Mersenne and Fermat primes

John T. Baldwin

October 8, 2004

Notice that if \( j \) is even, \( 2^j - 1 \) is composite unless \( j = 2 \). We want to find many generalizations of this fact. (Recall that a number is composite if it has a factor other than itself and 1.)

1. Consider numbers of the form \( b = 2^j - 1 \). A prime of this form is called a Mersenne prime.
   
   (a) Try to give a property of \( j \) (even, odd, prime, composite, etc) that guarantees \( b \) is prime or a different property that guarantees that \( b \) is composite.

   (b) Are there infinitely many primes of the form \( b = 2^j - 1 \). Yes, no, I don’t know.

2. Consider numbers of the form \( c = 2^j + 1 \).

   (a) Try to give a property of \( j \) (even, odd, prime, composite, etc) that guarantees \( c \) is prime or a different property that guarantees that \( c \) is composite.

   (b) Are there infinitely many primes of the form \( c = 2^j + 1 \). Yes, no, I don’t know.

   (c) Fermat primes are of a bit more special form than \( c = 2^j + 1 \). On the basis of answering the earlier questions you should be able to explain the terminology; do so.

This problem is open-ended in the sense that you have a lot of choices of what properties to choose and you have to decide whether you can ‘make’ the number prime or ‘make’ the number composite. But for this course a full solution of the problem will involve showing patterns of factoring that are a little more complex than usually taught is high school but readily accessible to high school seniors. A really full solution would make you famous!