Lecture 6.5 Saturation and homogeneity

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Assumption 1 *$\boldsymbol{K}$ is an abstract elementary class.*

The goal is to derive properties on embedding models from the realization of Galois types. We want to show that if M^1 realizes 'enough' types over M then any small extension N of M can be embedded into M^1. The idea is first published as 'saturation = model-homogeneity' in 3.10 of [6] (Theorem 8 below), where the proof is incomplete. Successive expositions in [5, 2], and by Baldwin led to this version, where the key lemma was isolated by Kolesnikov. In contrast to various of the expositions and like Shelah, we make no amalgamation hypothesis.

Whether we really gain anything by not assuming amalgamation is unclear. I know of no example where either λ-saturated or λ-model homogeneous structures are proved to exist without using amalgamation, at least in λ.

The key idea of the construction is that to embed N into M^1; we construct a $M^2 \prec_{\boldsymbol{K}} M^1$ and a $\boldsymbol{K}$-isomorphism f from M_2 onto an $N_2 \prec_{\boldsymbol{K}} N_3$ where $N \prec_{\boldsymbol{K}} N_3$. Then the coherence axiom tells us restricting f^{-1} to N, gives the required embedding. We isolate the induction step of the construction in the following lemma. We will apply the lemma in two settings. In one case $\overline{M}$ has the same cardinality as M and is presented with a filtration M_i. Then $\hat{M}$ will be one of the M_i. In the second, $\overline{M}$ is a larger saturated model and $\hat{M}$ will be chosen as a small model witnessing the realization of a type.

We work in the most general context with *no* amalgamation hypothesis. We state several definitions to indicate the exact context we are working in. The most appropriate background in Shelah in [5], not [3]. We use our own notation but the relation to his should be clear.

Definition 2
1. *For $M \prec_{\boldsymbol{K}} N_1 \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} N_2 \in \boldsymbol{K}$ and $a \in N_1 - M$, $b \in N_2 - M$, write $(M, a, N_1) \sim_{At} (M, b, N_2)$ if there exist strong embeddings f_1, f_2 of N_1, N_2 into some N^* which agree on M and with $f_1(a) = f_2(b)$.*
2. *Let $\sim$ be the transitive closure of $\sim_{AT}$ (as a binary relation on triples).*
3. *We say the Galois type a over M in N_1 is the same as the Galois type a over M in N_2 if $(M, a, N_1) \sim (M, b, N_2)$*

Exercise 3 *If $\boldsymbol{K}$ has amalgamation, $\sim_{AT}$ is an equivalence relation and $\sim = \sim_{AT}$.*

But we do *not* assume amalgamation.

Notation 4 *The set of Galois types over M is denoted* $\mathrm{ga-S(M)}$.

Definition 5 1. *We say the Galois type of a over M in N_1 is* strongly realized *in N with $M \prec_{\boldsymbol{K}} N$ if for some $b \in N$, $(M, a, N_1) \sim_{AT} (M, b, N)$.*

2. *We say the Galois type of a over M in N_1 is* realized *in N with $M \prec_{\boldsymbol{K}} N$ if for some $b \in N$, $(M, a, N_1) \sim (M, b, N)$.*

Now we need a crucial form of the definition of saturated from [5]

Definition 6 *The model M is μ-*Galois saturated *if for every $N \prec_{\boldsymbol{K}} M$ with $|N| < \mu$ and every Galois type p over N, p is* strongly *realized in M.*

Under amalgamation we could define saturation using realization and we would have an equivalent notion. Without amalgamation, the notion we have selected is obviously more restricted. For the moment we rely on the assertion in Definition 22 of [5] that in all 'interesting situations' we can use the strong form of saturation.

We use in this construction without further comment two basic observations. If f is a $\boldsymbol{K}$-isomorphism from M onto N and $N \prec_{\boldsymbol{K}} N_1$ there is an M_1 with $M \prec_{\boldsymbol{K}} M_1$ and an isomorphism f_1 (extending f) from M_1 onto N_1. (The dual holds with extensions of M.) Secondly, whenever $f_1 \circ f_2 : N \mapsto M$ and $g_1 \circ g_2 : N \mapsto M$ are maps in a commutative diagram, there is no loss of generality in assuming $N \prec_{\boldsymbol{K}} M$ and $f_1 \circ f_2$ is the identity.

Of course, under amalgamation of models of size $|M|$, we can delete the strongly in following hypothesis.

Lemma 7 *Suppose $M \prec_{\boldsymbol{K}} \overline{M}$ and $\overline{M}$* strongly realizes *all Galois-types over M. Let $f : M \mapsto N$ be a $\boldsymbol{K}$-isomorphism and $\tilde{N}$ a $\boldsymbol{K}$-extension of N. For any $a \in \tilde{N} - N$ there is a $b \in \overline{M}$ such that for any $\hat{M}$ with $Mb \subseteq \hat{M} \prec_{\boldsymbol{K}} \overline{M}$ and $|M| = |\hat{M}| = \lambda$, there is an N^* with $N \prec_{\boldsymbol{K}} N^*$ and an isomorphism $\hat{f}$ extending f and mapping $\hat{M}$ onto $\hat{N} \prec_{\boldsymbol{K}} N^*$ with $\hat{f}(b) = a$.*

Proof. Choose $\tilde{M}$ with $M \prec_{\boldsymbol{K}} \tilde{M}$ and extend f to an isomorphism $\tilde{f}$ of $\tilde{M}$ and $\tilde{N}$. Let $\tilde{a}$ denote $\tilde{f}^{-1}(a)$. Choose $b \in \overline{M}$ to strongly realize the Galois type of $\tilde{a}$ over M in $\tilde{M}$. Fix any $\hat{M}$ with $Mb \subseteq \hat{M} \prec_{\boldsymbol{K}} \overline{M}$ and $|M| = |\hat{M}| = \lambda$. By the definition of strongly realize, we can choose an extension M^* of $\tilde{M}$ and $h : \hat{M} \mapsto M^*$ with $h(b) = \tilde{a}$. Lift $\tilde{f}$ to an isomorphism f^* from M^* to an extension N^* of $\tilde{N}$. Then $\hat{f} = (f^* \circ h) \restriction \hat{M}$ and $\hat{N}$ is the image of $\hat{f}$. $\square_7$

A key point in both of the following arguments is that while the N_i eventually exhaust N, they are not required to be submodels (or even subsets) of N.

Here is the first application.

Theorem 8 *Assume $\lambda > \mathrm{LS}(\boldsymbol{K})$. A model M^2 is λ-Galois saturated if and only if it is λ-model homogeneous.*

Proof. It is obvious that λ-model homogeneous implies λ-Galois saturated. Let M^2 be λ-saturated. We want to show M^2 is λ-model homogeneous. So fix $M_0 \prec_{\boldsymbol{K}} M^2$ and N with $M_0 \prec_{\boldsymbol{K}} N$. Say, $|N| = \mu < \lambda$. We construct M^1 as a union of strong submodels M_i of M^2. At the same time we construct N^1 as the union of N_i' which are strong extensions of N and f_i mapping M_i onto N_i. Enumerate $N - M_0$ as $\langle a_i : i < \mu \rangle$. Let $N_0 = M_0$, $N_0' = N$ and f_0 be the identity. At stage i, f_i, N_i, M_i, N_i', are defined; we will construct N_{i+1}', f_{i+1}, N_{i+1}, M_{i+1}. Apply Lemma 7 with a_j as a for the least j with $a_j \notin N_i'$; take M_i for M; M_{i+1} is any submodel of M^2 with cardinality μ that witnesses the Galois type of b over M_i in M^2 and plays the role $\hat{M}$ in the lemma; N_i' is $\tilde{N}$ and N_i is N. The role of $\overline{M}$ is taken by M^2 at all stages of the induction. We obtain f_{i+1} as $\hat{f}$, N_{i+1} as $\hat{N}$ and N_{i+1}' as N^*. Finally f is the union of the f_i and N^1 is the union of the N_i'. $\square_8$

Just how general is Theorem 8? It asserts the equivalence of 'M is λ-model homogeneous' with 'M is λ-saturated' and we claim to have proved this without assuming amalgamation. But the existence of either kind of model is near to implying amalgamation on $\boldsymbol{K}_{<\lambda}$. But it is only close. Let ψ be a sentence of $L_{\omega_1,\omega}$ which has saturated models of all cardinalities and ϕ be a sentence of $L_{\omega_1,\omega}$ which does not have the amalgamation property over models. Now let $\boldsymbol{K}$ be the AEC defined by $\psi \vee \phi$ (where we insist that on each model either the $\tau(\psi)$-relations or the $\tau(\phi)$-relations are trivial but not both). Then $\boldsymbol{K}$ has λ-model homogeneous models of every cardinality (which are saturated) but does not have either the joint embedding or the amalgamation property (or any restriction thereof). However, with some mild restrictions we see the intuition is correct. First an easy back and forth gives us:

Lemma 9 *If $\boldsymbol{K}$ has the joint embedding property and $\lambda > \mathrm{LS}(\boldsymbol{K})$ then any two λ-model homogeneous models M_1, M_2 of power λ are isomorphic.*

Proof. It suffices to find a common strong elementary submodel of M_1 and M_2 with cardinality $< \lambda$ but this is guaranteed by joint embedding and $\lambda > \mathrm{LS}(\boldsymbol{K})$. $\square_9$

Definition 10 *For any AEC $\boldsymbol{K}$, and $M \in \boldsymbol{K}$ let $\boldsymbol{K}^M$ be the AEC consisting of all direct limits of strong substructures of M.*

Lemma 11 *Suppose M is a λ-model homogeneous member of $\boldsymbol{K}$.*

1. *$\boldsymbol{K}^M_{<\lambda}$ has the amalgamation property.*
2. *If $\boldsymbol{K}$ has the joint embedding property $\boldsymbol{K}_{<\lambda}$ has the amalgamation property.*

Proof. The first statement is immediate and the second follows since then by Lemma 9 we have $\boldsymbol{K}^M_{<\lambda} = \boldsymbol{K}_{<\lambda}$. $\square_{11}$

Now by Lemma 11 and Theorem 8 we have:

Corollary 12 *If $\boldsymbol{K}$ has a λ-saturated model and has the joint embedding property then $\boldsymbol{K}_{<\lambda}$ has the amalgamation property.*

The corollary, which is Remark 30 of [5], confirms formally the intuition that under mild hypotheses we need amalgamation on $\boldsymbol{K}_{<\lambda}$ to get saturated models of cardinality λ. But we rely on the basic equivalence, proved without amalgamation to establish this result.

Now we have a second application of the Lemma 7. This requires an amalgamation hypothesis. Theorem 14 is asserted without proof in 1.15 of [4]; another exposition of the argument is in [1].

Definition 13 *M_2 is σ-universal over M_1 if $M_1 \leq M_2 \leq N$ and whenever $M_1 \leq M_2'$, with $|M_2'| \leq \sigma$, there is a (partial isomorphism) fixing M_1 and taking M_2' into M_2.*

This is Definition 1.12 1) from [4]. Note that it does not require that all smaller models $\boldsymbol{K}$ imbed into M_2.

Theorem 14 *If $\boldsymbol{K}$ is λ-Galois stable and $\boldsymbol{K}_\lambda$ has the amalgamation property, then for every $M \in \boldsymbol{K}_\lambda$ there is an M^1 with cardinality λ that is λ-universal over M.*

Proof. Construct M^1 as a continuous union for $i < \lambda$ of M_i with $M_0 = M$, and each M_{i+1} realizes all Galois types over M_i. (The existence of the M_{i+1} is guaranteed by the amalgamation hypothesis.) Now fix any strong extension N of M. We will construct a $\boldsymbol{K}$-isomorphism f from M^1 into an extension N^1 of N with $N \subset \overline{N} \prec_{\boldsymbol{K}} N^1$, where $\overline{N}$ denotes the range of f. By the coherence axiom $f^{-1} \restriction N$ is the required map.

To construct f, enumerate $N - M$ as $\langle a_i : i < \lambda \rangle$. We construct a continuous increasing sequence of maps f_i. Let $f_0 = 1_M$. Suppose we have defined f_i, N_i and N'_i with f_i taking M_i onto $N_i \prec_{\boldsymbol{K}} N'_i$. Now apply Lemma 7 with a_j as a for the least j with $a_j \notin N'_i$; take M_i for M; M_{i+1} plays the role of both $\overline{M}$ and $\hat{M}$ in the lemma; N'_i is $\tilde{N}$ and N_i is N. We obtain f_{i+1} as $\hat{f}$, N_{i+1} as $\hat{N}$ and N'_{i+1} as N^*. Finally f is the union of the f_i and N^1 is the union of the N'_i. $\square_{14}$

The formulation of these results and arguments followed extensive discussions with Rami Grossberg, and Monica Van Dieren. Alexei Kolesnikov singled out Lemma 7.

References

[1] R. Grossberg and M. Van Dieren. Galois stability for tame abstract elementary classes. preprint.

[2] Rami Grossberg. Classification theory for non-elementary classes. In Yi Zhang, editor, *Logic and Algebra*, pages 165–204. AMS, 2002. Contemporary Mathematics 302.

[3] S. Shelah. Categoricity for abstract classes with amalgamation. *Annals of Pure and Applied Logic*, 98:261–294, 1999. paper 394.

[4] S. Shelah. Categoricity of abstract elementary classes: going up inductive step. preprint 600, 200?

[5] S. Shelah. Categoricity of abstract elementary class in two successive cardinals. *Israel Journal of Mathematics*, 126:29–128, 2001. paper 576.

[6] Saharon Shelah. Classification of nonelementary classes II, abstract elementary classes. In J.T. Baldwin, editor, *Classification theory (Chicago, IL, 1985)*, pages 419–497. Springer, Berlin, 1987. paper 88: Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; volume 1292 of *Lecture Notes in Mathematics*.