What is a complete theory?

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Outline

Themes

1. Logic for Logic's sake
2. Model theory as a tool for studying the methodology of mathematics.
   Each mathematical subject requires its own formalization.

A basic notion

A (consistent) theory T in a logic $\mathcal{L}$ is complete if for every $\mathcal{L}$-sentence ϕ,

$T \models \phi$
or
$T \models \neg\phi$.

Complete First Order Theories

1. dense linear order (w/o endpoints)
2. algebraically closed fields (of fixed characteristic)
3. true arithmetic
4. real closed fields

First order model theory

The main tool of first order model theory is the classification of complete theories by stability-like notions.

If complete theories have similar semi-syntactic theoretic properties: $\aleph_1$-categorical, ω-stable, o-minimal, strictly stable, then their class of models have similar algebraic properties: number of models, existence of dimension functions, interpretability of groups, existence of generic elements,

The Standard Example

$\mathrm{Th}(M)$ for any M.

Leitmotif

Study a mathematical structure M by studying $\mathrm{Th}(M)$.

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Thus, algebraic geometry is the model theory of $(\mathcal{C}, +, \cdot, 0, 1)$.

This philosophy underlies Hrushovsky's work on the geometric Mordell-Lang Conjecture.

$L_{\omega_1,\omega}$-completeness

For Δ a fragment of $L_{\omega_1,\omega}$, a Δ-theory T is complete if for every Δ-sentence ϕ,

$T \models \phi$
or
$T \models \neg\phi$.

Löwenheim Skolem properties

Downward: Every consistent countable set of $L_{\omega_1,\omega}$-sentences has a countable model.

No upward: There are sentences with maximal models in (that characterize) each $\aleph_\alpha$ and each $\beth_\alpha$.

What is the theory?

What is $\mathrm{Th}_{L_{\omega_1,\omega}}(\mathcal{C}, +, \cdot, 0, 1)$?

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What is $\mathrm{Th}_{L_{\omega_1,\omega}}(\mathcal{C},+,\cdot,0,1)$?

What is $\mathrm{Th}_{L_{\omega_1,\omega}}(\mathcal{R},+,\cdot,0,1)$?

Vaught's test

Let T be a set of first order sentences with no finite models, in a countable language.

If T is κ-categorical for some $\kappa \geq \aleph_0$,
then T is complete.

Small

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Definition

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Definition

An $L_{\omega_1,\omega}$-sentence ϕ is Δ-small if there is a set X countable of complete Δ-types over the empty set and each model realizes some subset of X.

'small' means $\Delta = L_{\omega_1,\omega}$

Small implies complet(able)

If M is small then M satisfies a complete sentence.

If ϕ is small then there is a complete sentence ψ_ϕ such that: $\phi \wedge \psi_\phi$ have a countable model.

So ψ_ϕ implies ϕ.

The $L_{\omega_1,\omega}$-Vaught test

Shelah If ϕ has an uncountable model M that is Δ-small for every countable Δ and ϕ is κ-categorical then ϕ is implied by a complete sentence ψ with a model of cardinality κ.

Keisler If ϕ has $< 2^{\aleph_1}$ models of cardinality $\aleph_!$, then for every countable Δ, ϕ is Δ not so big.
I.e. each model is Δ-small for every countable Δ.

So we effectively have Vaught's test.
But only in $\aleph_1$!
And only for completability!

Countable models I

Must an $\aleph_1$-categorical sentence have only countably many countable models?

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Must an $\aleph_1$-categorical sentence have only countably many countable models?

Trivially, no. Take the disjunction of a 'good' sentence with one that has $2^{\aleph_0}$-countable models and no uncountable models.

Countable models II

Is there a way to study the countable models of sufficiently nice incomplete sentences?

Must an $\aleph_1$-categorical sentence with the joint embedding property have only countably many countable models?

A direction: The Kesala-Hyttinen study of finitary abstract elementary classes.

Another direction: Kierstead's thesis using admissible model theory.

Two specific research questions

For ϕ a sentence in $L_{\omega_1,\omega}$:

Does categoricity in $\kappa > \aleph_1$ imply completeness (completeability)?

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Does categoricity in $\kappa > \aleph_1$ imply completeness (completeability)?

Is categoricity in $\aleph_1$ absolute?

Model Theory and Mathematics

Stability theory developed

1. abstractly with the stability classification
2. concretely by finding the stability class of important mathematical theories and using the techniques of the abstract theory.

The absoluteness of fundamental notions such as $\aleph_1$-categoricity and stability liberated first order model theory from set theory.

Infinitary Logic

At the same time and largely unnoticed, Shelah developed the fundamentals of stability theory for infinitary logic.

It was not until Zilber's exploration of complex exponentiation in the 1990's that the significance of this work for mainstream mathematics was realized.

More general questions

A successful strategy for first order

Study the complete first order theory of the mathematical structure of interest.

Infinitary logic

What structures in what languages benefit from analysis in infinitary languages?

What notion of complete is appropriate for such a study?

A simple example

Let $(V, +)$ be a $\mathbb{Q}$-vector space of cardinality $2^{\aleph_0}$.
Let h be a homomorphism from V to $(\mathcal{C}^*, \cdot)$ with kernel $\mathcal{Z}$.

Have I completely described a structure $(V, +, h, \mathcal{C}, +, \cdot)$?

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Zilber: Yes!

Acknowledgements

Most of the ideas here are reworking and reorganizing Zilber's Ravello paper. Some proofs and some emphases are different.

Covers of Algebraic Groups

Definition A cover of a commutative algebraic group $\mathbb{A}(\mathcal{C})$ is a short exact sequence

$$0 \to Z^N \to V \stackrel{\exp}{\to} \mathbb{A}(\mathcal{C}) \to 1. \quad (1)$$

where V is a $\mathbb{Q}$ vector space and $\mathbb{A}$ is an algebraic group, defined over k_0 with the full structure imposed by $(\mathcal{C}, +, \cdot)$.

Algebraic group

For any algebraically closed field F and any algebraic group $\mathbb{A}$ there is also a set of formulas $\mathbb{F}$ defining an algebraically closed field such that $\mathbb{F}(\mathbb{A}(F))$ and $\mathbb{A}(F)$ are interdefinable.

If (V, A) is such a model with $A = \mathbb{A}(F)$, we identify F with $\mathbb{F}(A)$.

Thus, definable in the group is the same as definable in the underlying field.

Axiomatizing Covers: first order

Let $\mathbb{A}$ be a commutative algebraic group over an algebraically closed field F.

Let T_A be the first order theory asserting:

1. $(V, +, f_q)_{q \in \mathbb{Q}}$ is a $\mathbb{Q}$-vector space.
2. The complete first order theory of $\mathbb{A}(F)$ in a language with a symbol for each k_0-definable variety (where k_0 is the field of definition of $\mathbb{A}$).
3. exp is a group homomorphism from $(V, +)$ to $(\mathbb{A}(F), \cdot)$.

Axiomatizing Covers: $L_{\omega_1,\omega}$

Add to T_A
$\Lambda = \mathcal{Z}^N$ asserting the kernel of exp is standard.

$$((\exists \mathbf{x} \in (\exp^{-1}(1))^N)(\forall y)[\exp(y) = 1 \rightarrow \bigvee_{\mathbf{m}\in\mathcal{Z}^N} \Sigma_{i<N} m_i x_i = y]$$

Some properties

For any $\mathbb{A}$:

$T_A + \Lambda = \mathcal{Z}^N$

1. has arbitrarily large models
2. has the amalgamation property

Associated Sequences

Let K be a field and $W \subset K^r$ a variety defined over K.

1. A *sequence* $\mathbf{W}$ *associated with* W over K is a family of varieties (defined over K) $W^{1/m}$ such that $(W^{1/mk})^k = W^{1/m}$, each $W^{1/m}$ is a minimal K-variety.
2. The sequence *stabilizes* with respect to $p(\mathbf{x})$, an r-type over the empty set if there is exists an ℓ such that for every m, there is a unique K-definable variety V with $V^m = W^{1/\ell}$ and such that $p(\mathbf{x})$ and $\langle \exp(x_1/m\ell), \ldots \exp(x_r/m\ell)\rangle \in V$ is consistent.

Pseudo-generating sequences

Let $\mathbb{V} = (V, A) \models T_A$. $\langle \tau_1, \ldots, \tau_N \rangle \in V$ is a *pseudogenerating tuple* of $\Lambda(V)$ if for each $m \in \mathcal{Z}$:

$$n_1\tau_1 + \ldots, +n_N\tau_N \in m\Lambda \text{ iff } \gcd(n_1, \ldots, n_N) \in m\mathcal{Z}.$$

Expanded Language

L is the basic language. Form L^* by adding predicates:
We expand L to L^* by adding the following formulas.

1. $\mathrm{Ind}^n(\mathbf{x})$ holds if $\mathbf{x}$ is a linearly independent n-tuple in V.
2. $\mathrm{PG}^\ell(\mathbf{x})$ holds if $\mathbf{x}$ is an ℓ-tuple from Λ that satisfies for each $m \in \mathcal{Z}$:

$$n_1\tau_1 + \ldots, +n_\ell\tau_\ell \in m\Lambda \text{ iff } \gcd(n_1, \ldots, n_\ell) \in m\mathcal{Z}.$$

3. $\mathrm{Gen}^W(\mathbf{x})$ holds if $\exp(\mathbf{x})$ satisfies the type of a generic point of the k-irreducible variety.
4. $R_m(v) \leftrightarrow (\exists y \in \exp^{-1}(1))[my = v]$.

Consequences I

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is small.

1. $T_A + \Lambda(V) = \mathcal{Z}^N$ admits elimination of quantifiers in L^*.
2. Every countable model of $T_A + \Lambda(V) = \mathcal{Z}^N$ + 'infinite dimension' is ω-homogeneous.

Consequences II

Let r be the type of a pseudogenerating sequence.

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is small.

If k is finitely generated any sequence $\mathbf{W}$ associated with any W over k stabilizes with respect to r.

Smallness and Completeness

[JB] $T_A + \Lambda(V) = \mathcal{Z}^N$ has a finite number of completions.

Aside: Characteristic p

[Bays, Zilber] Consider

$$0 \to Z[1/p] \to V \to F_p^* \to 0.$$

where $Z[1/p]$ is the localization at p and F_p^* is an infinite dimensional algebraically closed field of characteristic p.

$T_A + \Lambda(V) = \mathcal{Z}^N$ is not small. There are $2^{\aleph_0}$ completions - distinct minimal models.

The theories must be analyzed separately; each is categorical.

Choosing Roots

Definition

A *multiplicatively closed divisible subgroup* associated with $a \in \mathcal{C}^*$, is a **choice** of a multiplicative subgroup isomorphic to $\mathbb{Q}$ containing a .

Definition

$b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, *determine the isomorphism type* of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ over F if given subgroups of the form $c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ and ϕ_m such that

$$\phi_m : F(b_1^{\frac{1}{m}} \ldots b_\ell^{\frac{1}{m}}) \to F(c_1^{\frac{1}{m}} \ldots c_\ell^{\frac{1}{m}})$$

is a field isomorphism it extends to

$$\phi_\infty : F(b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}}) \to F(c_1^{\mathbb{Q}}, \ldots c_\ell^{\mathbb{Q}}).$$

An Algebraic Condition

For $\mathbb{A} = (\mathcal{C}^*, \cdot)$:

F-Thumbtack Lemma

Let F be a countable field. For any $b_1, \ldots b_\ell \subset \mathcal{C}^*$, there exists an m such that $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \ldots b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, *determine the isomorphism type* of $b_1^{\mathbb{Q}}, \ldots b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ over F.

Context

For $\mathbb{A} = (\mathcal{C}^*, \cdot)$, the thumbtack lemma is clearly stated and true.

As $\mathbb{A}$ varies, the exact formulation is not clear (at least to me) and truth will vary with the choice of $\mathbb{A}$.

Proving smallness and more

examining Zilber's arguments

Smallness is equivalent to F-thumbtack for finitely generated F.

ω-stability is equivalent to F-thumbtack for countable F.

Zilber

The (full) Thumbtack Lemma is equivalent to $T_A + \Lambda = \mathcal{Z}^N$ is excellent.

Mordell-Weil Theorem

For $\mathbb{A}$ a smooth elliptic curve,
If k is a finite algebraic extension $\mathbb{Q}$, $\mathbb{A}(k)$ is a finitely generated abelian group.

Smallness and Mordell-Weil

For any algebraic group $\mathbb{A}$:

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is small.

If k is finitely generated over $\mathbb{Q}$, $\mathbb{A}_{\mathrm{tor}}(k)$ is finite.

smallness implies finite torsion: boundedness

Definition

The algebraic group $\mathbb{A}$ is bounded if for every finitely generated extension k of the field of definition k_0 of $\mathbb{A}$, there is a d such that for every ℓ the Galois group of $\mathrm{Gal}(\tilde{k}, k)$ has only d-orbits on the set

$$X_\ell = \{\langle a_1, \ldots, a_N \rangle \in \mathbb{A}_\ell^N(\tilde{k}) : (\exists \mathbf{b})[\mathbf{a} = \exp(\mathbf{b}/\ell) \wedge \mathrm{PG}^{\,\ell}(\mathbf{b})]\ \}.$$

smallness implies bounded

Lemma

If $T_A + \Lambda(V)$ is small, the $\mathbb{A}$ is bounded.

Proof. Every sequence over k associated with the type $p = \mathrm{PG}^{\ell}(\mathbf{x})$ stabilizes.
Thus, there are only finitely many extensions of p to complete types over $(V(K), A(K))$ and by the homogeneity over the empty set we have a bound d on the number of orbits of pseudogenerating sets.
But since each automorphism of $\mathbb{V}$ induces an automorphism of $\mathbb{A}_\ell(\tilde{k})$ for each ℓ, we have the same bound in X_ℓ.

smallness implies finite torsion

Lemma

If $\mathbb{A}$ is bounded, then for every finitely generated extension k of the field of definition k_0, $\mathbb{A}_{\rm tors}(k)$ is finite.

Proof. We show that if $\phi(\ell) > d$, there is no element of $\mathbb{A}(k)$ that has order ℓ.

Suppose $a \in \mathbb{A}(k)$ is a counterexample. Then a can be taken as the first element in an N-tuple $\mathbf{a}$ from $\mathbb{A}_\ell(\tilde{k})$ with $\mathbf{a} = \exp(\mathbf{b}/\ell)$ and $\mathrm{PG}^N(\mathbf{b})$ For any m that is coprime to ℓ, a^m also has order ℓ and can be extended to a sequence $\mathbf{a}_m$, so that $\mathbf{a}_m = \exp(\mathbf{b}_m/\ell)$ with $\mathrm{PG}^N(\mathbf{b}_m)$.

Thus the sequences $\mathbf{a}_m$ for $m < \ell$ and $(m, \ell) = 1$ represent distinct orbits in X_ℓ under $\mathrm{Gal}(\tilde{k}, k)$ (the first elements of the sequences are distinct elements of k). So if $\phi(\ell) > d$, we have a contradiction.

ω-stability

Definition

ϕ is ω-stable if for every countable model of ϕ, there are only countably many types over M that are realized in models of ϕ.

If $T_A + \Lambda(V) = \mathcal{Z}^N$ is ω-stable.

If k is countable any sequence $\mathbf{W}$ associated with any W over k stabilizes with respect to r.

Quasiminimal Excellence

A class $(\mathbf{K}, \mathrm{cl})$ is *quasiminimal excellent* if cl is a combinatorial geometry which satisfies on each $M \in \mathbf{K}$:

1. there is a unique type of a basis,
2. a technical homogeneity condition: $\aleph_0$-homogeneity over $\emptyset$ and over models.
3. and the 'excellence condition' which follows.

Conditions 1 and 2 are sufficient for $\aleph_1$-categoricity.

Necessary Notation

In the following definition it is essential that $\subset$ be understood as proper subset.

Definition

1. For any Y, $\mathrm{cl}^{-}(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.
2. We call C (the union of) *an n-dimensional* cl*-independent system* if $C = \mathrm{cl}^{-}(Z)$ and Z is an independent set of cardinality n.

Essence of Excellence

There is a primary (unique prime) model over any finite independent system.

QM EXCELLENCE IMPLIES CATEGORICITY

QM Excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of finite sets is countable.

Almost Quasiminimal Excellence

Let **K** be a class of L-structures which admit a function cl_M mapping $X \subseteq M$ to $\mathrm{cl}_M(X) \subseteq M$ that satisfies the following properties.

1. cl_M satisfies is a monotone idempotent closure operator with $\mathrm{cl}_M(X) \in \mathbf{K}$ that satisfies 'excellence' (But not exchange).
2. cl_M induces a quasiminimal excellent geometry on a distinguished sort U.
3. $M = \mathrm{cl}_M(U)$.
4. We call the class Almost Quasiminimal if the 'excellence' is dropped.

Algebraic Formulations of Excellence

Let $\mathcal{S} = \{F_s : s \subset n\}$ be an independent n-system of algebraically closed fields contained in a suitable monster $\mathcal{M}$. Denote the subfield of $\mathcal{M}$ generated by $(\bigcup_{s \subset n} F_s)$ as k.

Canonical completions

$$\mathcal{A}(k) = A^n \oplus \prod_{s \subset n} \mathcal{A}(F_s)$$

where A^n is a free Abelian group.

Excellence

The following are equivalent under VWGCH ($2^{\aleph_n} < 2^{\aleph_{n+1}}$)

1. The cover of $\mathbb{A}$ is categorical in all uncountable κ.
2. The cover of $\mathbb{A}$ is categorical in all $\aleph_n$ for $n < \omega$.
3. The cover of $\mathbb{A}$ is almost quasiminimal excellent.
4. $\mathbb{A}$ satisfies the algebraic conditions ω-stability and homogeneity and has canonical completions.

AQE and covers

Claim/Conjecture

An almost quasiminimal class is $\aleph_1$-categorical.
Thus, an *omega*-stable cover is $\aleph_1$-categorical.

Are there $\mathbb{A}$ that are ω-stable but not excellent?

AQE and covers

There are important mathematical topics that can only be usefully formalized in infinitary logic.

There is a dynamic interplay between the study of such examples and the development of infinitary model theory.