

Geometry and Categoricity

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Zilber's Thesis

Geometry and
Categoricity

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Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Fundamental structures are canonical

Fundamental mathematical structures can be characterized in an appropriate logic.

Conversely, characterizable structures are 'fundamental'.

The relevant notion of 'characterize' is categoricity in power.

Whig History

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

If this term is unfamiliar see the wikipedia article.

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Specifying the thesis I

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Find an **axiomatization** for $\text{Th}(\mathcal{C}, +, \cdot, \text{exp})$.

Specifying the thesis II

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Zilber Conjecture

Every strongly minimal first order theory is

- 1 disintegrated
- 2 group-like
- 3 field-like

Specifying the thesis III

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Cherlin-Zilber Conjecture

Every simple ω -stable group is an algebraic group over an algebraically closed field.

This led to:

Is there an ω -stable field of finite Morley rank with a definable proper subgroup of the multiplicative group?

Challenge

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Hrushovski Construction I

There is a strongly minimal set which is not locally modular and not field-like. (Hrushovski)

Hrushovski Construction II

There is an ω -stable field of finite Morley rank with a definable proper subgroup of the multiplicative group. (Baudisch, Hils, Martin-Pizarro, Wagner)

Response

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Response I

Strengthen the hypotheses:

Extend first order to more powerful "Logics".

- 1 $L_{\omega_1, \omega}(Q)$
- 2 Zariski Structures

What is $L_{\omega_1, \omega}(Q)$?

Response II

Weaken the conclusion:

Replace first order interpretable in $(\mathcal{C}, +, \cdot)$ by 'analytically' definable.

Response

COMPLEX EXPONENTIATION

Geometry and
Categoricity

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Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Consider the structure $(\mathbb{C}, +, \cdot, e^x, 0, 1)$.

It is Godelian:

The integers are defined as $\{a : e^a = 1\}$.

The first order theory is undecidable and 'wild'.

ZILBER'S INSIGHT

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Maybe Z is the source of all the difficulty. Fix Z by adding the axiom:

$$(\forall x)e^x = 1 \rightarrow \bigvee_{n \in \mathbb{Z}} x = 2n\pi.$$

GEOMETRIES

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Definition

A **closure system** is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \rightarrow \mathcal{P}(G)$$

satisfying the following axioms.

- A1.** $cl(X) = \bigcup \{cl(X') : X' \subseteq_{fin} X\}$
- A2.** $X \subseteq cl(X)$
- A3.** $cl(cl(X)) = cl(X)$

(G, cl) is **pregeometry** if in addition:

- A4.** If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If points are closed the structure is called a geometry.

STRONGLY MINIMAL I

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

M is *strongly minimal* if every first order definable subset of **any elementary extension** M' of M is finite or cofinite.

$a \in \text{acl}(B)$ if for some $\mathbf{b} \in B$ and some $\phi(x, \mathbf{y})$:
 $\phi(a, \mathbf{b})$ and $\phi(x, \mathbf{b})$ has only **finitely many** solutions.

Exercise: If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from $\text{acl}(X)$ to $\text{acl}(Y)$.

STRONGLY MINIMAL II

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

A complete theory T is strongly minimal if and only if it has infinite models and

- 1 algebraic closure induces a pregeometry on models of T ;
- 2 any bijection between *acl*-bases for models of T extends to an isomorphism of the models

QUASIMINIMALITY I

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Tripl Definition M is '*quasiminimal*' if every first order $(L_{\omega_1, \omega}?)$ definable subset of M is countable or cocountable.

$a \in \text{acl}'(X)$ if there is a first order formula with **countably many** solutions over X which is satisfied by a .

Zilber

If M is quasiminimal then (M, acl') is a closure system.

First order case

Pillay, Bays (Itai, Tsuboi, Wakai) characterize when these closure systems are geometries in terms of symmetry properties of the generic type.

QUASIMINIMALITY II

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Exercise ? If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from $\text{acl}'(X)$ to $\text{acl}'(Y)$.

QUASIMINIMALITY II

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Exercise ? If f takes X to Y is an elementary isomorphism, f extends to an elementary isomorphism from $\text{acl}'(X)$ to $\text{acl}'(Y)$.

How is the geometry connected to the isomorphism type of M ?

Geometric Homogeneity

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

An infinite dimensional pregeometry (M, cl) is
Geometrically Homogenous if
for each finite $B \subset M$, $\{a \in M : a \notin \text{cl}(B)\}$ are the realizations
in M of a unique complete type in $S(B)$.

QUASIMINIMAL EXCELLENCE I

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Basic Conditions

Let \mathbf{K} be a class of L -structures such that $M \in \mathbf{K}$ admits a closure relation cl_M mapping $X \subseteq M$ to $\text{cl}_M(X) \subseteq M$ that satisfies the following properties.

- 1 Each cl_M defines a pregeometry on M .
- 2 For each $X \subseteq M$, $\text{cl}_M(X) \in \mathbf{K}$.
- 3 If f is a partial monomorphism from $H \in \mathbf{K}$ to $H' \in \mathbf{K}$ taking $X \cup \{y\}$ to $X' \cup \{y'\}$ then $y \in \text{cl}_H(X)$ iff $y' \in \text{cl}_{H'}(X')$.

Homogeneity over models

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

\aleph_0 -homogeneity over models

Let $G \subseteq H, H' \in \mathbf{K}$ with G empty or a countable member of \mathbf{K} that is closed in H, H' .

- 1 If f is a partial G -monomorphism from H to H' with finite domain X then for any $y \in \text{cl}_H(X)$ there is $y' \in H'$ such that $f \cup \{\langle y, y' \rangle\}$ extends f to a partial G -monomorphism.
- 2 If f is a bijection between $X \subset H \in \mathbf{K}$ and $X' \subset H' \in \mathbf{K}$ which are *separately* cl -independent (over G) subsets of H and H' then f is a G -partial monomorphism.

QUASIMINIMAL EXCELLENCE II

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

A class (\mathbf{K}, cl) is *quasiminimal excellent* if it admits a combinatorial geometry which satisfies on each $M \in \mathbf{K}$

- 1 Basic Conditions
- 2 \aleph_0 -homogeneity over countable models.
- 3 countable closure property (ccp)
- 4 and the 'excellence condition' which follows.

Towards Categoricity and Existence

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Beginning the induction

If (\mathbf{K}, cl) satisfies the basic conditions and \aleph_0 -homogeneity over countable models then

- 1 For any finite set $X \subset M$, if $a, b \in M - \text{cl}_M(X)$, a, b realize the same $L_{\omega_1, \omega}$ -type over X .
- 2 M is quasiminimal (for $L_{\omega_1, \omega}$) and (M, cl) is geometrically homogeneous.
- 3 There is one and only one model M in \aleph_1 .
- 4 Whence (by Shelah) there is an $L_{\omega_1, \omega}(Q)$ -extension in \aleph_2 .

Further Assumptions

To guarantee existence and uniqueness in larger cardinals we use ccp and excellence.

Notation for Excellence

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

In the following definition it is essential that \subset be understood as *proper* subset.

Definition

- 1 For any Y , $\text{cl}^-(Y) = \bigcup_{X \subset Y} \text{cl}(X)$.
- 2 We call C (the union of) *an n -dimensional cl -independent system* if $C = \text{cl}^-(Z)$ and Z is an independent set of cardinality n .

Elements of Excellence

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

There is a primary model over any finite independent system.

Let $C \subseteq H \in \mathbf{K}$ and let X be a finite subset of H . We say $\text{tp}_{\text{qf}}(X/C)$ is *defined over* the finite C_0 contained in C if it is determined by its restriction to C_0 .

Density of 'isolated' types

Let $G \subseteq H, H' \in \mathbf{K}$ with G empty or in \mathbf{K} . Suppose $Z \subset H - G$ is an n -dimensional independent system, $C = \text{cl}^-(Z)$, and X is a finite subset of $\text{cl}(Z)$. Then there is a finite C_0 contained in C such that $\text{tp}_{\text{qf}}(X/C)$ is defined over C_0 .

n-AMALGAMATION

Geometry and
Categoricity

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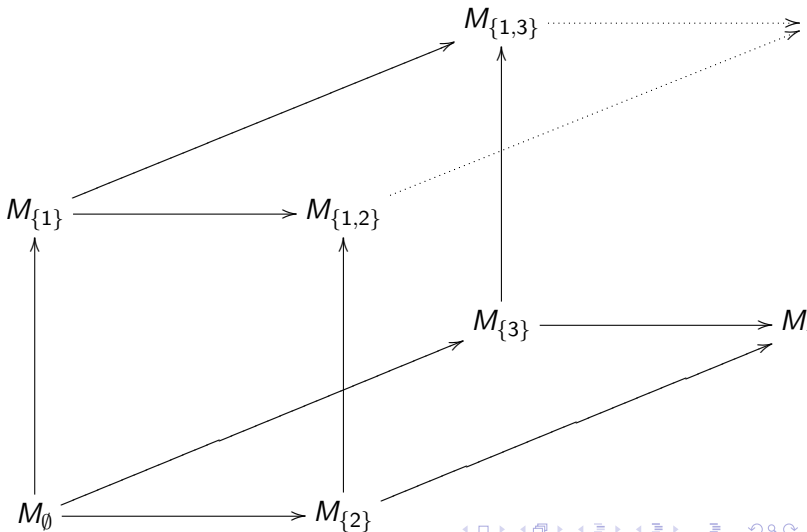
Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures



EXCELLENCE IMPLIES CATEGORICITY

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Excellence implies by a direct limit argument:

Lemma

An isomorphism between independent X and Y extends to an isomorphism of $\text{cl}(X)$ and $\text{cl}(Y)$.

This gives categoricity in all uncountable powers if the closure of each finite set is countable.

EXCELLENCE IMPLIES CATEGORICITY: Sketch

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Lemma

Suppose $H, H' \in \mathbf{K}$ satisfy the countable closure property. Let $\mathcal{A}, \mathcal{A}'$ be cl -independent subsets of H, H' with $\text{cl}_H(\mathcal{A}) = H$, $\text{cl}_{H'}(\mathcal{A}') = H'$, respectively, and ψ a bijection between \mathcal{A} and \mathcal{A}' . Then ψ extends to an isomorphism of H and H' .

Outline

We have the obvious directed union $\{\text{cl}(X) : X \subseteq \mathcal{A}; |X| < \aleph_0\}$ with respect to the partial order of finite subsets of X by inclusion.

And $H = \bigcup_{X \subseteq \mathcal{A}; |X| < \aleph_0} \text{cl}(X)$. So the theorem follows immediately if

for each finite $X \subseteq \mathcal{A}$ we can choose $\psi_X : \text{cl}_H(X) \rightarrow H'$ so that $X \subseteq Y$ implies $\psi_X \subseteq \psi_Y$.

We prove this by induction on $|X|$. If $|X| = 1$, the condition is immediate from \aleph_0 -homogeneity and the countable closure property. Suppose $|Y| = n + 1$ and we have appropriate ψ_X for $|X| < n + 1$. We will prove two statements.

- 1) $\psi_Y^- : \text{cl}^-(Y) \rightarrow H'$ defined by $\psi_Y^- = \bigcup_{X \subset Y} \psi_X$ is a monomorphism.
- 2) ψ_Y^- extends to ψ_Y defined on $\text{cl}(Y)$.

2) is immediate from excellence; 1) requires an argument.

CATEGORICITY

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Theorem Suppose the quasiminimal excellent (I-IV) class \mathbf{K} is axiomatized by a sentence Σ of $L_{\omega_1, \omega}$, and the relations $y \in \text{cl}(x_1, \dots, x_n)$ are $L_{\omega_1, \omega}$ -definable.

Then, for any infinite κ there is a unique structure in \mathbf{K} of cardinality κ which satisfies the countable closure property.

Note on Proof

The proof of existence of large models is inductive using categoricity in κ to obtain a model in κ^+ .

Any κ -categorical sentence of $L_{\omega_1, \omega}$ can be replaced (for categoricity purposes) by considering the atomic models of a first order theory. ($EC(T, Atomic)$ -class)

Shelah defined a notion of excellence; Zilber's is the 'rank one' case for $L_{\omega_1, \omega}$.

Zilber shows sufficiency for certain $L_{\omega_1, \omega}(Q)$ -sentences.

ω -stability

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

$(\mathbf{K}, \prec_{\mathbf{K}})$ is the class of atomic models of a first order theory under elementary submodel.

Definitions

$p \in S_{at}(A)$ if $a \models p$ implies Aa is atomic.

\mathbf{K} is ω -stable if for every countable model M , $S_{at}(M)$ is countable.

Theorem

[Keisler/Shelah]

$(2^{\aleph_0} < 2^{\aleph_1})$ If \mathbf{K} has $< 2^{\aleph_1}$ models of cardinality \aleph_1 , then \mathbf{K} is ω -stable.

Goodness

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Definition

A set A is *good* if the isolated types are dense in $S_{at}(A)$.

For countable A , this is the same as $|S_{at}(A)| = \aleph_0$.

Are there prime models over good sets?

Goodness

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Definition

A set A is *good* if the isolated types are dense in $S_{at}(A)$.

For countable A , this is the same as $|S_{at}(A)| = \aleph_0$.

Are there prime models over good sets?

YES, In \aleph_0 and \aleph_1

but not generally above \aleph_1 (J. Knight, Kueker,
Laskowski-Shelah).

Yes, if excellent.

Independence is now defined in terms of splitting.

Definition

1 \mathbf{K} is (λ, n) -good if for any independent n -system \mathcal{S} (of models of size λ), the union of the nodes is good.

That is, there is a prime model over any countable independent n -system.

2 \mathbf{K} is *excellent* if it is (\aleph_0, n) -good for every $n < \omega$.

Essence of Excellence

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Let \mathbf{K} be the class of models of a sentence of $L_{\omega_1, \omega}$.

\mathbf{K} is excellent

\mathbf{K} is ω -stable and any of the following equivalent conditions hold.

For any finite independent system of countable models with union C :

- 1 $S_{at}(C)$ is countable.
- 2 There is a unique primary model over C .
- 3 The isolated types are dense in $S_{at}(C)$.

Excellence implies large models

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Shelah proved:

Theorem

Let λ be infinite and $n < \omega$. Suppose \mathbf{K} has $(< \lambda, \leq n + 1)$ -existence and is (\aleph_0, n) -good. Then \mathbf{K} has (λ, n) -existence.

Excellence implies large models

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Shelah proved:

Theorem

Let λ be infinite and $n < \omega$. Suppose \mathbf{K} has $(< \lambda, \leq n + 1)$ -existence and is (\aleph_0, n) -good. Then \mathbf{K} has (λ, n) -existence.

This yields:

Theorem (ZFC)

If an atomic class \mathbf{K} is excellent and has an uncountable model then

- 1 it has models of arbitrarily large cardinality;
- 2 if it is categorical in one uncountable power it is categorical in all uncountable powers.

Enough categoricity implies Excellence

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

**Generalized
Amalgamation
and Existence**

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

VWGCH: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$.

Enough categoricity implies Excellence

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

VWGCH: $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$.

'Very few models in \aleph_n means at most $2^{\aleph_{n-1}}$.

VWGCH: Shelah 1983

An atomic class \mathbf{K} that has at least one uncountable model and very few models in \aleph_n for each $n < \omega$ is excellent.

Show by induction:

Very few models in \aleph_n implies $(\aleph_0, n - 2)$ -goodness.

Transition

Geometry and Categoricity

John T.
Baldwin

Introduction

Canonicity of Fundamental Structures

Quasiminimal
excellence

Generalized Amalgamation and Existence

Examples

Model
Homogeneity

Zariski Structures

Analytic Structures

I have discussed the general theory of categoricity for infinitary classes.

We now consider some examples.

Categoricity up to \aleph_ω is essential.

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1, \omega}$ sentence ϕ_k such that:

- 1 ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
- 2 ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.

Categoricity up to \aleph_ω is essential.

Theorem. [Hart-Shelah / Baldwin-Kolesnikov] For each $3 \leq k < \omega$ there is an $L_{\omega_1, \omega}$ sentence ϕ_k such that:

- 1 ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
- 2 ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$.
- 3 ϕ_k has the disjoint amalgamation property;
- 4 Syntactic types determine Galois types over models of cardinality at most \aleph_{k-3} ;
- 5 But there are syntactic types over models of size \aleph_{k-3} that split into $2^{\aleph_{k-3}}$ -Galois types.
- 6 ϕ_k is not \aleph_{k-2} -Galois stable;
- 7 But for $m \leq k - 3$, ϕ_k is \aleph_m -Galois stable;

\mathbb{Z} -Covers of Algebraic Groups

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Definition A \mathbb{Z} -cover of a commutative algebraic group $\mathbb{A}(\mathcal{C})$ is a short exact sequence

$$0 \rightarrow \mathbb{Z}^N \rightarrow V \xrightarrow{\exp} \mathbb{A}(\mathcal{C}) \rightarrow 1. \quad (1)$$

where V is a \mathbb{Q} vector space and \mathbb{A} is an algebraic group, defined over k_0 with the full structure imposed by $(\mathcal{C}, +, \cdot)$ and so interdefinable with the field.

Axiomatizing \mathbb{Z} -Covers: first order

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Let \mathbb{A} be a commutative algebraic group over an algebraically closed field F .

Let $T_{\mathbb{A}}$ be the first order theory asserting:

- 1 $(V, +, f_q)_{q \in \mathbb{Q}}$ is a \mathbb{Q} -vector space.
- 2 The complete first order theory of $\mathbb{A}(F)$ in a language with a symbol for each k_0 -definable variety (where k_0 is the field of definition of \mathbb{A}).
- 3 \exp is a group homomorphism from $(V, +)$ to $(\mathbb{A}(F), \cdot)$.

Axiomatizing Covers: $L_{\omega_1, \omega}$

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Add to T_A

$\Lambda = \mathbb{Z}^N$ asserting the kernel of exp is standard.

$$(\exists \mathbf{x} \in (\exp^{-1}(1))^N)(\forall y)[\exp(y) = 1 \rightarrow \bigvee_{\mathbf{m} \in \mathbb{Z}^N} \sum_{i < N} m_i x_i = y]$$

Categoricity Problem

Geometry and
Categoricity

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Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Is $T_A + \Lambda = \mathbb{Z}^N$ categorical in uncountable powers?

paraphrasing Zilber:

Categoricity would mean the short exact sequence is a reasonable 'algebraic' substitute for the classical complex universal cover.

What is known?

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

If \mathbb{A} is

\mathbb{Z} -covers

- 1 (\mathcal{C}, \cdot) then quasiminimal excellent (Zilber)
- 2 (\tilde{F}_p, \cdot) then **not small**.
Each completion is quasiminimal excellent. (Bays-Zilber)
- 3 elliptic curve w/o cm then ω -stable (Gavrilovich/Bays)
qme (Bays)
- 4 elliptic curve w cm then not ω -stable as \mathbb{Z} -module
but qme as $\text{End}(\mathbb{A})$ -module (Bays)

Geometry from Model Theory

Geometry and Categoricity

John T.
Baldwin

Introduction

Canonicity of Fundamental Structures

Quasiminimal excellence

Generalized Amalgamation and Existence

Examples

Model Homogeneity

Zariski Structures

Analytic Structures

Zilber has shown *equivalence* between certain ‘arithmetic’ statements about Abelian varieties (algebraic translations of excellence) and categoricity below \aleph_ω of the associated $L_{\omega_1, \omega}$ -sentence.

The equivalence depends on weak extensions of set theory and Shelah’s categoricity transfer theorem.

Hrushovski Construction

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

The dimension function

$$d : \{X : X \subseteq_{fin} G\} \rightarrow \mathbf{N}$$

satisfies the axioms:

D1. $d(XY) + d(X \cap Y) \leq d(X) + d(Y)$

D2. $X \subseteq Y \Rightarrow d(X) \leq d(Y).$

THE GEOMETRY

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Definition

For A, b contained M , $b \in \text{cl}(A)$ if $d_M(bA) = d_M(A)$.

Naturally we can extend to closures of infinite sets by imposing finite character. If d satisfies:

D3 $d(X) \leq |X|$.

we get a full combinatorial (pre)-geometry with exchange.

ZILBER'S PROGRAM FOR $(\mathcal{C}, +, \cdot, \exp)$

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Goal: Realize $(\mathcal{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1, \omega}$ -sentence discovered by the Hrushovski construction.

Objective A

Expand $(\mathcal{C}, +, \cdot)$ by a unary function f which behaves like exponentiation using a Hrushovski-like dimension function. Prove some $L_{\omega_1, \omega}$ -sentence Σ satisfied by $(\mathcal{C}, +, \cdot, f)$ is categorial and has quantifier elimination.

Objective B

Prove $(\mathcal{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

PSEUDO-EXPONENTIAL

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

E is a pseudo-exponential if for any n linearly independent elements over \mathbb{Q} , $\{z_1, \dots, z_n\}$

$$d_f(z_1, \dots, z_n, E(z_1), \dots, E(z_n)) \geq n.$$

Schanuel conjectured that true exponentiation satisfies this equation.

THE AXIOMS

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

$$L = \{+, \cdot, E, 0, 1\}$$

$$(K, +, \cdot, E) \models \Sigma \text{ if}$$

- 1 K is an algebraically closed field of characteristic 0.
- 2 E is a homomorphism from $(K, +)$ onto (K^\times, \cdot) and there is $\nu \in K$ transcendental over \mathbb{Q} with $\ker E = \nu\mathbb{Z}$.
- 3 E is a pseudo-exponential
- 4 K is strongly exponentially algebraically closed.

CONSISTENCY AND CATEGORICITY

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

For a finite subset X of an algebraically closed field k with a partial exponential function. Let

$$\delta(X) = d_f(X \cup E(X)) - \text{Id}(X).$$

Apply the Hrushovski construction to the collection of such (k, f) with

$\delta(X) \geq 0$ for all finite X
and with standard kernel.

The result is a quasiminimal excellent class.

ALGEBRA FOR OBJECTIVE A:

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Definition

A *multiplicatively closed divisible subgroup* associated with $a \in \mathcal{C}^*$, is a **choice** of a multiplicative subgroup isomorphic to \mathbb{Q} containing a .

ALGEBRA FOR OBJECTIVE A:

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Definition

$b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \dots, b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$, determine the isomorphism type of $b_1^{\mathbb{Q}}, \dots, b_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ over F if given subgroups of the form $c_1^{\mathbb{Q}}, \dots, c_\ell^{\mathbb{Q}} \subset \mathcal{C}^*$ and ϕ_m such that

$$\phi_m : F(b_1^{\frac{1}{m}} \dots b_\ell^{\frac{1}{m}}) \rightarrow F(c_1^{\frac{1}{m}} \dots c_\ell^{\frac{1}{m}})$$

is a field isomorphism it extends to

$$\phi_\infty : F(b_1^{\mathbb{Q}}, \dots, b_\ell^{\mathbb{Q}}) \rightarrow F(c_1^{\mathbb{Q}}, \dots, c_\ell^{\mathbb{Q}}).$$

Thumbtack Lemma

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Theorem (thumbtack lemma)

For any $b_1, \dots, b_\ell \in C^$, there exists an m such that $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \dots, b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset C^*$, determine the isomorphism type of $b_1^{\mathbb{Q}}, \dots, b_\ell^{\mathbb{Q}} \subset C^*$ over F .*

TOWARDS EXISTENTIAL CLOSURE

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

The Thumbtack Lemma (over finite independent systems of fields) implies

the consistency of the strong exponential existential closure axioms with the rest of Σ .

These existential closure axioms imply the models of Σ satisfy:

- 1 the homogeneity conditions and
- 2 excellence.

GENUINE EXPONENTIATION?

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Schanuel's conjecture: If x_1, \dots, x_n are \mathbb{Q} -linearly independent complex numbers then $x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}$ has transcendence degree at least n over \mathbb{Q} .

Theorem (Zilber)

$(\mathcal{C}, +, \cdot, \exp) \in \mathcal{EC}_{st}^*$. $(\mathcal{C}, +, \cdot, \exp)$ has the countable closure property.

assuming Schanuel

Marker extended by Günaydin and Martin-Pizarro have verified existential closure axioms for $(\mathcal{C}, \cdot, \exp)$ for irreducible polynomials $p(X, Y) \in \mathbb{C}[X, Y]$.

Axiomatizability/Characterizability

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence

Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Quasiminimal Excellence is defined semantically.

But

- 1 (Kirby) The class of models of a quasiminimal excellent class is axiomatizable in $L_{\omega_1, \omega}(Q)$.
- 2 Note the universal covers are axiomatized in $L_{\omega_1, \omega}$.

AMALGAMATION PROPERTY

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

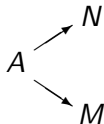
Examples

Model
Homogeneity

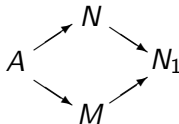
Zariski
Structures

Analytic
Structures

The class \mathbf{K} satisfies the *amalgamation property for models* if for any situation with $A, M, N \in \mathbf{K}$:



there exists an $N_1 \in \mathbf{K}$ such that



SET AMALGAMATION PROPERTY

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

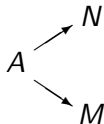
Examples

Model
Homogeneity

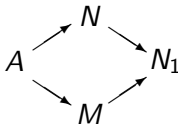
Zariski
Structures

Analytic
Structures

The class \mathbf{K} satisfies the *set amalgamation property* if for any situation with $M, N \in \mathbf{K}$ and $A \subset M, A \subset N$:



there exists an $N_1 \in \mathbf{K}$ such that



Is there a difference?

Geometry and
Categoricity

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Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

For a complete first order theory, Morley taught us:
There is no difference.

Is there a difference?

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

For a complete first order theory, Morley taught us:

There is no difference.

Tweak the language and we obtain set amalgamation.

(Tweak: put predicates for every definable set in the language)

There is a difference!

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Zilber's examples of quasiminimal excellent classes have amalgamation over models but the interesting examples do **not** have set amalgamation.

Model amalgamation is the natural notion for study in infinitary logic (AEC).

Challenge and Response

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

**Model
Homogeneity**

Zariski
Structures

Analytic
Structures

We return to the challenge of the Hrushovski construction.

Challenge and Response

Response: Strengthen the Hypotheses

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Zariski Structures: Motive

Generalize the notion of Zariski geometry.
Specialize the notion of strongly minimal set.

Zariski Structures: Technique

Define axioms on a set X and the powers X^n specifying a topology and relations between the powers to characterize the notion of 'smooth algebraic variety'.

Positive result: Gloss

Geometry and
Categoricity

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Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Hrushovski-Zilber

Every non-linear (ample) Noetherian Zariski structure is interpretable in an algebraically closed field.

Positive result: Hrushovski-Zilber (detail)

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

If M is an ample Noetherian Zariski structure then there is an algebraically closed field K , a quasiprojective algebraic curve $C_M = C_M(K)$ and a surjective map

$$p : M \mapsto C_M$$

of finite degree such that for every closed $S \subseteq M^n$, the image $p(S)$ is Zariski closed in C_M^n (in the sense of algebraic geometry);

if $\hat{S} \subseteq C_M^n$ is Zariski closed, then $p^{-1}(\hat{S})$ is a closed subset of M^n (in the sense of the Zariski structure M).

Whoops!

Geometry and
Categoricity

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Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

But if the ‘ample’ hypothesis is dropped there are finite covers of $P^1(\mathbf{K})$ that can **not** be interpreted in an Algebraically closed field.

Such counterexamples arise from finite covers of the affine line and of elliptic curves.

These structures induce an n th root quantum torus.

Response II: Weaken the conclusion

Geometry and
Categoricity

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Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Instead of interpreting the models in algebraically closed fields by first order formulas, find an analytic model.

This is impossible for the finite rank case but has interesting consequences for infinite rank.

Green Fields

Geometry and
Categoricity

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Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

The naughty or green field (Poizat)

Expand an algebraically closed field by a unary predicate for a proper subgroup of the multiplicative group.

$$\delta(X) = 2d_f(X) - \text{ld}(X \cap G).$$

This yields an ω -stable theory of rank $\omega \times 2$.

Concrete Models (Zilber/ Caycedo)

Assume Schanuel's conjecture and CIT; let $\epsilon = 1 + i$.

1 The naughty field.

$$(\mathcal{C}, +, \cdot, \mathbb{G})$$

where $G = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Q}\}$
and \exp is complex exponentiation.

Concrete Models (Zilber/ Caycedo)

Assume Schanuel's conjecture and CIT; let $\epsilon = 1 + i$.

- 1 The naughty field.

$$(\mathcal{C}, +, \cdot, \mathbb{G})$$

where $G = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Q}\}$
and \exp is complex exponentiation.

- 2 A superstable version (emerald field)

$$(\mathcal{C}, +, \cdot, \mathbb{G}')$$

where $G' = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Z}\}$
and \exp is complex exponentiation.

Non-Commutative Geometry

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

Three related objects

- 1 A structure
- 2 A finitely generated non-commutative C^* -algebra
- 3 A foliation

Analytic Zariski Structures

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Just as Zariski structures are defined axiomatically to generalize algebraic geometry,

Analytic Zariski structures are defined axiomatically to generalize the properties of analytic subsets of the complex numbers.

A key distinction from Zariski structures is the loss of the Noetherian property.

Non-Commutative Geometry and Model Theory I

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

Assume Schanuel's conjecture; let $\epsilon = 1 + i$.

Quantum Tori

The concrete superstable (emerald field) version
 \mathbb{R}^2/G' where

$$G' = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Z}\}$$

and \exp is complex exponentiation

is the leaf space of the Kronecker foliation, the quantum torus.

Non-Commutative Geometry and Model Theory II

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence

Generalized
Amalgamation
and Existence

Examples

Model
Homogeneity

Zariski
Structures

Analytic
Structures

a near Quantum Tori

The concrete naughty (green) field.

\mathbb{R}^2/\mathbb{G} where

$$\mathbb{G} = \{\exp(\epsilon t + q) : t \in \mathbb{R}, q \in \mathbb{Q}\}$$

and \exp is complex exponentiation. is apparently a 'new' structure to topologists, a quotient of the quantum torus by \mathbb{Q} .
(Baldwin-Gendron)

Consequences of Zilber's Thesis

Geometry and
Categoricity

John T.
Baldwin

Introduction

Canonicity of
Fundamental
Structures

Quasiminimal
excellence
Generalized
Amalgamation
and Existence
Examples
Model
Homogeneity

Zariski
Structures

Analytic
Structures

- 1 Deeper understanding of complex exponentiation
- 2 Specific conjectures in many areas of mathematics
- 3 Significance of Infinitary Logic
- 4 Broad Connections across mathematics and physics?!