CATEGORICITY AND U-RANK IN EXCELLENT CLASSES

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ABSTRACT. Let $\mathcal{K}$ be the class of atomic models of a countable first order theory. We prove that if $\mathcal{K}$ is excellent and categorical in some uncountable cardinal, then each model is prime and minimal over the basis of a definable pregeometry given by a quasiminimal set. This implies that $\mathcal{K}$ is categorical in all uncountable cardinals. We also introduce a U-rank to measure the complexity of complete types over models. We prove that the U-rank has the usual additivity properties, that quasiminimal types have U-rank 1, and that the U-rank of any type is finite in the uncountably categorical, excellent case. However, in contrast to the first order case, the supremum of the U-rank over all types may be ω (and is not achieved). We illustrate the theory with the example of free groups, and Zilber's pseudo analytic structures.

0. INTRODUCTION

A class of mathematical structures $\mathcal{K}$ is *categorical* in some cardinal λ if all the structures in $\mathcal{K}$ of size λ are isomorphic. The problem of categoricity can be roughly phrased as follows: Suppose $\mathcal{K}$ is categorical in some cardinal(s), is $\mathcal{K}$ also categorical in other (all) cardinals? The classical problem, when $\mathcal{K}$ is the class of models of a first order theory, has been a driving force in first order model theory, and it is difficult to overestimate the impact of Morley's theorem [Mo], Baldwin-Lachlan's Theorem [BaLa], and Shelah's generalisation to uncountable languages [Sh70] on its development.

The present paper is concerned with the categoricity of classes of models which may not be axiomatisable in first order logic. There are several natural extensions of first order logic, many of which are equivalent for this problem. We will focus on classes of models of a first order theory T omitting a prescribed set of types Γ. There are two extreme cases: When Γ is empty; this is the first order case. When Γ is the set of nonisolated types; this is the *atomic case*, the class of models omitting all nonisolated types is the class of atomic models of T. For simplicity, and without real loss of generality, we consider the atomic case; in Remark 1.4 we explain how to develop excellence for some more general Γ.

In the early 1970s, Keisler [Ke] and Shelah [Sh3] independently proved that if a class $\mathcal{K}$ of atomic models is categorical in some uncountable cardinal,

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then it is categorical in all uncountable cardinals, *provided* there are arbitrarily large homogeneous models in $\mathcal{K}$. Keisler asked at that time [Ke] whether categoricity (say in all uncountable cardinals) implies the proviso. Shelah answered negatively [Sh48] using an example of Marcus [Ma], and developed the theory of *excellence* [Sh48], [Sh87a], and [Sh87b]. He showed:

Theorem 0.1 (Shelah)**.** *Let $\mathcal{K}$ be the class of atomic models of a first order countable theory.*

(1) *Assume GCH. If $\mathcal{K}$ is categorical in all uncountable cardinals, then $\mathcal{K}$ is excellent.*
(2) *If $\mathcal{K}$ is excellent and categorical in some uncountable cardinal, then $\mathcal{K}$ is categorical in all uncountable cardinals.*

It follows from the work of Shelah [Sh3], that, in this case, the presence of uncountable homogeneous models implies that of arbitrarily large homogeneous models and that both imply excellence. Excellence is a form of strong amalgamation property (see Remark 2.23 for the precise definition). In this paper, we extract two main consequences of excellence and work only with these consequences (we also present an alternative proof directly from Shelah's definition of excellence for illustrative purposes in Remark 2.23). These consequences are:

(1) The amalgamation property over models.
(2) If p is a complete type over a model $M \in \mathcal{K}$ with the property that $p \upharpoonright C$ is realised in an extension of M for any finite subset $C \subseteq M$, then p is realised in an extension $N \in \mathcal{K}$ of M. Moreover, N can be chosen prime over M and a realisation of p.

Grossberg and Hart continued the classification for excellent classes in [GrHa]. They develop orthogonality calculus and prove the Main Gap, showing that DOP is a dividing line. All the results attributed to Grossberg-Hart in this paper are from [GrHa].

Section 1 of the paper is devoted to the basics of excellence. We present a very accessible description of those properties of excellence that are needed for our theorems. We also compare excellence with homogeneous model theory and Shelah's abstract elementary classes. The results on excellence in this section can be found in [Sh87a] and [Sh87b].

In Section 2 of the paper, we give a Baldwin-Lachlan proof of the categoricity theorem in the excellent case. We also remind the reader of some basic facts that can be found in [Sh87a], [Sh87b], and [GrHa]. We show:

Theorem 0.2. *Let $\mathcal{K}$ be excellent and categorical in some uncountable cardinal. Then each model is prime and minimal over the basis of a type-definable pregeometry given by a quasiminimal set. Moreover, the size of the basis determines the isomorphism-type of the model, so $\mathcal{K}$ is categorical in all uncountable cardinals.*

Quasiminimal sets are the natural extension of strongly minimal sets. They were first introduced by Shelah in a different context in [Sh48]. The name was coined by Zilber in the mid-1990s, where he used an equivalent notion in his work around the model theory of the field of complex numbers with exponentiation [Zi2].

In Section 3 of the paper, we introduce a natural U-rank to measure the complexity of complete types over models – excellence provides a good understanding of those types. The U-rank does not agree with Shelah's rank [Sh48] (Remark 3.7). Quasiminimal types over models are exactly those of U-rank 1

The main result of Section 3 is:

Theorem 0.3. *Let $\mathcal{K}$ be excellent and uncountably categorical. Then*

$$U(\mathrm{tp}(a/M)) < \omega,$$

for each $M \prec N$ and $a \in N \in \mathcal{K}$.

We also show that the U-rank has good additivity properties in the uncountably categorical, excellent case:

$$U(\mathrm{tp}(ab/M)) = U(\mathrm{tp}(a/M(b))) + U(\mathrm{tp}(b/M)),$$

where $M(b)$ is the primary model over $M \cup b$.

Finally, we examine the examples of free groups and one of Zilber's pseudo-analytic structures to illustrate the theory. Neither example is first order axiomatisable. The example of free groups shows that the supremum of the U-rank of types of elements may be ω, in contrast to the first order uncountably categorical case, where the supremum is always finite. This example also shows various limitations on possible generalisations of the theory of stable groups to nonelementary classes (free groups have no generics, and all their abelian subgroups are countable).

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1. Excellence

We consider the class $\mathcal{K}$ of atomic models of a complete first order theory T in a countable language L, *i.e.* $\mathcal{K}$ is the class of models of T which omit all the nonisolated types of T over the empty set. The atomicity implies that each type over finitely many parameters realised in a model of $\mathcal{K}$ is equivalent to a formula over the same parameters. It follows that each $M \in \mathcal{K}$ is $\aleph_0$-homogeneous and that $\mathcal{K}$ is $\aleph_0$-categorical. Recall that a model M is λ-*homogeneous* if for any elementary map $f : M \to M$ with $|f| < \lambda$ and $a \in M$, there is an elementary map $g : M \to M$ extending f such that $a \in \mathrm{dom}(g)$. Finally, notice that the downward

Löwenheim Skolem theorem holds for $\mathcal{K}$, *i.e.* if $M \in \mathcal{K}$ and $A \subseteq M$, there exists $M' \prec M$ (hence $M' \in \mathcal{K}$) containing A such that $\|M'\| = |A| + \aleph_0$.

In general, there may not be uncountable atomic models – for example the natural numbers $\mathbb{N}$ in the language $\{+, 0, 1\}$ is the only atomic model of its first order theory. Since we are interested in uncountable categoricity, we always assume that there exists arbitrarily large models (this also follows from excellence).

We now turn to *excellence*. In fact, rather than defining excellence, we isolate some basic consequences which are the only properties that we use in this paper. The first is the amalgamation property over models (recall that homogeneity is essentially the amalgamation property over sets).

(1) **Amalgamation property:** Let $\mathcal{K}$ be excellent. Let $M_\ell \in \mathcal{K}$ for $\ell = 0, 1, 2$ and $f_\ell : M_0 \to M_\ell$ elementary maps for $\ell = 1, 2$. Then there exist a model $N \in \mathcal{K}$ and elementary maps $g_\ell : M_\ell \to N$, such that $g_2 \circ f_2 \restriction M_0 = g_1 \circ f_1 \restriction M_0$.

Since the countable model is unique, we can use amalgamation and the fact that there are arbitrarily large models to show that each model $M \in \mathcal{K}$ has arbitrarily large elementary extensions in $\mathcal{K}$.

The next consequence has to do with our understanding of types. Given a complete type p over a model $M \in \mathcal{K}$, when do we know whether p is realised in an elementary extension $N \in \mathcal{K}$ of M? In the first order case, compactness provides an easy answer. Here the situation is a bit more involved. We certainly have the following *necessary* condition: If $a \in N \in \mathcal{K}$ realises p, then, since $Ma \subseteq N$, we have that $\mathrm{tp}(a/C)$ is isolated by a formula over C, for each finite $C \subseteq M$. Hence, $p \restriction C$ is realised in M for each finite subset $C \subseteq M$. Excellence implies that this is enough:

(2) **Type realisability:** Let $\mathcal{K}$ be excellent. Let p be a complete type over a model $M \in \mathcal{K}$ such that $p \restriction C$ is realised in M for each finite $C \subseteq M$. Then there exists $N \in \mathcal{K}$, with $M \prec N$, such that p is realised in N.

Notice that (2) is a form of weak compactness for complete types over models; knowing whether a complete type over a model is realised is a property which has finite character in the parameters. This criterion applies only to complete types over models. It is convenient to use the following definition.

Definition 1.1. Let $M \in \mathcal{K}$. We let $\mathrm{S}_{\mathrm{at}}(M)$ be the set of types $p \in S(M)$ such that $p \restriction C$ is realised in M for each finite $C \subseteq M$.

A consequence of (1) and (2) is that if $M \prec N \in \mathcal{K}$ and $p \in \mathrm{S}_{\mathrm{at}}(M)$, then there exists $q \in \mathrm{S}_{\mathrm{at}}(N)$ extending p (this property fails for atomic, nonexcellent $\mathcal{K}$ in general). We can now introduce the substitute to λ-saturated models.

Definition 1.2. A model $N \in \mathcal{K}$ is λ-*full* if N realises each complete type $p \in \mathrm{S}_{\mathrm{at}}(M)$, where $M \prec N$ has size less than λ.

This is not Shelah's original definition, but it is equivalent for uncountable λ, which are the ones we care about.

Using (1) and (2), we can construct λ-full models in $\mathcal{K}$ of size at least λ for arbitrarily large λ: Construct $(M_i : i < \lambda^+)$ increasing and continuous such that each M_{i+1} realises all types in $\mathrm{S_{at}}(M_i)$. Then $\bigcup_{i<\lambda^+} M_i$ is λ-full since λ^+ is regular and complete types over models can be extended.

Any λ-full model N functions as a *universal domain* for the class of models of $\mathcal{K}$ of size at most λ: Any such model model embed elementarily in N, and, by definition of λ-fullness, any type $p \in \mathrm{S_{at}}(M)$, where $M \prec N$ of size less than λ is realised in N.

Remark 1.3. Let us compare this context with Shelah's abstract elementary classes. Let $(K, <_K)$ be an abstract elementary class with amalgamation over models. (The class $\mathcal{K}$ of atomic models of a first order theory is an example of abstract elementary class.) Shelah defines a natural semantic notion of complete types over models, named *Galois types* by some: He considers triples of the form (a, M, N), where $a \in N$, $M, N \in K$ and $M <_K N$ and defines the relation $\sim$, where $(a_1, M_1, N_1) \sim (a_2, M_2, N_2)$ if $M_1 = M_2$ and there is $N \in K$ and $f_\ell : N_\ell \to N$ such that $f_1(a_1) = f_2(a_2)$. The relations $\sim$ is easily seen to be an equivalence relation using the amalgamation property. A type $\mathrm{tp}(a/M, N)$ is simply the equivalence class $(a, M, N)/\sim$. If $\mathcal{K}$ is atomic and excellent, type realisability ensures that the semantic notion of types $(a, M, N)/\sim$, for $M \prec N \in \mathcal{K}$, and $a \in N$, coincides with the syntactic notion of types in $\mathrm{S_{at}}(M)$, for $M \in \mathcal{K}$. The notion of λ-full that we use is what Shelah calls λ-saturation (for Galois types) in abstract elementary classes, which he showed equivalent to λ-model homogeneity. See Shelah's [Sh576] and [Sh600]. For an exposition of Abstract Elementary Classes, see [Gr1] or [Gr2], where a proof of the equivalence between λ-saturation and λ-model homogeneity is also presented.

The final consequence of excellence that we are going to use is an improvement of (2), which deals with the existence of prime models. Recall that a model M is *primary* over a set A, if $M = A \cup \{a_i : i < \lambda\}$ and $\mathrm{tp}(a_i/A \cup \{a_j : j < i\})$ is isolated, for each $i < \lambda$. If $M \in \mathcal{K}$ is primary over A, then it is *prime* over A in the class $\mathcal{K}$, *i.e.* each elementary map $f : A \to N$ extends to an elementary map from M into N. We can now state the improvement:

(3) **Existence of primary models:** Let $\mathcal{K}$ be excellent. Then the model N in (2) can be chosen primary over Ma, where a is any realisation of p.

Remark 1.4. Any uncountably categorical class of models, axiomatised by a complete sentence in $L_{\omega_1,\omega}$, can be axiomatised as the class of atomic models of a countable first order theory by expanding the language if necessary (see [Sh48]). However, in applications, we may not want to expand the language. Also, we may have a direct axiomatisation in terms of classes of models omitting a prescribed set of types, but realising some nonisolated types (Zilber's example of pseudo-analytic structure we present is initially of this nature). We describe here how to deal directly with this more general case. Let M be a model realising, over the empty set, only types inside a prescribed set D. The key point is that when D is the set of isolated types, then M is $\aleph_0$-homogeneous and realises all types inside D. Following Shelah, we say that a model is $(D, \aleph_0)$-homogeneous, if it is $\aleph_0$-homogeneous and realises exactly the types in D. We must replace $\mathrm{S}_{\mathrm{at}}(M)$ by $S_D(M)$, which is the collection of complete types $p \in S(M)$, such that for any $c \models p$, the set $M \cup c$ realises only types in D. We can define *excellence* for any class of models realising only types in D, where D is countable and all uncountable models are $(D, \aleph_0)$-homogeneous (this holds in Zilber's example below). In this case, there is also a countable $(D, \aleph_0)$-homogeneous model, and this model is unique up to isomorphism. Simply change the assumptions (1) and (2) with '$(D, \aleph_0)$-homogeneous' model, instead of 'model'. For (3), we use the notion of $(D, \aleph_0)$-primary model; M is $(D, \aleph_0)$-*primary over* A if it is $(D, \aleph_0)$-homogeneous and $M = A \cup \{a_i : i < \lambda\}$, and $\mathrm{tp}(a_i/A \cup \{a_j : j < i\})$ is implied by its restriction to finitely many parameters (so the notion of isolation is with respect to the number of parameters, rather than the number of formulas). A $(D, \aleph_0)$-primary model over A is prime in the class of $(D, \aleph_0)$-homogeneous models. We could, of course, consider other variations of what can be understood as excellence; the advantage of the one we just presented is that it follows from the existence of uncountably homogeneous models (see subsection below).

The simplest example of excellent atomic class is the class of models of an ω-categorical, ω-stable, countable, first order theory: All the models are atomic, amalgamation over models and type realisability are obvious, and the existence of primary models follows from ω-stability. We finish this section with another example of excellent classes which is not necessarily first order.

1.1. **Homogeneous model theory.** A natural hypothesis to assume on the class $\mathcal{K}$ is that it has arbitrarily large homogeneous models (obviously saturated models will not be in the class in general). This was done independently by Keisler [Ke] and Shelah [Sh3], and they both proved the categoricity theorem under this assumption (in two different but equivalent contexts). A Baldwin-Lachlan style proof of categoricity was given in [Le1]. (See also [Le2] for a simpler proof without using a rank.) A similar geometric proof (with a different statement) was also found independently by Hyttinen in [Hy].

Studying classes of models omitting a prescribed set of types under the assumption that there are large homogeneous models is now known as *homogeneous model theory*, in contrast to saturated (*i.e.* first order) model theory.

If $\mathfrak{C}$ is a (large) homogeneous model in $\mathcal{K}$, any atomic model, or indeed any atomic set, embeds elementarily in $\mathfrak{C}$, provided it has size at most $\|\mathfrak{C}\|$. Moreover, any complete type over an atomic set (of size less than $\|\mathfrak{C}\|$) realised in a model $M \in \mathcal{K}$ is realised in $\mathfrak{C}$. In fact, for a complete type p over an atomic set $A \subseteq \mathfrak{C}$

with $|A| < \|\mathfrak{C}\|$, we have the following *weak compactness* principle: p is realised in $\mathfrak{C}$ if and only if $p \upharpoonright C$ is realised in $\mathfrak{C}$ for each finite $C \subseteq A$.

Homogeneous model theory is very general; it includes the first order case, Robinson theories, e.c. models, Banach space model theory, classes of models with amalgamation over sets (infinitary, L^n, etc.), many generic constructions, and of course some concrete cases like Hilbert spaces, and free groups (see below). Homogeneous model theory is very well-behaved; weak compactness has a number of nice consequences, for example infinite indiscernible sequences can be extended.

The existence of arbitrarily large homogeneous models implies that the class is excellent: it is almost immediate for the amalgamation property and for type realisability. The existence of prime models depends on a form of ω-stability, which follows from uncountable categoricity in this case (see [Ke] or [Sh3]).

Fact 1.5 (Keisler, Shelah). *If $\mathcal{K}$ has arbitrarily large homogeneous models and is categorical in some uncountable cardinal, then over each countable atomic set A there are only countably many complete types realised by models of $\mathcal{K}$.*

Fact 1.6 (Shelah). *If $\mathcal{K}$ satisfies the conclusion of the previous fact, then there exists a prime (primary) model over each atomic set A.*

Hence, existence of primary models holds and the class is excellent. Uncountably categoricity for a homogeneous (not necessarily atomic) class implies that all the uncountable models are $(D, \aleph_0)$-homogeneous, and the existence of $(D, \aleph_0)$-primary models over any set (realising only types in D) follows also (see [Sh3] or [Le1]). Hence, the conditions we outlined in Remark 1.4 for the nonatomic case hold.

At present, homogeneous model theory has developed beyond categoricity, with good notions of ω-stability/total transcendence [Le1], superstability [HySh1], [HySh2], [HyLe], stability [Sh3], [Sh54], [GrLe], and even simplicity [BuLe]. Excellence, so far, lives in the realm of ω-stability.

Notice that the conclusion of Fact 1.5 is stronger than the conclusion we have in Proposition 2.1; both of which are natural notions of ω-stability. Keisler had asked whether categoricity implies the existence of arbitrarily large homogeneous models; Shelah answered negatively by giving a counterexample and developed excellence. It turns out that the difference between excellence and homogeneity in this context lies entirely in the strength of ω-stability [Le2]:

Fact 1.7 (Lessmann). *If $\mathcal{K}$ has an uncountable model and over each countable atomic A there are only countably many complete types realised by models of $\mathcal{K}$, then $\mathcal{K}$ has arbitrarily large homogeneous models.*

Thus, in the excellent, nonhomogeneous, uncountably categorical case, there may be countable atomic sets over which uncountably many types are realised. It follows that there cannot be any prime model over such sets, since only countably many types are realised over countable atomic models. This is one of the major difficulty of excellence, and one of the chief differences with the homogeneous case: there exists prime models only over certain sets. In this paper, we will only use the existence of prime models over sets of the form $M \cup a$, where $M \in \mathcal{K}$ and a realises a type $p \in \mathrm{S}_{\mathrm{at}}(M)$. Another difference with homogeneous model theory is that infinite indiscernible sequence cannot, in general, be extended.

Fact 1.7 implies also that if the class of atomic models of a first order theory T is excellent but not homogeneous, then T cannot be ω-stable. Zilber's example below has superstable first order theory.

2. Categoricity

We start this section with a few consequences of uncountable categoricity, which can be found in [Sh48], [Sh87a], and [Sh87b]. These properties follow from (1)–(3) only.

Fact 2.1 (Shelah). *Let $\mathcal{K}$ be excellent and categorical in some uncountable cardinal. Then $|\mathrm{S}_{\mathrm{at}}(M)| \leq \aleph_0$ for each countable $M \in \mathcal{K}$.*

We now capture the conclusion of the previous fact in a definition. We noted in the previous subsection other possible notions of ω-stability; in this paper, we will use:

Definition 2.2. $\mathcal{K}$ is *ω-stable* if $|\mathrm{S}_{\mathrm{at}}(M)| \leq \aleph_0$, for each countable $M \in \mathcal{K}$.

The next example shows that T may be unstable, even if $\mathcal{K}$ is ω-stable.

Examples 2.3. Consider the language containing a predicate N, binary function $+$, and constants 0 and 1. Let M be a model, where M^N is interpreted as the natural numbers $\mathbb{N}$ in the language $\{0, 1, +, \cdot\}$, and the complement of M^N is infinite. Then, M is an atomic model of its first order theory T. Moreover, any atomic model M' of T interprets N as $\mathbb{N}$, and has the complement of N the size of M'. This shows that the class of atomic models of T is categorical in all infinite cardinals. It is easy to see that the class is excellent (it is homogeneous). However, the first order theory T has the strict order property, and is thus unstable; yet the class of atomic models is ω-stable in the sense of the previous definition.

Recall that a complete type p over A *splits* over $B \subseteq A$, if there are $b, c \in A$ realising the same type over B and a formula $\phi(x, y)$ such that $\phi(x, b) \in p$ and $\neg\phi(x, c) \in p$. Nonsplitting is a dependence relation with amenable properties in the first order ω-stable case. It turns out to be quite robust in our case too:

Fact 2.4 (Shelah). *Assume that $\mathcal{K}$ is excellent and ω-stable. Let $p \in \mathrm{S}_{\mathrm{at}}(M)$, for $M \in \mathcal{K}$. Then there is a finite $C \subseteq M$ such that p does not split over C. Furthermore, if $M \prec N \in \mathcal{K}$, then there is a unique $q \in \mathrm{S}_{\mathrm{at}}(N)$ extending p which does not split over C.*

We now introduce a convenient notation.

Notation 2.5. We write $A \underset{B}{\downarrow} C$ for the property $\mathrm{tp}(a/B \cup C)$ does not split over B, for each finite sequence $a \in A$.

We will only use this notation when $B = M \in \mathcal{K}$. In addition to the previous fact, part of which is rephrased in (1), we have

Fact 2.6 (Shelah)**.** *Assume that $\mathcal{K}$ is excellent and ω-stable.*

(1) *(Extension) Let $\mathrm{tp}(a/M) \in \mathrm{S}_{\mathrm{at}}(M)$ and $M \prec N$. Then there is $b \models \mathrm{tp}(a/M)$ such that $b \underset{M}{\downarrow} N$.*
(2) *(Symmetry) Let $\mathrm{tp}(a/M), \mathrm{tp}(b/M) \in \mathrm{S}_{\mathrm{at}}(M)$. If $a \underset{M}{\downarrow} b$, then $b \underset{M}{\downarrow} a$.*
(3) *(Transitivity) Let $M_1 \subseteq M_2 \subseteq M_3$. Assume that $\mathrm{tp}(a/M_3) \in \mathrm{S}_{\mathrm{at}}(M_3)$. If $a \underset{M_1}{\downarrow} M_2$ and $a \underset{M_2}{\downarrow} M_3$, then $a \underset{M_1}{\downarrow} M_3$.*

For more on nonsplitting in excellent classes, see [Ko]. The next fact is essentially book-keeping when λ is regular; it uses nonsplitting in a nontrivial way for λ singular.

Fact 2.7 (Shelah)**.** *Assume that $\mathcal{K}$ is excellent and ω-stable. Then, for each uncountable cardinal λ there exists a λ-full model of size λ.*

Shelah's proof of categoricity in the excellent case follows Morley's argument using fullness instead of saturation: Any two uncountable full models of the same size are isomorphic; the unique model in the categoricity cardinal is full; if there is an uncountable model which is not full, then we can transfer the failure of fullness to construct a model of any uncountable cardinality which is not full.

We now consider the counterpart of nonalgebraic types.

Definition 2.8. We say that a type p over a subset of a model $M \in \mathcal{K}$ is *big* if for any $M' \in \mathcal{K}$ containing the parameters of p, there is $N \in \mathcal{K}$ containing M' such that p is realised in $N \setminus M'$.

Proposition 2.9. *Assume that $\mathcal{K}$ is excellent and ω-stable Let p be a type over $A \subseteq M \in \mathcal{K}$. The following conditions are equivalent:*

(1) *p is big;*
(2) *For some $M' \in \mathcal{K}$ containing A, there exists $N \in \mathcal{K}$ extending M' such that p is realised in $N \setminus M'$.*
(3) *There is a model $N \in \mathcal{K}$ containing M where there are more than $|A| + \aleph_0$ realisations of p.*

Proof. Clearly (1) implies (2) and (3). Also, (3) implies (2), as we can choose $M' \prec N$ containing A of size $|A| + \aleph_0$, which implies that some realisation of p is not in M'. Hence, it is enough to show that (2) implies (1): Assume that p is realised by some $c \in N \setminus M'$. Let $q = \mathrm{tp}(c/M') \in \mathrm{S}_{\mathrm{at}}(M')$. Then q is an extension of p. Let M_1 contain A. By amalgamating over the countable model if necessary, we may assume that M_1 contains M'. There is a finite subset C of M' such that q does not split over C. Let q' be the unique nonsplitting extension of q in $\mathrm{S}_{\mathrm{at}}(M_1)$. Let $N \in \mathcal{K}$ be an extension of M_1 and $c' \in N$ be a realisation of q'. Note that c' cannot be in M_1 by nonsplitting (otherwise $\{x = c'\}$ and $\{x \neq c\}$ are both in q', but c and c' have the same type over M', hence C). So p is realised outside M_1, and since M_1 is arbitrary, p is big. □

Notice in particular that a type over finitely many parameters is big if and only if it is realised uncountably many times in some model. It is not possible to change 'uncountably many' to 'infinitely many' as Example 2.3 demonstrates. Also, if p is big, then there exists a big extension $p' \in \mathrm{S}_{\mathrm{at}}(M)$, where $M \in \mathcal{K}$ contains the parameters of p. This naturally leads to the following definition, which is an extension of strong minimality.

Definition 2.10. A type q over a set $A \subseteq M$ is *quasiminimal* if it is big and has a unique big extension over each model $N \in \mathcal{K}$ containing A.

Observe that if a big type q over A fails to be quasiminimal, then there is a model N containing A and two contradictory big types extending q in $\mathrm{S}_{\mathrm{at}}(N)$. This implies that there is a formula $\phi(x, c)$, with $c \in N$ such that $q \cup \{\phi(x, c)\}$ and $q \cup \{\neg\phi(x, c)\}$ are both big. This makes the connection with strong minimality more explicit. Moreover, if $q(x, a)$ is big and $b \models \mathrm{tp}(a/\emptyset)$ then $q(x, b)$ is big, and similarly if $q(x, a)$ is quasiminimal then so is $q(x, b)$.

We now show that quasiminimal types exist and induce a pregeometry.

Proposition 2.11. *Let $\mathcal{K}$ be ω-stable. Let $M \in \mathcal{K}$. Then there exists a quasiminimal $q(x, c)$, with $c \in M$.*

Proof. Let $M \in \mathcal{K}$. If there exists a quasiminimal type over the unique prime model over the empty set, then there exists a quasiminimal type over M. Hence, we may assume that M is countable. Let $M = \{a_i : i < \omega\}$.

Assume, for a contradiction, that there are no quasiminimal type $q(x, c)$ over a finite subset c of M. We construct a family of big, complete types q_η over finitely many parameters a_η, for $\eta \in {}^{<\omega}2$, such that:

1. $q_\eta \subseteq q_\eta$,
2. q_η is big but not quasiminimal,
3. $a_i \in a_\eta$ if $i = \ell(\eta)$
4. There is ϕ_η such that $\phi_\eta \in q_{\eta\hat{}0}$ and $\neg\phi_\eta \in q_{\eta\hat{}1}$.

This is possible: For $\eta = \langle\rangle$, choose any big $q_{\langle\rangle}$ complete type over a_0 (any type realised outside M is big). Then $q_{\langle\rangle}$ is not quasiminimal, by assumption.

Now assume that q_η has been constructed. By assumption, q_η is big and is not quasiminimal. Hence, there is $\phi(x, b)$, with $b \in M'$, $M \prec M' \in \mathcal{K}$ such that both $q_\eta \cup \phi(x, b)$ and $q_\eta \cup \neg\phi(x, b)$ are big. Let $b_\eta \in M$ realise $\mathrm{tp}(b/a_\eta)$. Then, $q_\eta \cup \phi(x, b_\eta)$ and $q_\eta \cup \neg\phi(x, b_\eta)$ are big (since bigness depends only on the type of the parameters). Let $a_{\eta\hat{}\ell} = a_\eta \cup b_\eta \cup a_i$, where $i = \ell(\eta) + 1$. Choose complete big types $q_{\eta\hat{}0}$ and $q_{\eta\hat{}1}$ over $a_{\eta\hat{}0}$ extending $q_\eta \cup \phi(x, b_\eta)$ and $q_\eta \cup \neg\phi(x, b_\eta)$ respectively. Then, by assumption $q_{\eta\hat{}\ell}$ is not quasiminimal for $\ell = 0, 1$.

This is enough: For $\eta \in {}^{\omega}2$, we let $p_\eta = \bigcup_{n<\omega} q_{\eta\restriction n}$. Then, each $p_\eta \in \mathrm{S}_{\mathrm{at}}(M)$, and by (4) we have $|\mathrm{S}_{\mathrm{at}}(M)| = 2^{\aleph_0}$, which contradicts ω-stability. □

If $q(x, c)$ with $c \in M$ is quasiminimal, and $A \subseteq M$, there is a unique complete type over A extending q which is quasiminimal. This fact is the key to the next proposition.

Proposition 2.12. *Let $\mathcal{K}$ be excellent and ω-stable. Let $M \in \mathcal{K}$. Let $q(x)$ be a quasiminimal type over $C \subseteq M$. Let $N \in \mathcal{K}$ be a full model containing M such that $\|M\| < \|N\|$. For $a, A \subseteq q(N)$ and define $a \in \mathrm{cl}(A)$, if $\mathrm{tp}(a/AC)$ is not big. Then $(q(N), \mathrm{cl})$ satisfies the axioms of a pregeometry.*

Proof. Certainly if $a \in A$, then $a \in \mathrm{cl}(A)$. Also, if $a \in \mathrm{cl}(A)$, then $\mathrm{tp}(a/AC)$ is not big and thus differ from the only big type over AC extending q; there is a finite witness for this, so there is a finite $B \subseteq A$ such that $a \in \mathrm{cl}(B)$.

Now suppose that $a \in \mathrm{cl}(A)$ and $A \subseteq \mathrm{cl}(B)$. Without loss of generality, we may assume that A and B are finite. Let $M' \prec N$ containing BC. We must show that $a \in M'$. But since $\mathrm{tp}(A/BC)$ is not big, we have that $A \subseteq M'$. Hence, since $\mathrm{tp}(a/AC)$ is not big, it cannot be realised outside of M', so $a \in M'$.

Finally, let us assume that $a \in \mathrm{cl}(Bb) \setminus \mathrm{cl}(B)$. We must show that $b \in \mathrm{cl}(Ba)$. Without loss of generality, we may assume that B is finite. Write $p(x, y) = \mathrm{tp}(a, b/BC)$. By assumption $p(x, b)$ is not big. Assume, for a contradiction, that $p(a, y)$ is big. Let $\{a_i : i < \lambda\} \subseteq N$ be distinct realisations of $\mathrm{tp}(a/BC)$ (which is big since $a \notin \mathrm{cl}(B)$), where $\lambda > \|M\| + |B| + \aleph_0$. Let $M' \prec N$ contain $M \cup B \cup \{a_i : i < \lambda\}$ and let b' realise the unique big extension of q over M'. Since $\mathrm{tp}(a/MB) = \mathrm{tp}(a_i/MB)$, each type $p(a_i, y)$ is big extending q, and so by uniqueness we have that b' realises $p(a_i, y)$ for each $i < \lambda$. This shows that $p(x, b')$ is big, which is a contradiction, since $\mathrm{tp}(b/BC) = \mathrm{tp}(b'/BC)$ (by uniqueness of big extensions). □

We now show the existence of prime models over bases; an alternative proof using Shelah's original definition of excellence is given in Remark 2.23. For this, we remind the reader of a few concepts and facts belonging to $\mathop{\smile\hspace{-0.9em}\vert}$-calculus; we will use these notions again in the next section.

Definition 2.13 (Grossberg-Hart)**.** We say that a *dominates* C over a model M, if whenever $a \underset{M}{\mathop{\smile\hspace{-0.9em}\vert}} D$, then $C \underset{M}{\mathop{\smile\hspace{-0.9em}\vert}} D$.

Fact 2.14 (Grossberg-Hart)**.** *Let $M \in \mathcal{K}$ and M' be a primary model over $M \cup a$. Then a dominates M' over M.*

Proposition 2.15. *Let $\mathcal{K}$ be excellent and ω-stable. Let $M \in \mathcal{K}$ be full and $q(x) \in \mathrm{S}_{\mathrm{at}}(M_0)$ be quasiminimal, with $M_0 \prec M$. If $I \subseteq q(M)$ is independent over M_0, then there is a primary model over $I \cup M_0$.*

Proof. Let $I = \{a_i : i < \lambda\}$. Construct an increasing and continuous chain of models $(M_i : i < \lambda)$, $M_i \prec M$, M_{i+1} is primary over $M_i \cup a_i$, and

$$M_{i+1} \underset{M_i}{\mathop{\smile\hspace{-0.9em}\vert}} I \setminus \{a_j : j < i\}, \quad \text{for } i < \lambda.$$

This is possible by excellence; the independence follows from the previous fact. Let $M' = \bigcup_{i<\lambda} M_i$. Then, the independence requirement ensures that pasting together the constructions of all the M_{i+1} over $M_i \cup a_i$ gives a construction of M' over $M_0 \cup \{a_i : i < \lambda\}$. It follows that M' is the desired primary model. $\square$

We recall some more generalisations of Shelah's orthogonality calculus in this context.

Fact 2.16 (Grossberg-Hart)**.** *Let $\mathcal{K}$ be excellent and ω-stable. If $\mathrm{tp}(a/Mb)$ is isolated and $a \underset{M}{\mathop{\smile\hspace{-0.9em}\vert}} b$, then $a \in M$.*

Definition 2.17 (Grossberg-Hart)**.** Let $p, q \in \mathrm{S}_{\mathrm{at}}(M)$, where $M \in \mathcal{K}$. We say that p is *perpendicular* to q, written $p \perp q$, if for all $M' \in \mathcal{K}$, with $M \prec M'$, and $a \models p$, $b \models q$ with $a \underset{M}{\mathop{\smile\hspace{-0.9em}\vert}} M'$ and $b \underset{M}{\mathop{\smile\hspace{-0.9em}\vert}} M'$, then $a \underset{M'}{\mathop{\smile\hspace{-0.9em}\vert}} b$.

Fact 2.18 (Grossberg-Hart)**.** *Let $\mathcal{K}$ be excellent and ω-stable. Let $p, q \in \mathrm{S}_{\mathrm{at}}(M)$.*

(1) *Then $p \perp q$ if and only if $a \underset{M}{\mathop{\smile\hspace{-0.9em}\vert}} b$, for all $a \models p$ and $b \models q$.*
(2) *Let $M' \in \mathcal{K}$ such that $M \prec M'$. Let $p', q' \in \mathrm{S}_{\mathrm{at}}(M')$ be nonsplitting extensions of p and q respectively. Then $p \perp q$ if and only if $p' \perp q'$.*

Definition 2.19. Let $\mathcal{K}$ be excellent and ω-stable. $\mathcal{K}$ is *unidimensional* if whenever $M \prec N$, and $q(x, c)$ is a quasiminimal type with $c \in M$ and $q(M) = q(N)$, then $M = N$.

Unidimensionality has the following consequence, which we will use in the next section.

Lemma 2.20. *Let $\mathcal{K}$ be excellent, ω-stable, and unidimensional. Let $p, q \in \mathrm{S}_{\mathrm{at}}(M)$, where p is big and q is quasiminimal. Then $p \not\perp q$.*

Proof. Let $a \models p$ such that $a \notin M$. Let M' be primary over $M \cup a$. Then $M' \neq M$, so by unidimensionality, there is $b \models q$ with $b \in M' \setminus M$. Then $\mathrm{tp}(b/M \cup a)$ is isolated and $b \notin M$, so $b \underset{M}{\not\downarrow} a$ by Fact 2.16. This shows that $p \not\perp q$. □

Proposition 2.21. *Let $\mathcal{K}$ be excellent and categorical in some uncountable cardinal. Then $\mathcal{K}$ is unidimensional.*

Proof. Suppose, for a contradiction, that $q(M) = q(N)$, but $M \neq N$. We may assume that $q \in \mathrm{S}_{\mathrm{at}}(M_0)$ for some countable $M_0 \prec M$.

We first show that we may assume M, N are countable. Let $b \in N \setminus M$. Construct increasing sequences $(M_i : i < \omega)$ and $(N_i : i < \omega)$ of countable models such that:

(1) $b \in N_0$;
(2) $M_i \prec M$ and $N_i \prec N$;
(3) $M_i \prec N_i$;
(4) $q(N_i) \subseteq q(M_{i+1})$.

This is easy to do and we have $q(\bigcup_{i<\omega} M_i) = q(\bigcup_{i<\omega} N_i)$, yet $\bigcup_{i<\omega} M_i \neq \bigcup_{i<\omega} N_i$.

So, we may assume that M, N are countable, and again let $b \in N \setminus M$. Let $p = \mathrm{tp}(b/M_0)$. Then p is big and stationary. Furthermore, for all M' containing M_0 and $a' \models q$ such that $a' \underset{M_0}{\downarrow} M'$ and $b' \models p$ such that $b' \underset{M_0}{\downarrow} M'$, then $a' \underset{M'}{\downarrow} b'$.

Construct $(N_i : i < \mu)$ an increasing and continuous sequence of models such that:

(1) $N_0 = N$.
(2) $N_i \neq N_{i+1}$
(3) $q(N_i) = q(N_0)$.
(4) N_{i+1} is primary over $N_i b_i$, where $b_i \notin N_i$ realises the unique free extension of p in $\mathrm{S}_{\mathrm{at}}(N_i)$.

Let us see that this is possible. For $i = 0$ and i a limit, there is no problem. At the successor stage, notice that there exists $b_i \notin N_i$ realising the unique free extension of p in $\mathrm{S}_{\mathrm{at}}(N_i)$, since p is big. By excellence, there is N_{i+1} primary over $N_i b_i$. Now $q(N_{i+1}) = q(N_i)$. Otherwise, there is $c \in N_{i+1} \setminus N_i$ realising q. Hence $\mathrm{tp}(c/N_i b_i)$ is isolated (since N_{i+1} is primary over $N_i b_i$). Since $c \notin N_i$, we must have $c \underset{N_i}{\not\downarrow} b_i$ by the previous fact. But this contradicts our first claim about p and q.

This is enough: The models $\bigcup_{i<\lambda} N_i$ has size λ and is not full, as it omits any stationary extension of $q \in \mathrm{S}_{\mathrm{at}}(M)$. Since there is a full model of size λ, $\mathcal{K}$ cannot be categorical in λ, and so $\mathcal{K}$ is not uncountably categorical since λ was arbitrary. $\square$

We can now prove the main theorem of this section. We say that a set is *quasiminimal* if it is the set of realisations of a quasiminimal type.

Theorem 2.22. *Let $\mathcal{K}$ be excellent and categorical in some uncountable cardinal. Then each model is prime and minimal over the basis of a pregeometry given by a quasiminimal set (and its parameters). Moreover, the size of the basis determines the isomorphism-type of the model and $\mathcal{K}$ is categorical in all uncountable cardinals.*

Proof. Let $M \in \mathcal{K}$ be uncountable. Let $M_0 \prec M$ be the prime model over the empty set. Let M_1 be a full, uncountable model extending M (by Fact 2.7). By Proposition 2.11, we can find a quasiminimal $q(x) \in \mathrm{S}_{\mathrm{at}}(M_0)$. Then $q(M_1)$ is a pregeometry by Proposition 2.12.

Notice that $q(M) \subseteq q(M_1)$ is closed in the sense of the pregeometry $q(M_1)$: Let $a \in q(M_1)$ be such that $a \in \mathrm{cl}(q(M))$. Then $\mathrm{tp}(a/q(M) \cup M_0)$ is not big by definition, so cannot be realised outside M by Proposition 2.9, hence $a \in M$, so $a \in q(M)$.

Since $q(M)$ is closed, we can choose $I = (a_i : i < \lambda)$ a basis for it.

By Proposition 2.15 we can find a primary model $M' \prec M$ over $I \cup M_0$. But $q(M')$ is closed and $I \subseteq q(M') \subseteq q(M)$, so $q(M') = q(M)$. Hence $M = M'$ by unidimensionality (Proposition 2.21). This shows that M is primary over $I \cup M_0$. Another application of unidimensionality shows that M is minimal over $I \cup M_0$. Notice also that $|I| = \|M\|$, since $\|M'\| = |I| + \aleph_0$.

For M countable, notice that by Proposition 2.15 there is a prime model M' over $M_0 \cup I$, where I is any countable independent set in $q(M_1)$, and $q(M')$ is closed and has countable dimension. By $\aleph_0$-categoricity, M is isomorphic to M', so M is prime over the infinite basis of a quasiminimal set and its parameters. Minimality follows also from unidimensionality (Proposition 2.21).

We can now show categoricity in all uncountable cardinals: Let $M, N \in \mathcal{K}$ be models of size $\mu > \aleph_0$. Let M_0 be the prime model over the empty set. Without loss of generality $M_0 \prec M$ and $M_0 \prec N$. Let $q(x) \in \mathrm{S}_{\mathrm{at}}(M_0)$ be quasiminimal. Let I be a basis for $q(M)$ and J a basis for $q(N)$. Then $|I| = |J|$ and any bijection f from I to J extending the identity on M_0 is elementary. Since M is prime over $I \cup M_0$, there an elementary map $g : M \to N$ extending f. Then $g(M) \prec N$ and contains $J \cup b$. This implies that $g(M) = N$ since N is minimal over $J \cup M_0$. Hence, g is an isomorphism between M and N. $\square$

Remark 2.23. One of the key issues is the existence of prime (primary) models over various sets. Excellence is a condition that ensures the existence of prime models over certain countable atomic sets.

Consider n-dimensional directed systems of countable models of $\mathcal{K}$:

$$(M_s : s \subsetneq n), \quad \text{for } n < \omega;$$

i.e. each $M_s \in \mathcal{K}$ is countable and $s \subseteq t$ implies $M_s \prec M_t$. We say that an n-dimensional system is *independent* if, in addition,

$$M_s \underset{M_{s\cap t}}{\perp} M_t, \qquad \text{for each } s, t \subsetneq n.$$

Shelah defines $\mathcal{K}$ to be *excellent* if, for any $n < \omega$, and for any n-dimensional independent directed system of countable models $(M_s : s \subsetneq n)$, there exists a primary model over $\bigcup_{s\subsetneq n} M_s$.

Let $M \in \mathcal{K}$ be uncountable. Let us see how excellence can be used directly to show the existence of a prime model over $M_0 \cup I$, where $M_0 \prec M$ is countable, and I is a basis for $q(M)$, where $q \in \mathrm{S}_{\mathrm{at}}(M_0)$ is quasiminimal. We construct a directed system of models

$$(M_s : s \subseteq I, |s| < \aleph_0),$$

with $M_s \prec M$ as follows:

- For $s = \emptyset$, we let $M_\emptyset = M_0$.
- For $s = \{a\}$ a singleton, we let $M_{\{a\}} \prec M$ be the prime model over $M_0 \cup a$.
- For $s = \{a, b\}$, then $a \underset{M_0}{\perp} b$, and hence $M_{\{a\}} \underset{M_0}{\perp} M_{\{b\}}$ (see Fact 2.14). Thus,
$$(M_0, M_{\{a\}}, M_{\{b\}})$$
forms a 2-dimensional independent system and by excellence, there exists $M_{\{a,b\}} \prec M$ prime over $M_{\{a\}} \cup M_{\{b\}}$.
- For $s = \{a, b, c\}$, one notices similarly that
$$(M_t : t \subsetneq \{a, b, c\})$$
is a 3-dimensional independent system, so by excellence there is a prime model $M_{\{a,b,c\}} \prec M$ over $M_{\{a,b\}} \cup M_{\{b,c\}} \cup M_{\{a,c\}}$.
- Continue in this way inductively.

It is not difficult to check that

$$M' := \bigcup_{s\subseteq I, |s|<\aleph_0} M_s$$

is an elementary submodel of M which is prime over $M_0 \cup I$. The proof of categoricity continues as before.

This method allows us to construct arbitrarily large models (from the assumption that there exists an uncountable model). An inductive process is used to show the existence of primary models over any n-dimensional independent system of models of $\mathcal{K}$ (that is, not necessarily countable), by decomposing each n-dimensional independent system into an $(n+1)$-dimensional system of models of smaller size (see [Sh87a] and [Sh87b] for details).

Hence, excellence implies the amalgamation property. To see that property (3) holds (which implies (2)), express $M \cup a$, where $a \models p \in \mathrm{S}_{\mathrm{at}}(M)$, as $\bigcup_{i<\|M\|} M_i \cup a$, where $M \underset{M_0}{\perp} a$. Choose M_0' primary over $M_0 \cup a$. Then, inductively, choose M_{i+1}' prime over $M_{i+1} \cup M_i'$, using excellence by noticing that (M_{i+1}, M_i, M_i') forms an independent 2-dimensional system (at limits, take unions).

Before we consider the examples, we state a final result. It is proved easily by constructing, for each $\beta \leq \alpha$, an $\aleph_\beta$-full model $M \in \mathcal{K}$ of size $\aleph_\alpha$ with a quasiminimal set of dimension exactly $\aleph_\beta$. We leave this to the reader. It is well-known from the first order case that the lower-bound cannot be improved.

Theorem 2.24. *Let $\mathcal{K}$ be excellent, ω-stable, and not uncountably categorical. Then in each cardinal $\aleph_\alpha$, there are at least $|\alpha+1|$ nonisomorphic models.*

We now describe two examples which are not first order. The first one fits within homogeneous model theory, and the second is excellent but not homogeneous.

2.1. **Free groups.** We consider the class of free groups $F(X)$, where X is an infinite set of generators. The language is the language of groups with an extra predicate for the set of generators X. This class is not first order axiomatisable, but it can be axiomatised easily by omitting a type – the first order axioms state that $F(X)$ is a group, that X is infinite, and if two products of elements of X (and their inverse) are equal, then the constituents of the product, their order, and number are equal; finally, omit the type of an element which is not a product of elements (and inverses of elements) of X.

It is a basic fact of algebra that two free groups with the same number of generators are isomorphic. One deduces easily that all free groups on *infinitely* many generators are elementary equivalent, so that they are all models of the same complete, countable, first order theory. It turns out that the free groups correspond to the atomic models of this theory. Hence, infinitely generated free groups form an atomic class of models, which is categorical in all infinite cardinals.

It was noticed by Keisler already that any infinitely generated group in this language is homogeneous, so this class belongs to homogeneous model theory. As we noted, it implies that this class is excellent.

The quasiminimal set predicted by the Theorem 2.22 is X; it is actually strongly minimal and carries a trivial pregeometry. If $F(X)$ is the free group generated by X, then $F(X)$ is clearly prime and minimal over X; it is the definable closure of X.

One can see directly that there are no generic elements in free infinitely generated free groups. In fact, although the class of free groups is categorical in all infinite cardinals, free groups only have countable abelian subgroups. This is in sharp contrast to the theorem of Baur-Cherlin-Macintyre [BCM] asserting that any infinite group, whose first order theory is categorical in all infinite cardinals, has a definable abelian subgroup of finite index.

2.2. **Zilber's pseudo-analytic structures.** This is a summary of [Zi1] in the language of omitting types, rather than $L_{\omega_1,\omega}$. Consider

$$0 \to \mathbb{Z} \to^i H \to^{ex} F^* \to 1,$$

where F^* is the multiplicative group of an algebraically closed field of characteristic 0, H is a torsion-free, divisible, abelian group, and the sequence is exact.

The canonical example, the simplest among Zilber's pseudo-analytic structures [Zi2], is when $F = \mathbb{C}$, $H = \mathbb{C}^*$, and ex is the exponentiation map

$$exp : \mathbb{C}_+ \to \mathbb{C}^*.$$

Zilber represents this as a one-sorted structure H, whose universe is the universe of the torsion-free divisible abelian group H, in the language of abelian groups $+$. He adds two basic relations: a basic equivalence relation E, whose interpretation is

$$E(h_1, h_2) \quad \text{if and only if} \quad ex(h_1) = ex(h_2),$$

and a ternary relation S with interpretation

$$S(h_1, h_2, h_3) \quad \text{if and only if} \quad ex(h_1) + ex(h_2) = ex(h_3).$$

So the nonzero elements of the field F are the equivalence classes h/E, for $h \in H$. Multiplication in F can then be defined via E and $+$ and addition in F is defined via S. The kernel of ex, which is simply the class corresponding to the unit 1 of the field, is then definable.

This class can be axiomatised as follows:

- H is a torsion free, divisible, abelian group,
- $H/E \cup \{0\}$ is an algebraically closed field of characteristic 0

We also omit a type to express:

- For all x_1 and x_2 in the kernel of ex, there are $z_1, z_2 \in \mathbb{Z} \setminus \{0\}$ such that $z_1x_1 + z_2x_2 = 0$.

This class is categorical in all uncountable cardinals, *i.e.* the fields are isomorphic, the abelian groups are isomorphic, and these isomorphisms commute with ex). It follows (as above) that all the models have the same complete first order theory in this language. Zilber further proves that every uncountable model is $(D, \aleph_0)$-homogeneous, where D is the set of complete types over the empty set realised by the models in the class and that the class is excellent. However D is not the set of atomic types for a trivial reason: The type of an n-tuple which has transcendence degree n is realised by all models, yet is not atomic. We outlined in Remark 1.4 how this fits in our general framework. As we pointed out, a countable expansion where all the models are atomic can always be found. Note that, the trivial reason why the class is not atomic is essentially the only reason; Zilber in [Zi3] works in an expansion with extra predicates for linearly and algebraically independent tuples over $\mathbb{Q}$. The class is then atomic in this language. Both the atomicity in this language and excellence have field-theoretic significance (see [Zi1] and [Zi3]).

In this example, the universe H is quasiminimal and $a \in \mathrm{cl}(B)$ if

$$ex(a) \in acl(ex(B)),$$

where acl is the algebraic closure operator in the sense of the field structure (on $ex(H) \cup \{0\}$).

This example does not belong to homogeneous model theory. A variation, due to Zilber, where $\mathbb{Z}$ is replaced by its completion in the profinite topology, does belong to homogeneous, as well as other variations due to Hyttinen with an added random logarithm.

3. U-rank

In this section, we assume that $\mathcal{K}$ is excellent and ω-stable, though we may repeat this fact in some statements for emphasis.

We consider here complete types over models. Excellence gives us a good understanding of such types; they are realised by models in $\mathcal{K}$ if and only if they belong to $\mathrm{S_{at}}(M)$, for some $M \in \mathcal{K}$. Notice that each $p \in \mathrm{S_{at}}(M)$ is *stationary*, *i.e.* has a unique nonsplitting extension to any $N \in \mathcal{K}$ with $M \prec N$. For $p \in \mathrm{S_{at}}(M)$ and $q \in \mathrm{S_{at}}(N)$, we define

$$q < p,$$

if $M \prec N$, $p \subseteq q$, and q is a splitting extension of p.

Note that if $p_i = \mathrm{tp}(a_i/M_i) \in \mathrm{S_{at}}(M_i)$, for $i = 0, 1, 2$ and $p_1 \subseteq p_0$ and $p_1 < p_2$, then p_0 is a splitting extension of p_2, which gives the transitivity of $<$.

Since $\mathcal{K}$ is ω-stable, the order $<$ is well-founded: Suppose that $p_i \in \mathrm{S_{at}}(M_i)$ and $(p_i : i < \omega)$ forms an infinite strictly descending chain. Then $M = \bigcup_{i<\omega} M_i \in \mathcal{K}$ and $p = \bigcup_{i<\omega} p_i \in \mathrm{S_{at}}(M)$. Let $a \models p$. Then $\mathrm{tp}(a/M) \in \mathrm{S_{at}}(M)$

does not split over some finite set by Fact 2.4. Hence, there exists $i < \omega$ such that $\mathrm{tp}(a/M)$ does not split over M_i. Since $a \models p_{i+1}$, we have $p_{i+1} \not> p_i$, a contradiction. We can therefore define the foundational rank for $<$:

Let $p \in S(M)$.

- $U(p) \geq 0$ if and only if $p \in \mathrm{S_{at}}(M)$;
- $U(p) \geq \alpha + 1$, if there exists $N \in \mathcal{K}$, $M \prec N$ and $q \in \mathrm{S_{at}}(N)$ such that $q < p$ and $U(q) \geq \alpha$;
- $U(p) \geq \delta$, for δ a limit ordinal, if $U(p) \geq \alpha$, for all $\alpha < \delta$.

As usual, we let $U(p) = \alpha$ if $U(p) \geq \alpha$, but it is not the case that $U(p) \geq \alpha + 1$.

Thus, every $p \in \mathrm{S_{at}}(M)$ is given an ordinal by the U-rank (types not realised by models of the class can be thought of having rank -1). The U-rank is invariant under elementary maps. It will be convenient to use the following two pieces of notation.

Notation 3.1. We will write $U(a/M)$ for $U(\mathrm{tp}(a/M))$ and always assume that $\mathrm{tp}(a/M) \in \mathrm{S_{at}}(M)$.

Notation 3.2. We denote by $M(a)$ the primary model over Ma, where $\mathrm{tp}(a/M) \in \mathrm{S_{at}}(M)$.

We can now prove two easy lemmas.

Lemma 3.3. *Let $M \prec N$. Then $U(a/M) \geq U(a/N)$ with equality if and only if $a \underset{M}{\perp} N$.*

Proof. Everything is clear except, possibly, that $U(a/M) = U(a/N)$ if $a \underset{M}{\perp} N$. To see this, it is enough to show that $U(a/M) \geq \alpha$ implies that $U(a/N) \geq \alpha$ when $a \underset{M}{\perp} N$, by induction on α and for all $M \prec N \in \mathcal{K}$. For $\alpha = 0$ or a limit, there is nothing to prove. Suppose that $U(a/M) \geq \alpha + 1$. Let $M' \in \mathcal{K}$, with $M \prec M'$ be such that $U(a/M') \geq \alpha$ and $a \underset{M}{\not\perp} M'$. By using an elementary map which is the identity on $M(a)$, we may assume that $M' \underset{M(a)}{\perp} N$, which implies that $M' \underset{M}{\perp} N$ (since $M(a) \underset{M}{\perp} N$, as a dominates $M(a)$ over M and $a \underset{M}{\perp} N$). Let $N' \in \mathcal{K}$ be an extension of M' and N such that $a \underset{M'}{\perp} N'$ (this exists by excellence). By induction hypothesis, we have that $U(a/N') \geq \alpha$. But, $a \underset{N}{\not\perp} N'$ by transitivity, so we must have $U(a/N) \geq \alpha + 1$. □

Lemma 3.4. *Suppose that $U(a/M) = \alpha$ and $0 \leq \beta < \alpha$. Then there exists $N \in \mathcal{K}$, $M \prec N$ such that $U(a/N) = \beta$.*

Proof. We prove this by induction on $U(a/M) = \alpha$, for all $M \in \mathcal{K}$. It is trivial for $\alpha = 0$. Suppose that $U(a/M) = \alpha + 1$. By definition, there is $\mathrm{tp}(b/N) \in \mathrm{S_{at}}(N)$ such that $\mathrm{tp}(b/N)$ is a splitting extension of $\mathrm{tp}(a/M)$ and $U(b/N) = \alpha$. By using an elementary map if necessary, we may assume that $b = a$. Either $\beta = \alpha$ and we are done, or $\beta < \alpha$ and we are done also by induction applied to $\mathrm{tp}(a/N)$ using transitivity of $<$. Now suppose that $U(a/M) = \alpha$ is a limit ordinal and $\beta < \alpha$. By definition, there exists a splitting extension $\mathrm{tp}(a/N)$ of $\mathrm{tp}(a/M)$ such that $U(a/N) \geq \beta$. We must have $U(a/N) < \alpha$, so we can use the induction hypothesis on $U(a/N)$ to get the conclusion. □

We now observe the following easy fact, where (2) is simply a restatement of (1).

Remark 3.5. (1) $U(b/M) = 0$ if and only if $b \in M$.
(2) $U(b/M) \geq 1$ if and only if $\mathrm{tp}(b/M) \in \mathrm{S_{at}}(M)$ is big.
(3) $U(ab/M) \geq U(b/M)$ with equality if $b \in M$.

This now gives the desired correspondence between the U-rank and quasiminimal types.

Lemma 3.6. *Let $q \in \mathrm{S_{at}}(M)$. Then q is quasiminimal if and only if $U(q) = 1$. Moreover, if q is quasiminimal and $a \in q(N)$, where N is a full model extending M of size greater than $\|M\|$, then $U(a/M) = \dim(a)$, in the sense of the pregeometry.*

The previous lemma, together with Lemma 3.4 gives another proof that quasiminimal types over models exist. We now give an example to illustrate the difference between the U-rank and Shelah's rank [Sh48].

Remark 3.7. Consider a strongly minimal, $\aleph_0$-categorical, countable, first order theory T. The class of models of T is excellent. Complete types in one variable over a model either define a singleton (and have U-rank 0) or are strongly minimal (and have U-rank 1). Shelah introduced a rank for the ω-stable, atomic case, which is based on the 2-rank. If we compute Shelah's rank in this case, we find that singletons have rank 0, finite sets which are not singletons have rank 1, and one dimensional nonalgebraic sets have rank 2. Hence, complete types over models in one variable have Shelah's rank 0 when they define singletons, or 2 otherwise (there is no complete type over a model with Shelah rank 1).

Fact 2.14 quickly leads to a proof of the next lemma using $\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise0.2ex\hbox{$\mid$}}$-calculus. We provide the details for completeness.

Lemma 3.8. *Let $M \prec N$ and $M(b) \prec M(ab), N(b)$. Then $M(ab) \underset{M}{\smile} N$ if and only if $M(ab) \underset{M(b)}{\smile} N(b)$ and $M(b) \underset{M}{\smile} N$.*

Proof. We only prove the left to right direction, as the converse follows immediately from transitivity and symmetry.

Assume $M(ab) \underset{M}{\downarrow} N$. Then $ab \underset{M}{\downarrow} N$, so $b \underset{M}{\downarrow} N$ and therefore $M(b) \underset{M}{\downarrow} N$. Also, $M(ab) \underset{M(b)}{\downarrow} N$ by monotonicity, which implies that $M(ab) \underset{M(b)}{\downarrow} Nb$. To show that $M(ab) \underset{M(b)}{\downarrow} N(b)$ for the primary model $N(b)$, we use finite character: Suppose that $d \in N(b)$. Let $c \in N$ such that $\mathrm{tp}(d/Nb)$ is isolated by some formula over cb. By excellence, there is a primary model M' over $M(b) \cup c$, which we may assume contains d. Since $M(ab) \underset{M(b)}{\downarrow} c$, we have $M(ab) \underset{M(b)}{\downarrow} M'$ by dominance, and so $M(ab) \underset{M(b)}{\downarrow} d$. Thus $M(ab) \underset{M(b)}{\downarrow} N(b)$, since $d \in N(b)$ was arbitrary. □

We can now establish the usual additivity properties of the U-rank. The proofs are as in the first order case using the previous lemma, instead of the so-called Pairs Lemma for forking. Recall the meaning of the natural sum of two ordinals, written $\alpha \oplus \beta$. We define $\alpha \oplus \beta$ inductively by

$$\alpha \oplus \beta = \sup(\{\alpha' \oplus \beta + 1 : \alpha' < \alpha\} \cup \{\alpha \oplus \beta' + 1 : \beta' < \beta\}).$$

The key property used in the next theorem is that if $\alpha_1 < \alpha_2$ *or* $\beta_1 < \beta_2$ then

$$\alpha_1 \oplus \beta_1 < \alpha_2 \oplus \beta_2.$$

We will use later that $\oplus$ agrees with ordinal addition and regular addition on finite ordinals.

Theorem 3.9. *Let $\mathcal{K}$ be excellent and ω-stable. Then*

$$U(a/M(b)) + U(b/M) \leq U(ab/M) \leq U(a/M(b)) \oplus U(b/M).$$

Proof. We first show that $U(a/M(b)) + U(b/M) \leq U(ab/M)$ by induction on $\alpha = U(a/M(b)) + U(b/M)$, for all $M \in \mathcal{K}$. When $\alpha = 0$ or is a limit ordinal, it is easy. Assume that α is a successor. Then $U(b/M)$ must be a successor ordinal. By Lemma 3.4, we can choose $N \in \mathcal{K}$ with $M \prec N$ such that $U(b/N) + 1 = U(b/M)$. We may assume, without loss of generality, that $N \underset{M(b)}{\downarrow} a$, so $U(a/M(b)) = U(a/N(b))$ by Lemma 3.3 and symmetry, where $N(b)$ is chosen so $M(b) \prec N(b)$. Now by induction hypothesis we have

$$U(a/N(b)) + U(b/N) \leq U(ab/N). \tag{1}$$

But $ab \underset{M}{\not\downarrow} N$ by Lemma 3.8 since $b \underset{M}{\not\downarrow} N$. Hence

$$U(ab/N) + 1 \leq U(ab/M). \tag{2}$$

Then (1) and (2) and the choice of N give $U(a/M(b)) + U(b/M) \leq U(ab/M)$.

We now prove $U(ab/M) \leq U(a/M(b)) \oplus U(b/M)$ by induction on $\alpha = U(ab/M)$, for all $M \in \mathcal{K}$. Again for $\alpha = 0$ or a limit ordinal, it is easy. Assume that α is a successor. By Lemma 3.4, we can choose $N \in \mathcal{K}$, $M \prec N$, such that $U(ab/N) + 1 = U(ab/M)$. By induction hypothesis,

$$U(ab/N) \leq U(a/N(b)) \oplus U(b/N), \tag{3}$$

were $N(b)$ is primary chosen so that $M(b) \prec N(b)$. Since $ab \not\downarrow_M N$, then either $a \not\downarrow_{M(b)} N(b)$, or $b \not\downarrow_M N$, by Lemma 3.8. Hence, either $U(a/N(b)) < U(a/M(b))$ or $U(b/N) < U(b/M)$. In any case, we have:

$$U(a/N(b)) \oplus U(b/N) < U(a/M(b)) \oplus U(b/M). \tag{4}$$

Then $U(ab/M) \leq U(a/M(b)) \oplus U(b/M)$ follows from (3) and (4) and the choice of N. □

We can now prove the following finiteness result.

Theorem 3.10. *Let $\mathcal{K}$ be excellent and uncountably categorical. Then $U(p) < \omega$, for each $p \in \mathrm{S_{at}}(M)$.*

Proof. Assume, for a contradiction, that $U(a/M) \geq \omega$ for some $\mathrm{tp}(a/M) \in \mathrm{S_{at}}(M)$. By Lemma 3.4, there exists $N \in \mathcal{K}$ such that $U(a/N) = \omega$ (in particular, $\mathrm{tp}(a/N)$ is big). Let $\mathrm{tp}(b/N) \in \mathrm{S_{at}}(N)$ be quasiminimal. Then $U(b/N) = 1$. By unidimensionality (Lemma 2.20), we may assume that $a \not\downarrow_N b$. Thus,

$$U(a/N(b)) = n < \omega = U(a/N).$$

By Theorem 3.9 we have

$$U(ab/N) = U(a/N(b)) + U(b/N) = n + 1 < \omega.$$

Hence, $U(a/N) \leq U(ab/N) < \omega$, a contradiction. □

We state the following useful corollary.

Corollary 3.11. *Let $\mathcal{K}$ be excellent and uncountably categorical. Then*

$$U(ab/M) = U(a/M(b)) + U(b/M).$$

We can now define what we mean by computing the U-rank of an excellent class. In the next definition, it is enough to consider only types over countable models.

Definition 3.12. Let $\mathcal{K}$ be excellent, and ω-stable. The *U-rank of $\mathcal{K}$* is the supremum of $U(a/M)$, where $M \in \mathcal{K}$ and a is an element such that $\mathrm{tp}(a/M) \in \mathrm{S_{at}}(M)$.

For example, the universe of Zilber's pseudo-analytic structure is quasi-minimal. This implies that $U(a/M)$, when a is an element is either 0 (when the type is not big) or 1, when the type is big. Thus, the U-rank of Zilber's pseudo-analytic structure is 1.

The example of free groups is more pathological: We show below that the U-rank of the class of infinitely generated free groups is ω. However, this supremum is not achieved. This contrasts with the first order uncountably categorical case, where the supremum is always finite (and therefore achieved). This is another way of seeing that there are no generics in free groups; the types of maximum rank are omitted.

3.1. **Free groups.** Let $F = F(X)$ be any uncountable free group generated by X. We want to compute $\sup U(a/M)$, where $a \in F$, and $M \prec F(X)$ is countable. It is not difficult to see that $F(X)$ is supersimple in the sense of [BuLe] (see that paper for more details). Concretely, this means that we can extend the U-rank to all complete types and that it is enough to compute the supremum of $U(\mathrm{tp}(a/\emptyset))$ for $a \in F$.

Let $a \in F$. Then $a = a_1^{\epsilon_1} \cdot \dots \cdot a_n^{\epsilon_n}$, where $a_i \in X$ and $\epsilon_i \in \{-1, 1\}$. Furthermore, the a_i's are uniquely determined. This implies that an automorphism of F fixes a if and only if it fixes $a_1, \dots, a_n$. It follows that $U(\mathrm{tp}(a/\emptyset)) = U(\mathrm{tp}(a_1, \dots, a_n/\emptyset))$. Now since $a_i \in X$ and X carries a trivial pregeometry, it is not difficult to see that $U(\mathrm{tp}(a_1, \dots, a_n/\emptyset)) = |\{a_1, \dots, a_n\}| \leq n$. Hence, for each $n < \omega$, there are elements $a \in G$ of U-rank n, so

$$\sup_{a \in F} U(\mathrm{tp}(a/\emptyset)) = \omega.$$

Yet, no element $a \in F$ has U-rank ω by Theorem 3.10.

References

[BaLa] J.T. Baldwin and A. H. Lachlan, On strongly minimal sets, *Journal of Symbolic Logic*, **36** (1971) 79–96.

[BCM] Walter Baur, Greg Cherlin, Angus Macintyre, Totally categorical groups and rings. *J. Algebra* **57** (1979), no. 2, 407–440.

[BuLe] Steve Buechler and Olivier Lessmann, Simple homogeneous models, *Journal of the AMS* **16** (2003), no. 1, 91–121.

[Gr1] Rami Grossberg, A course on abstract elementary classes. Class notes from a course given at Carnegie Mellon University in 1997.

[Gr2] Rami Grossberg, Classification theory for abstract elementary classes. Logic and algebra, Contemp. Math., Vol 302, 165–204, Amer. Math. Soc., Providence, RI, 2002.

[GrHa] Rami Grossberg and Bradd Hart, The classification theory of excellent classes, *Journal of Symbolic Logic* **54** (1989) 1359–1381.

[GrLe] Rami Grossberg and Olivier Lessmann, Shelah's stability spectrum and homogeneity spectrum in finite diagrams. *Arch. Math. Logic* **41** (2002), no. 1, 1–31.

[Hy] Tapani Hyttinen, Generalizing Morley's Theorem, *Mathematical Logic Quarterly*, **44**, (1998), 176–184.

[HyLe] Tapani Hyttinen and Olivier Lessmann, A rank for the class of elementary submodels of a superstable homogeneous model. *Journal of Symbolic Logic*, **67** (2002), no. 4, 1469–1482.

[HySh1] Tapani Hyttinen and Saharon Shelah, Strong splitting in stable homogeneous models, *Annals of Pure and Applied Logic*, **103**, (2000), 201–228

[HySh2] Tapani Hyttinen and Saharon Shelah, Main gap for locally saturated elementary submodels of a homogeneous structure. *J. Symbolic Logic* **66** (2001), no. 3, 1286–1302.

[Ke] H. Jerome Keisler, **Model theory for infinitary logic** , North-Holland, 1971, Amsterdam.

[Ko] Alexei Kolesnikov, Dependence relations in nonelementary classes. Preprint

[Le1] Olivier Lessmann, Ranks and pregometries in finite diagrams, *Annals of Pure and Applied Logic*, **106**, (2000), pages 49–83.

[Le2] Olivier Lessmann, Homogeneous model theory: Existence and Categoricity, in Logic and Algebra, ed. Yi Zhang, Contemporary Mathematics, Vol 302, AMS 149–164, Providence, RI, (2002).

[Ma] Leo Marcus, A prime minimal model with an infinite set of indiscernibles, *Israel Journal of Mathematics*, **11**, (1972), pages 180–183.

[Mo] Michael Morley, Categoricity in power, *Trans. of the A.M.S* **114** (1965), pages 514–538.

[Sh3] Saharon Shelah, Finite diagrams stable in power, *Annals Math. Logic* **2**, (1970), pages 69–118.

[Sh70] Saharon Shelah, Solution to Łoš Conjecture for uncountable languages, *Notices of the A.M.S*, **17** (1970), 968.

[Sh48] Saharon Shelah, Categoricity in $\aleph_1$ of sentences in $L_{\omega_1\omega}(Q)$, *Israel Journal of Math.* **20** (1975), pages 127–148.

[Sh54] Saharon Shelah, The lazy model theorist's guide to stability, **Proc. of a Symp. in Louvain**, March 1975, ed. P. Henrand, *Logique et Analyse*, 18eme annee, **71-72** (1975), pages 241–308.

[Sh87a] Saharon Shelah. Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1\omega}$. Part A. *Israel Journal of Mathematics*, **46**:212–240, 1983.

[Sh87b] Saharon Shelah. Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1\omega}$. Part B. *Israel Journal of Mathematics*, **46**:241–273, 1983. In *Classification theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Mathematics*, pages 264–418. Springer, Berlin, 1987. Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; ed. Baldwin, J.T.

[Sh576] Saharon Shelah. Categoricity of an abstract elementary class in two successive cardinals. 115 pages, preprint.

[Sh600] Saharon Shelah, preprint.

[Zi1] Boris Zilber, Covers of the multiplicative group of an algebraically closed field of characteristic 0. Preprint.

[Zi2] Boris Zilber, Analytic and pseudo-analytic structures. Preprint

[Zi3] Boris Zilber, Covers of the multiplicative group under $\aleph_1$-categoricity assumption. Preprint.

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