are given $\{a_i : i \leq \omega\}$ and $f_{\omega,i}$ which fixes M_i and maps a_ω to a_i. We can construct g_i as in the original proof. The difficulty is to find g_ω which extends all the g_i and maps a_ω to a_0. In the argument for Lemma 3.11, we found a map h and an element (which we will now call a'_ω such that h takes a'_ω to a_0 while h extends all the g_i. We would be done if a_ω and a'_ω realized the same galois type over $M = M_\omega$. In fact, a_ω and a'_ω realized the same Galois type over each M_i. So the following *locality* condition (for chains of length ω) would suffice for this special case. Moreover, by a further induction locality would give Lemma 3.11 for chains of arbitrary length. Locality does not hold for all AEC with amalgamation; it would be interesting to find a concrete example. Locality is defined in Definition 24 of [**15**].

Definition 3.12. *$\boldsymbol{K}$ has local Galois types if for every $M = \bigcup_{i<\kappa} M_i$ in a continuous increasing chain of members of $\boldsymbol{K}$ and for any $p, q \in \mathrm{ga-S(M)}$: if $p \restriction M_i = q \restriction M_i$ for every i then $p = q$.*

We have sketched the proof of:

Lemma 3.13. *Suppose $\boldsymbol{K}$ has local Galois types. If $M = \bigcup_{i<\kappa} M_i$ in an increasing chain of members of $\boldsymbol{K}$ and $\{p_i : i < \kappa\}$ satisfies $p_{i+1} \restriction M_i = p_i$, there is a $p_\kappa \in \mathrm{ga-S(M)}$ with $p_\kappa \restriction M_i = p_i$ for each i.*

Locality provides a key distinction between the general AEC case and homogenous structures. In homogeneous structures, types are syntactic objects and locality is trivial. Thus, as pointed out by Shelah, Hyttinen, and Buechler-Lessmann, Lemma 3.13 applies in the homogeneous context.

4. Getting stability

In this section we show that a countable λ-categorical AEC is μ-stable for μ above the Löwenheim number and below λ. The key idea is that for a linear order I and model $EM(I, \Phi)$, automorphisms of I induce automorphisms of $EM(I, \Phi)$. And, automorphisms of $EM(I, \Phi)$ preserve types in *any* reasonable logic; in particular, automorphisms of $EM(I, \Phi)$ preserve Galois types. Note that a model N is (defined to be) stable if few types are realized *in* N. So if N is a brimful model (Definition 4.2) then the model N is σ-stable for every $\sigma < |N|$.

Since we deal with reducts and will consider several structures with the same universe; it is crucial to keep the vocabulary of the structure in mind. The AEC under consideration has vocabulary τ; it is presented as reducts of models of theory T' (which omit certain types) in a vocabulary τ'. In addition, we have the class of linear orderings (LO) in the background.

We really have three AEC's: $(LO, \subset)$, $\boldsymbol{K}'$ which is $Mod(T')$ with submodel as τ'-closed subset, and $(\boldsymbol{K}, \preceq_{\boldsymbol{K}})$. We are describing the properties of the EM-functor between $(LO, \subset)$ and $\boldsymbol{K}'$ or $\boldsymbol{K}$. $\boldsymbol{K}'$ is only a tool that we are singling out to see the steps in the argument. The following definitions hold for any of the three classes and I write $\leq$ for the notion of substructure. In this section of the paper I am careful to use $\leq$ when discussing all three cases versus $\preceq_{\boldsymbol{K}}$ for the AEC.

Definition 4.1. *M_2 is σ-universal over M_1 in N if $M_1 \leq M_2 \leq N$ and whenever $M_1 \leq M_2' \leq N$, with $|M_1| \leq |M_2'| \leq \sigma$, there is a $\leq$-embedding fixing M_1 and taking M_2' into M_2.*

I introduce one term for shorthand. It is related to Shelah's notion of *brimmed* in [**13**] but here the brimful model is bigger than the models it is universal over while brimmed models may have the same cardinality.

Definition 4.2. *M is* brimful *if for every $\sigma < |M|$, and every $M_1 \leq M$ with $|M_1| = \sigma$, there is an $M_2 \leq M$ with cardinality σ that is σ-universal over M_1 in M.*

The next notion just makes it easier to write the proof of the following Lemma.

Notation 4.3. *Let $I \subset J$ be linear orders. We say a and b in J realize the same cut over I and write $a \sim_I b$ if for every $i \in I$, $a < i$ if and only if $b < i$.*

Claim 4.4 (Lemma 3.7 of [**16**]). *The linear order $I = \lambda^{<\omega}$ is brimful.*

Proof. Let $J \subset I$ have cardinality $\theta < \lambda$. Without loss of generality we can assume $J = A^{<\omega}$ for some $A \subset \lambda$. Note that $\sigma \sim_J \tau$ if and only if for the least n such that $\sigma \restriction n \neq \tau \restriction n$, neither is in J and $\sigma(n) \sim_A \tau(n)$. Thus there are only θ cuts over J realized in I. For each cut C_α, $\alpha < \theta$, we choose a representative $\sigma_\alpha \in I - J$ of length n such that $\sigma_\alpha \restriction n-1 \in J$, so a cut in J is isomorphic to $\{\sigma_\alpha \hat{\ } \tau : \tau \in \lambda^{<\omega}, \alpha < \theta\}$. We can assume any J^* extending J is $J^* = B^{<\omega}$ for some $B \subset \lambda$, say with $\mathrm{otp}(B) = \gamma$. Thus, the intersection of J^* with a cut in J is isomorphic to a subset of $\gamma^{<\omega}$. We finish by noting for any ordinal $|\gamma| = \theta$, $\gamma^{<\omega}$ can be embedded in $\theta^{<\omega}$. Thus, the required θ-universal set over J is $J \cup \{\sigma_\alpha \hat{\ } \tau : \tau \in \theta^{<\omega}, \alpha < \theta\}$.

Qing Zhang has provided the following elegant argument for the last claim. First show by induction on γ there is a map g embedding γ in $\theta^{<\omega}$. (E.g. if $\gamma = \lim_{i<\theta} \gamma_i$, and g_i maps γ_i into $\theta^{<\omega}$, let for $\beta < \gamma$, $g(\beta) = i \hat{\ } g_i(\beta)$ where $\gamma_i \leq \beta < \gamma_{i+}$.) Then let h map $\gamma^{<\omega}$ into $\theta^{<\omega}$ by, for $\sigma \in \gamma^{<\omega}$ of length n, setting $h(\sigma) = \langle g(\sigma(0)), \ldots, g(\sigma(n-1)) \rangle$. $\square_{4.4}$

The argument for Claim 4.4 yields:

Corollary 4.5. *Suppose $\mu < \lambda$ are cardinals. Then for any $X \subset \mu^{<\omega}$ and any Y with $X \subseteq Y \subset \lambda^{<\omega}$ and $|X| = |Y| < \mu$, there is an order embedding of Y into $\mu^{<\omega}$ over X.*

Since every τ'-substructure N of $EM(I,\Phi)$ is contained in a substructure $EM(I_0,\Phi)$ for some subset I_0 of I with $|I_0| = |N|$, we have immediately:

Claim 4.6. *If I is brimful as a linear order, $EM(I,\Phi)$ is brimful as an τ'-structure.*

Now using amalgamation and categoricity, we move to the AEC $\boldsymbol{K}$. There are some subtle uses here of the 'coherence axiom': $M \subseteq N \preceq_{\boldsymbol{K}} N_1$ and $M \preceq_{\boldsymbol{K}} N_1$ implies $M \preceq_{\boldsymbol{K}} N$.

Claim 4.7. *If I is brimful as a linear order, $EM_\tau(I,\Phi)$ is brimful as a member of $\boldsymbol{K}$.*

Proof. Let $M = EM(I,\Phi)$; we must show $M \restriction \tau$ is brimful as a member of $\boldsymbol{K}$. Suppose $M_1 \preceq_{\boldsymbol{K}} M \restriction \tau$ with $|M_1| = \sigma < |M|$. Then there is $N_1 = EM(I',\Phi)$ with $|I'| = \sigma$ and $M_1 \subseteq N_1 \leq M$. By Lemma 2.7.5, $N_1 \restriction \tau \preceq_{\boldsymbol{K}} M \restriction \tau$. So $M_1 \preceq_{\boldsymbol{K}} N_1 \restriction \tau$ by the coherence axiom. Let M_2 have cardinality σ and $M_1 \preceq_{\boldsymbol{K}} M_2 \preceq_{\boldsymbol{K}} M \restriction \tau$. Choose a τ'-substructure N_2 of M with cardinality σ containing N_1 and M_2. Now, N_2 can be embedded by a map f into the σ-universal τ'-structure N_3 containing N_1 which is guaranteed by Claim 4.6. But $f(N_2) \restriction \tau \preceq_{\boldsymbol{K}} N_3 \restriction \tau$ by the coherence axiom so $N_3 \restriction \tau$ is the required σ-universal extension of M_1. $\square_{4.7}$

Definition 4.8. (1) *Let $N \subset \mathbb{M}$. N is λ-Galois-stable if for every $M \subset N$ with cardinality λ, only λ Galois types over M are realized in N.*
(2) *$\boldsymbol{K}$ is λ-Galois-stable if $\mathbb{M}$ is. That is $\mathrm{aut}_M(\mathbb{M})$ has only λ orbits for every $M \subset \mathbb{M}$ with cardinality λ.*

Since we are usually working in an AEC, we will frequently abuse notation and write stable rather than Galois-stable.

Since for brimful I, a $M = EM(I,\phi)$ is brimful, and for $M_0 \preceq_{\boldsymbol{K}} M_1 \preceq_{\boldsymbol{K}} M$, each Galois type over M_0 realized in M is represented by an M_1 with $|M_1| = |M_0|$, Claim 4.7 implies immediately:

Claim 4.9. *If $\boldsymbol{K}$ is λ-categorical, the model M with $|M| = \lambda$ is σ-Galois stable for every $\sigma < \lambda$.*

Theorem 4.10. *If $\boldsymbol{K}$ is categorical in λ, then $\boldsymbol{K}$ is σ-Galois-stable for every $\sigma < \lambda$.*

Proof. Suppose $\boldsymbol{K}$ is not σ-stable for some $\sigma < \lambda$. Then by Löwenheim-Skolem, there is a model N of cardinality σ^+ which is not σ-stable. Let M be the σ-stable model with cardinality λ constructed in Claim 4.9. Categoricity and joint embedding imply N can be embedded in M. The resulting contradiction proves the result. $\square_{4.10}$

Corollary 4.11. *Suppose $\boldsymbol{K}$ is categorical in λ and λ is regular. The model of power λ is saturated and so model homogeneous.*

Proof. Choose in $M_i \preceq_{\boldsymbol{K}} \mathbb{M}$ using $< \lambda$-stability and Löwenheim-Skolem, for $i < \lambda$ so that each M_i has cardinality $< \lambda$ and M_{i+1} realizes all types over M_i. By regularity, it is easy to check that M_λ is saturated. $\square_{4.11}$

The same argument gives saturated models in smaller regular cardinals; more strongly we can demand that the saturated model be an Ehrenfreuht-Mostowski model.

Corollary 4.12. *Suppose* $\boldsymbol{K}$ *is an AEC with vocabulary* τ *that is categorical in* λ *and* λ *is regular. Then for every regular* μ, $\mathrm{LS}(\boldsymbol{K}) < \mu < \lambda$ *there is a model* $M_\mu = EM_\tau(I_\mu, \Phi)$ *which is saturated. In particular, it is* μ*-model homogeneous.*

Proof. For any ordered set J of cardinality λ, let $M = EM_\tau(J, \phi)$ be the model of cardinality λ. We construct an alternating chain of $\boldsymbol{K}$-submodels of length μ. $M_0 \preceq_{\boldsymbol{K}} M$ is arbitrary with cardinality μ. $M_{2\alpha+1}$ has cardinality μ and realizes all types over $M_{2\alpha}$ (possible by Corollary 4.10. $M_{2\alpha+2}$ has cardinality μ, $M_{2\alpha+1} \preceq_{\boldsymbol{K}} M_{2\alpha+2}$ and $M_{2\alpha+2}$ is $EM_\tau(I_{\alpha+1}, \Phi)$ where $I_\alpha \subset I_{\alpha+1} \subset J$ and all I_α have cardinality μ. Then $EM_\tau(I_\mu, \Phi) = \bigcup_{\alpha<\mu} EM_\tau(I_\alpha, \Phi)$ is saturated by regularity. $\square_{4.12}$

Now using stability we can get a still stronger result, eliminating the hypothesis that μ is regular. We show the proofs of both Corollary 4.12 and Corollary 4.13 since in the first case we constructed a saturated model directly and in the second a model homogeneous structure.

Corollary 4.13. *Suppose* $\boldsymbol{K}$ *is an AEC with vocabulary* τ *that is categorical in* λ *and* λ *is regular. Then for every* μ, $\mathrm{LS}(\boldsymbol{K}) < \mu < \lambda$ *there is a model* $M_\mu = EM_\tau(\mu^{<\omega}, \Phi)$ *which is* μ*-model homogeneous.*

Proof. Represent the categoricity model as $M^* = EM_\tau(\lambda^{<\omega}, \Phi)$. We show $M_\mu = EM_\tau(\mu^{<\omega}, \Phi)$ is model homogenous. Suppose $M_1 \preceq_{\boldsymbol{K}} M_\mu$ with $|M_1| = \sigma < |M_\mu|$. Then there is $N_1 = EM_\tau(I_1, \Phi)$ with $|I_1| = \sigma$, $M_1 \subset N_1$ and $I_1 \subset \mu^{<\omega}$. Let M_2 have cardinality σ and $M_1 \preceq_{\boldsymbol{K}} M_2$. By amalgamation, choose $N_2 \in \boldsymbol{K}$ which is an amalgam of N_1 and M_2 over M_1. By the λ-model homogeneity of M^*, there is an embedding of N_2 into M^* over N_1 say with image N_2'. Then $N_2' \subset EM_\tau(J, \Phi)$ for some J with $I_1 \subset J \subset \lambda^{<\omega}$ and $|J| = \sigma$. Now by Corollary 4.5 and an argument like that in Claim 4.7, there is an embedding of $EM_\tau(J, \Phi)$ into $M = EM_\tau(\mu^{<\omega}, \Phi)$ over N_1, and *a fortiori* over M_1 and we finish. $\square_{4.13}$

Remark 4.14.
(1) *Note that for each* σ *less than the categoricity cardinal* λ*, the* σ*-universal model that is constructed has the form* $EM_\tau(I', \Phi)$ *for some* I'.
(2) *Compare Claim 4.13 to I.3.1 in* [**18**]*, which has the same conclusion but weakening the amalgamation propery to: there are no maximal models. There are two uses of the amalgamation property in the argument for Claim 4.13. The first requires only that* M_1 *be an amalgamation base for models in* $\boldsymbol{K}$ *of size* μ *and so extends easily to prove the analogous result where* $\boldsymbol{K}$ *has amalgamation is replaced by* $\boldsymbol{K}$ *has no maximal models. The second is that* M *is* $< \lambda$ *model homogenous. This step is done in quite a different way in the proof of I.3.1 in* [**18**]*; stability is not used but GCH is.*

5. Morley's method for Galois Types

Now we prove 'Morley's method' for Galois types.

LEMMA 5.1. *[II.1.5 of Sh394] If $M_0 \leq M$ and M is substantial* with respect to *$|M_0|$, we can find an EM-set Φ such that the following hold.*

(1) *The τ-reduct of the Skolem closure of the empty set is M_0.*
(2) *For every I, $M_0 \leq EM(I, \Phi)$.*
(3) *If I is finite, $EM_\tau(I, \Phi)$ can be embedded in M.*
(4) *$EM_\tau(I, \Phi)$ omits every galois type over M_0 which is omitted in M.*

Proof. Let τ_1 be the Skolem language given by the presentation theorem and consider M as the reduct of τ_1 structure M^1. Add constants for M_0 to form τ_1'. Now apply Lemma 2.6 to find an EM-diagram Φ (in τ_1') with all τ-types of finite subsets of the indiscernible sequence realized in M. Now 1) and 2) are immediate. 3) is easy (using clause 5 of Theorem 2.7) since we chose Φ so all finite subsets of the indiscernible set (and so their Skolem closures) are realized in M.

The omission of Galois types is more tricky. Consider both M and $N = EM_\tau(I, \Phi)$ embedded in $\mathbb{M}$. Let N^1 denote the τ_1'-structure $EM(I, \Phi)$. We need to show that if $a \in N$, $p = \mathrm{ga-tp(a/M_0)}$ is realized in M. For some $\mathbf{e} \in I$, a is in the τ_1-Skolem hull $N_{\mathbf{e}}$ of $\mathbf{e}$. (Recall the notation from the presentation theorem.) By 3) there is an embedding α of $N_{\mathbf{e}}$ into M^1 over M_0. α is also an isomorphism of $N_{\mathbf{e}} \restriction \tau$ into M. Now, by the model homogeneity, α extends to an automorphism of $\mathbb{M}$ fixing M_0 and $\alpha(a) \in M$ realizes p. $\square_{5.1}$

This has immediate applications in the direction of transferring categoricity.

THEOREM 5.2. *Suppose $M \in \boldsymbol{K}$ omits a Galois type p over a submodel M_0 with $|M| \geq \mu(|M_0|)$. Then there is no regular cardinal $\lambda \geq |M|$ in which $\boldsymbol{K}$ is categorical.*

Proof. By Lemma 5.1, there is a model $N \in \boldsymbol{K}$ with cardinality λ which omits p. But by Lemma 4.11, the unique model of power λ is saturated. $\square_{5.2}$

6. Tameness and Downwards Categoricity

In [**12**] Shelah asserts the following result:

THEOREM 6.1. *If $\boldsymbol{K}$ is categorical in a regular cardinal λ and $\lambda > \mu(\mu(|\tau|))$ then $\boldsymbol{K}$ is categorical in every θ with $\mu(\mu(|\tau|)) \leq \theta \leq \lambda$.*

Here is a sketch of the argument. We have shown that there are saturated models of power θ for every $\theta < \lambda$. The obstacle to deducing downward categoricity is that Theorem 5.1 only allows us to transfer the omission of types when the model omitting the type is much bigger than the domain of the type. The first step in remedying this problem is to show that all types are determined by 'relatively small' subtypes. More precisely, we need the notion that Grossberg and Van Dieren [**6**] have called χ-tame and Shelah [**12**] refers to has 'having χ-character'. We add an extra parameter to be careful.

Definition 6.2. *We say* $\boldsymbol{K}$ *is* (χ, μ)*-tame if for any saturated* $N \in \boldsymbol{K}$ *with* $|N| = \mu < \lambda$ *if* $p, q, \in \mathrm{ga} - \mathrm{S(N)}$ *and for every* $N_0 \leq N$ *with* $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ *then* $q = p$.

Shelah asserts the following in Sections II.1 and II.2.3 of the published version of [**12**]. The published proof is incomplete; I haven't yet seen the corrections. But it seems to use only Ehrenfeucht-Mostowski type methods.

Theorem 6.3. *Suppose* $\boldsymbol{K}$ *is* λ*-categorical for* $\lambda \geq \mu(\tau)$ *and* λ *is regular. Then* $\boldsymbol{K}$ *is* (χ, χ_1)*-tame for some* χ *and any* χ_1 *with* $\chi < \mu(\tau) \leq \chi_1 \leq \lambda$.

The naive argument would give $\chi = \mu(\tau)$ since one is omitting types. But omitting in every cardinal below $\mu(\tau)$ is as good as in $\mu(\tau)$ so the conclusion becomes for some χ with $\chi < \mu(\tau)$.

The remainder of the argument for Theorem 6.1 uses such technologies as splitting and minimal types that are beyond the scope of this paper.

Since we were expounding [**12**] we assumed, as there, that $\boldsymbol{K}$ has arbitrarily large models and the amalgamation and joint embedding properties. We used amalgamation heavily to get monster models and thus get the group theoretic definition of Galois-type. By using the more complicated definition of a Galois type as an equivalence relation on triples, many of these notions can be extended to classes without amalgamation. And one can even prove [**15, 1**], saturation equals model homogeneity with no amalgamation hypothesis whatsoever. However, I don't know anyway to prove the existence of either saturated model homogeneous models in general AEC without at least some amalgamation hypothesis.

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