3 Red Herrings Around Vaught's Conjecture
Notre Dame

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March 29, 2013

Today's Topics

Section 1: Context for this seminar

Vaught's Conjecture

An $L_{\omega_1,\omega}$-sentence has 1, $\aleph_0$, or $2^{\aleph_0}$ countable models.

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Apparently using descriptive set theory,

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Strategy

Prove any counterexample to Vaught's conjecture has a model in $\aleph_2$.

Made more plausible by

Harrington's Theorem

If there is a counterexample to Vaught's conjecture there models in $\aleph_1$ with arbitrarily high Scott ranks below $\aleph_2$.

Sources

Joint work with S. Friedman, Hyttinen, Koerwien, Laskowski

Building on J. Knight, Hjorth, Laskowski-Shelah, and Souldatos.

The 3 Red Herrings

1. Hjorth's proof is pure model theory.
2. The real result is that every model in $\aleph_1$ is maximal.
3. Harrington's proof tells us about complexity of models and the real issue is the structure of the embeddability relation.

Section 1: The first red herring
Model theory vrs descriptive set theory

The key ideas

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Definition

1. Let θ be a complete τ_2 sentence of $L_{\omega_1,\omega}$ and suppose M is the countable model of θ and $N(M)$ is a set of absolute indiscernibles in M such $M - N$ projects onto N. We will say θ is a receptive sentence.
2. For any sentence ψ of $L_{\omega_1,\omega}$, the merger of ψ and θ is the sentence $\chi = \chi_{\theta,\psi}$ obtained by conjoining with θ, $\psi \restriction N$.
3. For any model M_1 of θ and N_1 of ψ we write $(M_1, N_1) \models \chi$ if there is a model with such a reduct.

Models in $\aleph_1$ of a receptive sentence

$\#(\chi, \lambda)$ denotes the number of models of χ in λ.

Theorem

Let θ be a complete sentence of $L_{\omega_1,\omega}$ with a receptive countable model and ψ a sentence of $L_{\omega_1,\omega}$.

1. *There is a 1-1 isomorphism preserving function between the countable models of ψ and the models of the merger $\chi_{\theta,\psi}$.*
2. $\#(\chi, \lambda) = \max(\#(\theta, \lambda), \#(\psi, \lambda))$.
3. *If $(M_1, N_1) \models \chi$, $|M_1| \geq |N_1|$.*

Varying Fraissé: setup

Definition

A generalized Fraissé class is a collection $\boldsymbol{K}$ of finite structures along with a notion $\prec_{\boldsymbol{K}}$ of strong substructure with the following properties.

- **A1**. If $A \in \boldsymbol{K}$ then $A \prec_{\boldsymbol{K}} A$.
- **A2**. If $A \prec_{\boldsymbol{K}} B$ then $A \subseteq B$.
- **A3**. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} B$, and $B \prec_{\boldsymbol{K}} C$ then $A \prec_{\boldsymbol{K}} C$.
- **A4**. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$.

We will fix a class $\boldsymbol{K}^0$ of closed structures such that for every $A \in \boldsymbol{K}$, there is a finite $B \in \boldsymbol{K}^0$ with $A \subseteq B$.

Hjorth's variation

In the context here we fix a class of closed submodels in advance we are assuming $A \in \boldsymbol{K}_0$ and in the examples in this paper we will verify that any member of $\boldsymbol{K}$ expands to a member of $\boldsymbol{K}^0$ with the same universe. We may then assume that $B_1, B_2 \in \boldsymbol{K}^0$.

These will be the 'algebraically closed substructures'.

Varying Fraissé: The theorem

Theorem

Let $\boldsymbol{K}$ be a collection of countably many finite τ-structures closed under substructure, satisfying joint embedding and amalgamation over closed sets. Then there is unique countable generic τ-structure with Scott sentence $\phi_{\boldsymbol{K}}$.

We haven't built in local finiteness. The first order theory may not be $\aleph_0$-categorical. But the generic will be atomic.

Duplicating Finite Structures

Definition

$\boldsymbol{K}$ satisfies

1. Amalgamation over closed sets if $A \prec_{\boldsymbol{K}} B_1$ and $A \prec_{\boldsymbol{K}} B_2$ there isa $C \in \boldsymbol{K}$ with $B_1 \prec_{\boldsymbol{K}} C$ and $B_2 \prec_{\boldsymbol{K}} C$.
2. Strong disjoint amalgamation if for $A \prec_{\boldsymbol{K}} B_1, B_2$ with $B_1 \cap B_2 = A$, there is an expansion of $B_1 \cup B_2$ which is a closed structure in $\boldsymbol{K}$.
3. duplication of finite structures if for every $A \prec_{\boldsymbol{K}} B$ and any n there is a strong disjoint amalgamation of n copies of B over A.

Duplication of finite substructures is what we are after.
Strong disjoint amalgamation is a sufficient condition.

Constructing Absolute Indiscernibles: setup

Notation

Fix a vocabulary τ. τ_1 is obtained by adding a unary predicate S, τ_2 is obtained by adding unary predicates M, N and a binary relation symbol P. τ_3 is obtained by adding a unary predicate S to τ_2.

If $\mathcal{M}$ is τ_2 structure, we say it is a (κ, λ)-model if $|M(\mathcal{M})| = \kappa$ and $|N(\mathcal{M})| = \lambda$

Constructing Absolute Indiscernibles: Theorem

Theorem

Let $\boldsymbol{K}$ be a collection of countably many finite τ-structures closed under substructure, satisfying joint embedding, amalgamation over closed sets and duplication of finite structures. For an appropriate expansions of the τ-structures in $\boldsymbol{K}$ to τ_3 -structure we obtain a $\boldsymbol{K}'$-generic τ_2-structure $\mathcal{M}$ with

1. There is a projection function p from M onto a set N such that the structure $\mathcal{M} = (M, N, p, \ldots)$ is a τ_2-full structure. $N(\mathcal{M})$ is a set of absolute indiscernibles in $\mathcal{M}$ and $M(\mathcal{M}) \restriction \tau$ is isomorphic to the generic structure for $\boldsymbol{K}$.
2. Further, there is a proper elementary extension of $\mathcal{M}$ fixing $N(\mathcal{M})$.

The Descriptive Set Theory

Definition

S_∞ divides the group H if there is a homomorphism from a closed subgroup of H onto S_∞.

Theorem

Let X is a set of absolute indiscernibles in a model $\mathcal{M}$ $\hat{\mathcal{M}}$ is the relativized reduct of $\mathcal{M}$ to $M(\mathcal{M})$ (so a τ-structure). In particular, if the structure $\mathcal{M}$ is built as above, $\mathrm{aut}(\mathcal{M})$ projects onto S_∞ and also S_∞ divides $\mathrm{aut}(\hat{\mathcal{M}})$, where $\hat{\mathcal{M}}$ is the relativized reduct of $\mathcal{M}$ to $M(\mathcal{M})$ (so a τ-structure).

Question:

Apparent DST theorem

S_∞ divides $\mathrm{aut}(\mathrm{N})$ for some countable τ-structure N then it is possible to expand N to a receptive τ_2 structure.

Section 2: The Second Red Herring

$\aleph_2$ or $\aleph_1$

Extendible models

Definition

M is an extendible atomic model in $\aleph_1$ of T_ϕ if $|M| = \aleph_1$ and there is a proper elementary extension of M which satisfies ϕ and is also atomic.

'No extendible model in $\aleph_1$' is the same as 'all models in $\aleph_1$ are extendible.'

Each of the three known ur-examples of theories with no model in $\aleph_2$ have all models in $\aleph_1$-maximal and (not accidentally) $2^{\aleph_1}$ models in $\aleph_1$.

The three examples

Examples

Complete Sentence of $L_{\omega_1,\omega}$ with no model in $\aleph_2$
aka Complete first order theories with no atomic model in $\aleph_2$

1. J. Knight (1977) ad hoc construction –$\aleph_1$-like linear order
2. Laskowski-Shelah (1993) Fraissé – dimension bound
3. Hjorth (2007) Fraissé – combinatorial

Why all models are maximal I: Setup

Definition

Let $f : \mathcal{P}_\omega(X) \mapsto \mathcal{P}(X)$.
We say $A \in \mathcal{P}_\omega(X)$ is independent (for f) if for every $A' \subseteq A$ and $a \in A'$, $a \notin f(A' - \{a\})$.

Somewhat tricky induction yields:

Lemma

Suppose f maps finite sets of $\mathcal{P}_\omega(X)$ to sets of cardinality strictly less $\aleph_m$. If $|X| = \aleph_{m+k}$ there is an independent set of size $k+1$ in X.

Why all models are maximal I: Theorem

The proof actually yields:

Theorem

Suppose $\boldsymbol{K}$ is a class of models that admits a uniformly definable function $(f^M : \mathcal{P}_\omega(M) \mapsto \mathcal{P}(M))$ for $M \in \boldsymbol{K}$. By uniform we mean if $M \subset N$, $f^N \restriction M = f^M$.

Suppose for all M and $A \in \mathcal{P}_\omega(M)$, $|f^M(A| \leq \aleph_n$ and no $M \in \boldsymbol{K}$ admits an independent set of $r+1$ elements. If $|M| = \aleph_{m+r}$ then M is not extendible.

Knight Example

In these two examples, cl is closure under functions in the vocabulary.

Knight's example

Julia Knight constructed by an ad hoc procedure a complete sentence ϕ_K in $L_{\omega_1,\omega}$ such that if $M \models \phi_K$, M is linearly ordered and all predecessors of any $a \in M$ are in $\mathrm{cl}(a)$ so the order is $\aleph_1$-like.

By our last theorem with $r = 1$ since there is no pair of independent elements every model in $\aleph_1$ is maximal.

Laskowski-Shelah Example

Laskowski-Shelah example

Laskowski-Shelah constructed by a Fraissé construction, a structure such that cl is locally finite on models of ϕ_{LS} (i.e. atomic models of the first order theory) and the sentence implies that there is no cl-independent set of cardinality 3.

By our last theorem with $r = 2$ since there is no pair of independent elements every model in $\aleph_1$ is maximal.

All $\aleph_1$ models are maximal II

Hjorth example

Hjorth constructed by a Fraissé construction two complete (see below) sentences that each characterize $\aleph_1$. The vocabulary τ_1 contains binary relations S_n, $k+2$-ary relations $T_k(x_0, x_1, y_0, \ldots y_{k-1})$.
We require a function $f : M^2 \mapsto \mathbb{N}$ (which is not in the formal language) such that:

1. each model M of ϕ_H satisfies for every pair a, b there is an n such that $M \models S_n(a, b)$ and
2. that a generic model $M \models T_k(a, b, c_0, c_{k-1})$, exactly if $\{c_0, \ldots c_{k-1}\}$ is the set of points on which $f(a, *) = f(b, *)$.

Clearly, there cannot be a model in $\aleph_1$ which is properly extended.

Strengthening the result

Theorem

If all atomic models in $\aleph_1$ of a complete first order theory are maximal there are $2^{\aleph_1}$ models in $\aleph_1$.

This follows easily from an early result of Shelah, chapter 7 in my monograph.

If all models in $\aleph_1$ are maximal, there is a maximal triple in $\aleph_0$ and this implies $2^{\aleph_1}$ models in $\aleph_1$.

Section 4: Automorphisms and receptive models

Finding receptive models

We discuss Hjorth's example; the construction was imbedded but not noticed in Laskowski-Shelah. We show the class supports finite duplication of structures.
Define $\boldsymbol{K}^0$ to be the finite structures that satisfy both conditions 1) and 2) demanded of the generic. Note that any member of $\boldsymbol{K}$ can be expanded to such a structure by first adding instances of new S_n to guarantee 1) and then defining T_k to satisfy 2) for each pair in the finite structure.

Since all 'algebraicity' has been pushed into the base, the class satisfies strong disjoint amalgamation over closed structures.

Hjorth's two examples

1. The first has the combinatorics but not the projection. The absolute indiscernibles are in T^{eq}.
2. The second has the projection and is receptive as defined above.

Dividing by S_∞

Clearly if M is receptive S_∞ divides $\mathrm{aut(M)}$.
But Knight's example is linearly ordered so S_∞ does not divide $\mathrm{aut(M)}$.

However the other two cases are receptive. What more can we say about the models of a receptive sentence?
Hjorth says the automorphism group of Knight's conjecture satisfies Vaught's conjecture even on analytic sets.
I don't know what this really means.

Models in $\aleph_1$ of a receptive sentence

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1. *There is a 1-1 isomorphism preserving function between the countable models of ψ and the models of the merger $\chi_{\theta,\psi}$.*
2. $\#(\chi, \lambda) = \max(\#(\theta, \lambda), \#(\psi, \lambda))$.
3. *If $(M_1, N_1) \models \chi$, $|M_1| \geq |N_1|$.*

It is by no means obvious (and probably false in $\aleph_1$) that if $M_1 \models \theta$ and $N_1 \models \psi$ then $(M_1, N_1) \models \chi$.

Section 5: A new version Harrington's construction

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Harrington's construction

Sy has a somewhat more direct argument. The main point is that the construction tell us nothing about the embedability of the models and so nothing really germane to Vaught's conjecture.
A goal would be to enhance the argument to show there is a pair of models in $\aleph_1$ with one contained in the other. But this is basically a problem of amalgamation of countable models.