Three Mathematical Cultures

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The title is inspired by the famous discussion of 'two cultures' by C.P. Snow[6]. He spoke in the late 50's of the divergence between the scientific and humanistic cultures in Great Britain. Our topic is much more localized. The transmission of mathematics to the younger generation in the United States is largely determined by three groups: teachers in the schools, mathematics educators, and university mathematicians. These groups differ in several respects. The working conditions and responsibilities of the university faculty differ from those of teachers in the schools. The views that mathematics teachers and mathematics educators have of mathematics can be contrasted with those of research mathematicians. All share certain views of the centrality of mathematics that distinguish them from the rest of the world. All have certain strengths that they can contribute to the improvement of mathematics education. I discuss here some of these differences and similarities in the hopes of promoting more effective interactions among these communities.

By a Mathematics Educator I mean a person, usually but not exclusively holding a Ph.D. in Mathematics Education, whose primary intellectual activity is the understanding of how mathematics is learned and transmitting this understanding to others. The predominant theme of Mathematics Education (as represented in scholarly journals) is to disseminate this understanding to others who are interested in theoretical issues. A second literature and active oral tradition is concerned with translating this understanding into practical recommendations for teaching and conveying this insight to teachers. The

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latter work seems to be distinctly undervalued in the profession. In my view, the predominance of the first group is due to the insistence of the university on the publication of 'scholarly' work and the accepted definition of scholarship.

Several subcultures are being ignored in this discussion. There is, for example, a much wider difference between the elementary school teachers for whom mathematics is one of several subjects and high school teachers than any of the distinctions analyzed here. I focus on high school teachers of mathematics out of sheer personal ignorance of the elementary schools. Less severe but still important is the distinction between the active research mathematician (our topic here) and other university mathematicians. Similarly, general statements here about the thoughts of mathematics educators should be taken with a grain of salt. They also come in many subspecies and my contacts with them are much sparser than with the other groups. I know a few well, I have attended several conferences and heard 10-15 talks by mathematics educators and have read fairly carefully several books and no more than 50 papers in the field. Nevertheless, I will continue in a fairly didactic style so the reader can be amused or outraged rather than bored.

My own background is predominantly that of a research mathematician. My main focus in the twenty years since my doctorate has been on mathematical logic. Only in the last 4 or 5 years have I seriously worried about either undergraduate or high school education. In working with high school teachers for three years in a major teacher enhancement project, I have learned more of their ethos. My partner on this project is the only mathematics educator with whom I have had really extended conversations on the material here. This paucity of contact with the Mathematics Education community is reflected in the less thorough treatment of their viewpoint in this article.

During my work with the high schools I have observed that a number of misunderstandings were caused by people with different backgrounds using the same words in quite different ways or having different expectations of the normal way that things are done. It is this cultural difference that I hope to explain here. In some cases I hope to explain some of the research mathematicians prejudices to the other two cultures. In the other direction, I hope to help other research mathematicians avoid some mistakes I have made. Most centrally, we cannot rely on our memory or introspection to describe what happens in the schools. First, we must discount the value of recalling how we, as talented mathematicians, learned mathematics as a guide to the behavior of the average student. Further, we must discount our observation of our current college students who are the upper rank of contemporary students. Finally, we must realize that there have been major changes in methods and content of teaching and in the society at large since we were in elementary school that affect the knowledge of the current student. I have found visiting high school classrooms to be the best way to discover how high school students actually understand mathematics.

1 Attitudes to Mathematics

We analyze below some of the different views of mathematics of the different groups. Of course, these views are formed by how and what mathematics they learn. Most high school teachers will have had a slightly specialized undergraduate degree in mathematics. That is, they will have studied the first two years of mathematics with engineers, scientists and other mathematics majors. Generally, this would cover calculus and introductory linear algebra and differential equations. They probably would have one or two years of courses which are aimed at future high school teachers (usually one or two semesters of geometry). Sometimes they take analysis or advanced calculus with other mathematics majors; sometimes there is a special sequence. In addition, they would have taken several electives, chosen from courses such as probability, discrete mathematics, number theory, abstract algebra, logic, and point set topology. A future Ph.D. is likely to have taken courses of the same sort but a full year of analysis and abstract algebra. Frequently, research mathematicians will have taken at least one graduate sequence in mathematics as an undergraduate. But while the teacher begins to work, the future research mathematician takes three to four mathematics courses per year for three or four years and then begins research in mathematics. This education is completed by writing the first research paper, the Ph.D. thesis. The prospective high school teacher will have had several courses in the College of Education and a semester of supervised teaching experience. The researcher will have had almost no formal training in teaching but will have several years of part-time experience with varying levels of responsibility and rather minimal supervision.

Mathematics Educators come in many varieties and it is hard to generalize about their backgrounds. Most will have a mathematics background somewhat stronger than most teachers - perhaps through a Master's degree in mathematics. Some will have taught in the schools. A few have Ph.D.'s in mathematics and research experience. But most will have graduate work in the College of Education with courses in Psychology, Applied Statistics, and advanced work in the area of Mathematics Education.

In addition to quantity, there is a major difference in focus between the mathematics-oriented and the education-oriented programs. All but three or four of the future teachers' courses regard mathematics as a series of techniques for solving problems. In contrast most of the mathematician's work after the first two years of college focus on the systematic development of one area or another. Learning to prove results and to recognize that results must be proved is the core of this training. Law Schools recognize explicitly that their students must learn a completely new method of analyzing arguments. They instill new standards of rigor by vigorously rooting out sloppy reasoning with ridicule and constant pressure. The graduate school in mathematics has to instill an even more precise notion of rigor. The job is somewhat easier because the abstraction of mathematics makes it easier to separate the new reasoning from the old. But the same shift in precision of language and analysis is the goal and in general research mathematicians have made this shift when dealing with mathematics. Thus, the mathematician is likely to regard 'mathematics' as dealing solely with this sort of reasoning.

Three examples of mathematician teacher interaction may illuminate these differences in viewpoint. The following problem was posed at an in-service for teachers in our College Preparatory Mathematics Program (CPMP): How many squares (composed of contiguous squares of the given board) are there on an 8 by 8 checkerboard? By chance, our group consisted of two researchers and two high school teachers. One of the teachers rather quickly provided us with the correct answer ($\sum_{i=0}^{8} i^2$) and was astonished when the researchers pressed him about how he knew it was right. The question was not about the formal induction to do this for an arbitrary size checkerboard but about what is necessary to describe the intutive step of counting the number of $k \times k$ subsquares.

A second example arose at the Regional Geometry Institute. By means of paper folding the group 'saw' that the lines connecting the midpoints of any quadrilateral form a parallelogram. I and one of the teachers spent 10 unsuccessful minutes trying to prove this theorem. The other three teachers in the group measured to verify that it was a parallelogram and went on to explore a number of other interesting properties of the figure. One of these eventually provided the clue for proving the original proposition. This incident reinforces a reservation about the wonders of Geometric Sketchpad. How will students learn to mistrust the picture (surely one of the most important lessons of elementary geometry) if the picture is always right?

A third example concerns the following problem that appeared on one of the commercially available worksheets. A butcher has one each of a 1,3,9 and 27 kilogram weight. Show that he can weigh with a balance scale each integer weight up to 40 kg. (This is illustrated with some pictures that show sometimes you put weights on both sides.) One of the teachers said they had solved the problem; they had exhibited a solution for each integer less than 40. Of course, that is all the sheet asked for, but it is hard for me to see the point of even high school freshmen doing a problem if that is considered a full explanation. In my view, this is a rich and wonderful problem if students see that they are looking at all polynomials in 3^i for $i \leq 3$ and coefficients ± 1. (I don't mean they should understand this terminology.) After writing this paragraph, I attended a ninth grade class where the students did the problem with little trouble. I remarked to the teacher that I thought the problem was about base three arithemic and she replied that for these students the point was to understand the balancing of equations. This is certainly a valid point and we have not had an opportunity to explore it further. I remain hopeful that further use of the setting to emphasize the powers of three would be useful.

2 Attitudes towards teaching

Two of these three cultures regard the actual teaching of mathematics as an essential part of their duties: the mathematician and the high school teacher. In contrast, the teaching function of the mathematics educator is primarily directed to the teaching of 'methods' or 'theory of learning'. Some are based in Departments of Mathematics and a portion of their teaching load is then similar to the research mathematician's. In addition, many mathematics educators regularly teach and observe in the schools. This provides both a background for formulating and a laboratory for testing their hypotheses on learning. The high school teacher and the research mathematician live near two poles on the continuum of research/teaching. The high school teacher displays little interest in mathematics as the mathematician views it but is very interested in specific mathematical problems which are relevant for conveying ideas to his students.

On the income tax form the high school teacher lists his profession as teacher; the university professor as mathematician. Thus, for the high school teacher, teaching is paramount and the specific subject is secondary; for the university professor very much the opposite is the case. This distinction has important implications for classroom behavior. Classically, university professors regard their task as the presentation of the mathematics. If there is any attempt to motivate the student, it is only intellectual: In terms of what we have learned before, why should the present material be studied? High school teachers are intimately connected with the lives of their students. For example, keeping students in school and steering them out of gangs, are often far more important and pressing questions than the internal logic of the mathematics. The high school teacher knows quite well what mathematics students do and do not know. The professor is only much more vaguely aware of this. The high school teacher self-consciously pursues various teaching strategies to deal with different problems and different motivations of students. The university professor has both a much more limited arsenal of such techniques and much less interest in using them. Here the professor has much to learn.

The following aphorism is well-known to high school teachers and practically unknown to university faculty. 'The neophyte teaches material; the experienced teacher teaches students'. A high school teacher explained to me that this really meant that only the second was really teaching. Further she asked, 'Do you think that well-prepared students, who come to the university prepared for calculus, do not need to be taught but are able to teach themselves?' It seems that this question really comes to grips with the distinction. My reaction is that this student is now ready to be taught. One does not have to be concerned with motivating the student to work but only with motivating the particular result or viewpoint. But, contrary to the assumption of the high school teacher, there is a great deal left. Each time I teach an upper division or graduate course, I revise the order of the subject, give a different proof of a major result, choose new examples, or emphasize different aspects of the subject. And these are essential teaching techniques which have little to do with methods of presentation such as cooperative learning or concrete problems. They are all what it means to really teach the material. The crux of learning is viewed not as an interpersonal relation but a common relation of the teacher and the student to the mathematics. Now, the basic fact is that for many students in high school and the early years of university this kind of relationship with mathematics does not exist and the teacher must both try to develop the relationship and in the meantime find substitutes for it. This is teaching in the second sense.

A second teacher made this point even clearer. He said, "There is the first level of teaching where the teacher acts as encourager, is supportive, etc. But, students really only concentrate on activities that are 'cool'. The art is to convince those students who set the tone that math is cool." Now teaching is really an advanced form of social engineering. And many high school teachers and almost all research mathematicians would assert that this is not part of their job.

These different meanings for 'teach' represent a paradigm of the cultural distinction I am trying to describe. There are two aspects of teaching: subject oriented and person oriented. The two cultures emphasize the two distinct meanings to such an extent that they may talk past each other if the word 'teach' is used without explanation.

The high school teacher and the university mathematician are teaching different populations – not just the same students four years older. Even the students in the university remedial courses come from the upper half of the high school class and the mathematics majors represent the most mathematically talented 1% of students in the country. This fact profoundly influences the kind of teaching that is appropriate.

The actual teaching situation of the university and high school teacher is also quite different. Both may teach 150-200 students at a time. But the high school teacher will have 5 classes of 30; the university faculty members may teach one class of over 150 (learning high school mathematics or at best calculus) and one of three to ten graduate students in their speciality. But more often the researcher will have two classes of under 30 students: one in the research area and one other upper division course. The high school teacher has 20-25 contact periods per week, the professor 6. The professor expects to spend several hours preparing for each hour of lecture in an advanced course - perhaps learning the material for the first time. An elementary course may demand up to an hour organizing a lecture on well known material. The well-qualified high school teacher rarely teaches unfamiliar material. With four or five preparations a day it is is impossible to spend an hour on each class. But what preparation time can be found is concerned with motivating students to learn the material. High school teacher do all their own grading. While university professors grade their advanced courses, homework in the elementary courses is graded (if at all) by teaching assistants and possibly undergraduates. At least once and perhaps several times a semester a pool of instructors and teaching assistants join in the 5 or 6 hour ordeal of grading the hundreds of responses to an exam they have set for an elementary course. Neither high school teachers nor university professors spend much time discussing teaching methods. High school teachers have little time for discussion; university faculty spend their time discussing their research interests and the administration of the university.

3 Professsional situation

University faculty in any discipline have a probationary period of approximately six years. At this time a tenure decision is based primarily on quality and quantity of published research. This is a real decision; the percentage of individuals who are actually denied tenure is lowered by those who see the handwriting on the wall. The decision is made by a sequence of committees beginning with members of the faculty member's department and reviewed by various further faculty committees in the university. Letters from external reviewers (that are not available to the candidate) play a major factor in both hiring and tenure decisions. High school faculty are hired by principals or District personnel officers based on transcripts, interviews, and letters of reference (which are often available to the candidate). Tenure may be awarded after a probationary period of approximately three years. The major factor in the award is evaluation by the principal. Elaborate procedures to regulate this power are built into the union contract.

One crucial distinction among the cultures is really a distinction between the governance of high schools and universities. Major decisions in professional life are routinely taken by university faculty: considerable choice in courses to teach, free choice of texts in many courses, vast flexibility in daily schedule. Further important decisions are made at the departmental level (with various degrees of democracy depending on the department): salary, recruitment of colleagues, choice of which courses to offer, major design of the curriculum. While the high school teacher has some input into issues such as specific courses and texts, most of these issues are decided by Department Head, Principal, School Board, or State Department of Education. The right to make these kinds of decisions is an important aspect of the professional respect of the university faculty. One of the major contributions the professor can make to a high school-university collaboration is to extend the same respect to high school teachers and thereby encourage the teachers' participation in educational decisions. Our experience has shown that teachers who take the initiative can influence the climate of instruction in their schools far more than their formal role would seem to permit.

No one in any of these groups is 'in it for the money'. All are employed by state or not-for-profit organizations which are not known for high pay. Concern for students or subject matter is a prime motivation for members of all three cultures. The financial situation of members of the three cultures depends on many different factors. The salaries of high school teachers vary greatly (by seniority and the school district in which they are employed). The salaries of university faculty vary even more widely, both by university and within the department by perceived merit. Average salaries in the most highly paid high school districts exceed the average of university faculty at many universities. The most highly paid university faculty make far more than the highest paid high school teachers. Actual teaching performance is virtually irrelevant to the salary in either situation. The chief determinant of salary in the research university is quantity and quality of published research; the chief components in the high schools are seniority and highest degree (or additional coursework). In general mathematics educators at research universities are relatively badly paid; at non-research schools this may be reversed. The recent influx of grant money for mathematics education affects this issue at nonresearch universities and may in the future change the situation at research universities.

Salaries for both high school teachers and university faculty are generally for 9 months of teaching. At major research universities from 1/4 to 2/3 of the faculty have been supported on research grants (2 months pay) during the summer. The number and size of these grants is decreasing because of the federal budget problems. Perhaps another 1/4 of the faculty will teach summer school at the university. Other summer employment is unheard of. The time must be spent on research if the faculty member has any hope of advancement. Some high school teachers teach summer school; others attend summer schools or institutes; others work at non-education related jobs.

The center of professional life for research mathematician is not the university where they teach but the small group of mathematicians scattered around the country and around the world who understand the details of their work. The mathematician is primarily interested in a specific research program. Publication of several original results each year is essential for advancement. Usually, no more than two or three members of the same department are able to read this work. Many of these theorems may be jointly proved with another mathematician – often at another university or even in another country.

A description of how these results are discovered is an entirely different topic (addressed by such mathematicians as Polya, Hadamard, and Poincairé). It fits well into the general task of Mathematics Educators. But there are some exterior signs that can be enumerated here. Several hours are spent each week in research seminars where a faculty member or graduate student presents a recent result. In general, these are lectures – not discussions. An audience of 10 is exceptionally large. Those who are particularly interested in a topic supplement these formal presentations with hours of informal discussion. The main content of professional meetings in a specific subject are 6-8 lectures a day on new work. Researchers have read tens of research papers fairly closely related to their speciality. Such papers are usually read carefully at a rate of a page or so an hour. Sometimes from conversation or inspired browsing one is able to focus immediately on the key paragraphs but understanding these paragraphs may require days of effort. Such papers suggest specific problems or general areas for further investigation. Hours of contemplation or discussion with another expert will bring small steps of progress. Almost any significant result will be thought both proved and disproved several times before a final correct argument emerges. Sometimes breakthroughs occur in the midst of work; more often one wakes up with the correct idea (maybe).

Mathematics Educators have studied a similar literature relating to their research. There are some papers which make very technical analyses, especially those with sophisticated statistics. However, in general, the papers are less dense but many more need to be read. In fact, a specific genre exists within the education literature of papers which systematically review and report the results of many papers on the same subject. This is very different from a survey paper in mathematics. A survey paper in mathematics reviews a field and in passing makes historical remarks about contributing papers; it is not a literature review.

But much research in Mathematics Education is in many ways closer to research in the physical sciences than to traditional mathematics. That is, it requires observation if not experimentation with a group of students. Just as a Biologist requires several years to set up a laboratory before fruitful work can be expected, the Mathematics Educator must spend several years developing a relationship with schools, teachers and students before it is possible to draw reasonable conclusions. Working with persons instead of molecules makes this a much more difficult task. If the project involves a massive implementation of change in the schools the difficulties increase exponentially.

Mathematics faculty are widely believed to have one of the least stressful of occupations [5][1]. High school teaching is certainly one of the most stressful. A mathematics educator involved in a major project inherits the administrative stress. This stess is exacerbated by the exigencies of applying for and justifying grants while balancing the competing bureaucracies of university and school system. There is however, a more subtle strain on the Calvinist mathematician. Administrative tasks are never done but are always being done; at the end of each day some small steps have been taken. In the midst of a proof one may go days with no advance or the advance is to discover that last week's work was wrong.

There are a number of ways in which members of these cultures participate in mathematical activities beyond teaching. There are professional organizations of all three groups which hold state, regional, national, and international meetings. Most research mathematicians regard attendance at conferences, either general mathematics, or very specialized conferences organized around their research area as central to their professional lives, and I could omit professional. A normally active research mathematician would attend one or two weekend meetings and at least one week long meeting each year. Research grants routinely include support for attending such meet-

[1] I cite this volume as evidence that mathematicians are thought to have low stress jobs; not as evidence that they do. The methodology of the book would make such a claim hazy. Three different occupations are considered. Mathematicians have the 47th lowest stress rating, high school teachers are 170th and college professors are 201st lowest. But 2/3 of mathematicians are said be college professors. Sorting out this contradictory information would be difficult but the stress on college professors is explicitly related to tenure anxiety. So, if one restricted to tenured faculty, the original point would hold.

ings. Faculty without research grants may get some reimbursement for the expenses of attending meeting but this varies greatly by university. Major speakers receive nominal honorariums. Mathematical conferences charge low registration fees relative to most professions: Engineering, Computer Science, Law, Medicine and Education charge substantially more.

The most active teachers are involved in the professional organizations. There are many well-attended state and regional conferences; more than 14,000 teachers attended the 1992 national meeting of the NCTM (National Council of Teachers of Mathematics) in Seattle. The NCTM has about 90,000 members. About 3000 attend the combined annual meeting of the AMS (American Mathematical Society) and the MAA (Mathematical Association of America). The AMS has about 25,000 members and the MAA 30,000 with a significant dual membership of about 8,000. Broadly speaking, the AMS draws its membership from research mathematicians and the MAA from small college and junior college faculty but the number of dual members illustrates the ambiguity of these characterizations. Travel support is much more available to college faculty although some teacher travel is supported by their local district or by federal Eisenhower funds.

In addition to these conferences of professional associations, high school teachers often attend various kinds of in-service activities paid for by the school district or some granting agency. These in-service activities frequently try to reach all teachers and usually pay \$8-\$10 per hour for attendance. There is a significant opportunity for mathematics educators to work as consultants leading such in-services. The going rate is approximately \$300/day and goes considerably higher for extremely popular speakers.

4 Mathematics and mathematics education

During the 80's one of the tides in mathematics education was an emphasis on problem solving. As a mathematician I have two reactions to this movement. On the one hand, the stress on problems arising in applications and open-ended problems as opposed to context-free drill is very welcome. On the other, perusal of many of the problems in use (notably on published worksheets) shows a lack of coherence that is dismaying. Isolated problems are presented because they are thought-provoking with little concern for the overall development of the students' mathematical knowledge.

The different cultures seem to have different intuitions of the role of 'problems' in mathematics. There certainly are some mathematicians who enter the field precisely because they like to solve problems. The tradition of the Mathematics Olympiad, the Putnam exam, and so on witnesses this. There are, however, many mathematicians who find such context free puzzles abhorrent (and usually difficult). A second major type of mathematician is driven to organize and explain a collection of results. A solution of a particular problem is not useful unless it fits into a coherent framework. The relative importance of these viewpoints among mathematicians is indicated by the following fact. Absolutely first rate mathematicians have been dismissed as contenders for the Fields Medal as 'mere problem solvers'. Here a 'mere problem solver' is a person who has solved a number of open problems but not displayed a unified program for developing an area of mathematics.

I was surprised to learn when I first began to referee NSF proposals that the crucial issue in evaluating a research proposal is not, 'Is the proposer likely to progress on the question?' but 'Is this a good problem?' The former is a necessary but secondary condition. Thus the evaluation not of mathematical truth but of mathematical significance is central to the professional life of a mathematician. A major challenge to researchers who work with teachers is: Convey to the teachers first the fact that mathematical researchers are constantly concerned with this question of significance, secondly that it should be a major factor in the high school teacher's selection of materials.

The three cultures perceive quite differently the role of abstraction in mathematics and even more its role in teaching. The university mathematician views earlier mathematics as the most important source for the development of mathematics. Mathematics Educators and high school teachers stress the relationship of mathematics to the 'real world'. This difference is to some extent caused by the specific mathematics being dealt with. School mathematics is in fact more closely related to the physical world than that taught in the university. But this truism often seems to obscure the real nature of high school mathematics. Most of the problems of 9th grade algebra arise from the intellectual curiosity of previous generations. They are attempts to solve more and more problems, which are suggested by analogy with physical problems. Attempts to reformulate them as physical problems (an endeavor I regularly engage in) are always somewhat artificial and occasionally ludicrously so. (Many university mathematicians regard the entire enterprise of teaching systems of linear equations to business students as such an example. Certainly the texts are a good source of silly examples.)

Both teacher education and modern school curriculum are heavily influenced by the significant developments in mathematics education during the last few decades. A research mathematician who is interested in high school education must become acquainted with this work. I will discuss two areas of Mathematics Education constructivism and the analysis of hierarchies of abstraction. These ideas have influenced curriculum development and classroom practice.

Constructivism holds that each student must construct the meanings of each concept that is encountered. This notion has been elaborated through both introspective reconstruction of the process of knowing (as in traditional epistemological analysis) and by systematic investigation of student behavior [3]. The application of these ideas to curriculum development makes an implicit assertion that most students are not yet ready for the kind of abstraction that earlier high school students succeeded at. This statement is not contradictory because the success of the earlier generations was for 5 or 10 percent of the 18 year olds; we now would like to teach serious mathematics to most of the 18 year olds. Thus there is a push to concretize high school mathematics, to emphasize the solution of particular concrete problems, and to use manipulatives rather than to just elucidate the general schemes for solving a large family of problems.

This proposal is contrasted with much of current high school teaching that focuses on the rote learning of algorithms for the simplification of algebraic expressions and the solutions of equations. These exercises are to be replaced by more concrete and open-ended problems. Many excellent problems have been constructed in carrying out this program. However, it seems to me a full diagnosis of this difficulty requires careful consideration of the exact material involved.

Consider, for example, the treatment of conic sections in precalculus textbooks. The connections to geometry are given short shrift. Rather students are forced to memorize certain standard forms and the way to read off from these standard forms certain parameters (e.g. location of the foci) which given the previous short treatment of the definitions are meaningless to the students. This is more abstract than finding the parameters for a specific conic given in a 'real world' situation and seeing the meaning of it. But the difficulty is not solely the abstraction but the lack of discussion of the internal mathematical motivation, which of course could be given more concrete expression with examples like the focus of a parabolic mirror. However, the real significance of these standard forms could be demonstrated by considering what Descartes actually wrote [4]. One of the motives he explicitly advances for the introduction of analytic geometry is to determine a good hierarchy (degree) to classify curves. To show the hierarchy is appropriate, he must locate the conics at the first level (i.e. quadratic functions) and he derives these parameters in the process. Using his analysis, Descartes solves several construction problems posed by the Greeks which would make excellent activities with Geometric Sketchpad. Placing the study of the analytic geometry of conics in this context would give the students a reasonable basis for understanding motivation for the subject as well as illustrating the historical development of mathematics.

There are a number of educational theories descended from Piaget that try to identify various levels of abstraction for students (or specific responses of students) [1], [7]. Without denying the possible efficacy and correctness of these distinctions, they seem very fine and vastly incomplete distinctions to a mathematician who views the levels of abstraction as e.g. High School Algebra, Abstract Algebra, metamathematics. It is abundantly clear from observation that students can understand the rudiments of linear algebra (and thus attain the highest levels in these scales) and yet be unable to prove results about vector spaces. Moreover, this same student may do proof in Euclidean geometry quite well so the difficulty is not simply a failure to reach the 'proof stage'. This provides one example of how massive confusion can occur between the mathematician and the learning theorist. The analysis of mathematical learning is still primarily focused on younger children. This is not an overriding objection, the theory has to start there. The difficulty arises with prescriptions that seem to overgeneralize from such a narrow base.

In the last two paragraphs I have discussed two strains of research in Mathematics Education which have contributed a great deal to the current reform movement. In each case I suggest a difficulty which arises, in my view, from the (albeit natural) development of this research from studies of younger students and general rather than specifically mathematical learning. More contact between researchers in mathematics and mathematics education could enrich the effect of the Mathematics Educator's insights on curriculum development.

The aims of the Mathematics Reform 'movement" are codified in a document written by Mathematics Educators and teachers and endorsed by the

AMS and MAA. The Curriculum and Evaluation Standards for School Mathematics [2] address both a process approach to mathematics (Standard 1: Mathematics as Problem Solving, Standard 2: Mathematics as Communication, Standard 3: Mathematics as Reasoning) and changes in the content of the curriculum. Major changes in both pedagogy and material are recommended. One important theme is the distinction between a 'teacher centered' and a 'student centered' classroom. A mathematician who wants to understand these recommendations must not only read the Standards but visit classrooms to see the current situation.

The practical insights of high school teachers into student motivations and abilities, the psychological studies of mathematics educators, and the overview of mathematics of the research mathematicians can all play crucial roles in the current reform movement. The mathematics educator who is able to transform these insights into real changes in teachers is even more essential. In this article I have attempted to outline some distinctions between these cultures in the hope that avoiding some misunderstandings can further this collaboration. In that respect I have tried to speak to all three cultures. Each culture needs be aware of the different ways the others may interpret words and situations. But I can only really sensibly advise the research mathematician who wants to participate in the reform of mathematics education in the schools.

To this research mathematician I say: Inform yourself. Relying on your recollection and introspection as a guide to how current American students learn is folly. Read the Mathematics Education literature and talk with high school teachers. But most important, visit the schools. Attend both traditional classes and those taught by teachers who are strongly influenced by the Standards. These activities will not only inform you about mathematics education but improve your own teaching.

References

[1] J.B. Biggs and K.F. Collis. *Evaluating the Quality of Learning.* Academic Press, 1982.

[2] National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics.* NCTM, Reston, Va., 1989.

[3] Robert B. Davis. *Learning Mathematics.* Ablex Publishing Co., 1984.

[4] Rene Descartes. *The Geometry of Rene Descartes.* Dover, 1954. Translated by David Eugene Smith and Marcia L. Latham: 1924; originally published 1637 in French.

[5] Les Krantz. *Jobs Rated Almanac.* Pharos Books, 1992.

[6] C.P. Snow. *The Two Cultures and a Second Look.* Cambridge, 1964.

[7] Zalman Usiskin. Van Hiele levels and achievement in secondary school geometry, CDASSG project. 1982.