The Stability spectrum for classes of atomic models

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February 20, 2012

Abstract
We prove two results on the stability spectrum for $L_{\omega_1, \omega}$. Here $S^m(M)$ denotes an appropriate notion (at or mod) of Stone space of $m$-types over $M$. Theorem A. (unstable case) Suppose that for some positive integer $m$ and for every $\alpha < \delta(T)$, there is an $M \in K$ with $|S^m(M)| > |M|^{2^{\alpha}(|T|)}$. Then for every $\lambda \geq |T|$, there is an $M$ with $|S^m(M)| > |M| = \lambda$. Theorem B. (strictly stable case) Suppose that for every $\alpha < \delta(T)$, there is $M_\alpha \in K$ such that $\lambda_\alpha = |M_\alpha| \geq 2^{\alpha}$ and $|S^m(M_\alpha)| > \lambda_\alpha$. Then for any $\mu$ with $\mu^{\aleph_0} > \mu$, $K$ is not $\iota$-stable in $\mu$. These results provide a new kind of sufficient condition for the unstable case and shed some light on the spectrum of strictly stable theories in this context. The methods avoid the use of compactness in the theory under study. In the Section 4, we expound the construction of tree indiscernibles for sentences of $L_{\omega_1, \omega}$. Further we provide some context for a number of variants on the Ehrenfeucht-Mostowski construction.

1 Context
For many purposes, e.g., the study of categoricity in power, the class of models of a sentence $\phi$ of $L_{\omega_1, \omega}$ can be profitably translated to the study of the class of models of a first order theory $T$ that omit a collection $\Gamma$ of first order types over the empty set. In particular, if $\phi$ is complete (i.e. a Scott sentence) $\Gamma$ can be taken as the collection of all non-principal types and the study is of the atomic models of $T$. This translation dates from the 60’s; it is described in detail in Chapter 6 of [Bal09].

∗ We give special thanks to the Mittag-Leffler Institute where this research was conducted. This is paper 959 in Shelah’s bibliography. Baldwin was partially supported by NSF-0500841. Shelah thanks the Binational Science Foundation for partial support of this research.
diagrams (see below) is equivalent to studying sentences of \( L_{\omega_1 \omega} \); the study of atomic models of a first order theory is equivalent to studying complete sentences of \( L_{\omega_1 \omega} \).

The stability hierarchy provides a crucial tool for first order model theory. Shelah \[She78a\] and Keisler \[Kei76\] show the function \( f_T(\lambda) = \sup \{ |S(M)| : |M| = \lambda, M \models T \} \) has essentially only six possible behaviors (four under GCH). In \[She70\], Shelah establishes a similar result for homogeneous finite diagrams. The homogeneity assumption is tantamount to assuming amalgamation over all sets. This is a strong hypothesis that is avoided in Shelah’s further investigation of categoricity in \( L_{\omega_1 \omega} \) ([She83a, She83b]), which is expounded as Part IV of \[Bal09\]. Important examples, due to Marcus and Zilber, which do not satisfy the homogeneity hypothesis are also described in [Bal09]. As we explain below, this investigation begins by identifying the appropriate notion of type over a set (and thus of \( \omega \)-stability). Shelah ([She83a, Bal09]) showed that \( \omega \)-stability implies stability in all powers. And assuming \( 2^{\aleph_0} < 2^{\aleph_1} \), \( \omega \)-stability was deduced from \( \aleph_1 \)-categoricity. But further questions concerning the stability hierarchy for this notion of type for arbitrary sentences of \( L_{\omega_1 \omega} \) had not been investigated. We do so now.

Remark 1.1 In fact our results hold for arbitrary finite diagrams, the class of models of first order theory that omit a given set of types over the empty set. Thus they generalize some of the results of \[She70\]. But our results are by no means as complete as in homogeneous case. We have chosen to formulate the results for atomic classes because of the connection with the strong results of \[She83a, She83b\]. But we are using only the property that atomic classes are defined by omitting all non-principal types over the empty set. We do not in this paper use the stronger properties of atomic classes exploited in the arguments of \[She83a, She83b\].

There are (at least) two a priori reasonable notions of Stone space for studying atomic models of a first order theory. (As noted, we could more generally replace ‘atomic’ by ‘finite diagram’.) Recall that for a first order theory \( T \) (with a monster model \( M \)) \( A \subset M \) is an atomic set if each finite sequence from \( A \) realizes a principal type over the empty set. An atomic set is an atomic model if it is also a model of the theory \( T \).

Definition 1.2 Let \( K \) be the class of atomic models of a complete first order theory.

1. Let \( A \) be an atomic set; \( S_{\text{at}}(A) \) is the collection of \( p \in S(A) \) such that if \( a \in M \) realizes \( p \), \( Aa \) is atomic.

2. Let \( A \) be an atomic set; \( S_{\text{mod}}(A) \) is the collection of \( p \in S(A) \) such that \( p \) is realized in some \( M \in K \) with \( A \subseteq M \).

In \[Bal09\] we wrote \( S^* \) for the notion called \( S_{\text{mod}} \) here. The latter notation is more evocative. We will simultaneously develop the results for both notions of Stone space and indicate the changes required to deal with the two cases. We will write \( S_i(M) \) where \( i \) can be either at or mod.

We sometimes write \(|T|\) for \(|\tau|\) where \( \tau \) is the vocabulary of \( T \). \( K = K_T \) is the class of atomic models of \( T \). We write \( H = H(\mu) \) for the Hanf number for atomic
models of all theories with $|T| = \mu$. By [She78a] $H$ equals $\beth_\delta(T)$, where $\delta(T)$, the well-ordering number of the class of models of a theory $T$ omitting a family of types, is defined in VII.5 of [She78a]. It is also shown there that if $T$ is countable, $H$ evaluates as $\beth_{\omega_1}$ while for uncountable $T H = \beth_{(2|T|)^+}$. Fix $\mu_\alpha = \beth_{\alpha(|T|)}$.

**Remark 1.3** In [She70], Shelah’s definition of stability makes a stronger requirement; it implies by definition the existence of homogeneous models in certain cardinals. We do not make that assumption here so we are considering a larger class of theories.

**Definition 1.4**
1. $K$ is $i$-stable in $\lambda$ (for $i = \text{at}$ or \text{mod}) if for every $m < \omega$, and $M \in K$ with $|M| = \lambda$, $|S^m_i(M)| = \lambda$.

2. **Stability classes.** For either $i = \text{at}$ or \text{mod},
   - (a) $K$ is $i$-stable if it is $i$-stable in some $\lambda$.
   - (b) $K$ is $i$-superstable if it is $i$-stable in all $\lambda \geq H$.
   - (c) $K$ is strictly $i$-stable if it is $i$-stable but not $i$-superstable.

For any $M$, $S_{\text{at}}(M)$ contains $S_{\text{mod}}(M)$ so at-stability in $\lambda$ implies mod-stability in $\lambda$. Thus for both notions $\omega$-stability implies stability in all powers by results of [She83a, She83b], expounded in [Bal09].

We prove Theorem A in Section 2 and Theorem B in Section 3. The proof of Theorem B uses an application of omitting types in Ehrenfeucht-Mostowski models generated by trees of the form $<\omega \lambda$. This is by no means new technology but we weren’t able to locate an explicit statement of the result so we include a proof in Section 4.

We acknowledge helpful discussions with Tapani Hyttinen, Ali Enayat, Alexei Kolesnikov, and Lynn Scow.

### 2 Unstable $K$

We first show that if there are cardinals $\lambda_\alpha$ in which $K$ is ‘sufficiently unstable’, then $K$ is not stable in any cardinal. This is a partial response to the following question.

**Question 2.1** Must an atomic class that is unstable in all $\lambda$ have the order property?

**Definition 2.2** A class of atomic models has the order property if in large enough models $M$ in the class there is some first order $\phi$:

$$M \models (\phi(c_t, a_s) \equiv \phi(c_t, b_s)) \text{ iff } s <_t t.$$  

and the set $\{c_t : t \in I\}$ is an atomic set.

Theorem 2.3 falls short of answering this question in two ways: the hypothesis is stronger than mere instability; the sequence constructed in conditions 3) and 4) of Lemma 2.7 does not have all the sequences contained in a single atomic set but only requires each triple to be atomic.

The argument with splitting in Theorem 2.3 derives from Theorems I.2.6/I.2.7 of [She78b] showing that if in a stable first order theory, $\kappa(T)$ is finite in a stability cardinal it is finite in all cardinals.
**Theorem 2.3** Suppose that for some positive integer \( m \) and for every \( \alpha < \delta(T) \), there is an \( M_{\alpha} \in K \) with \( |S^n(T)(M_{\alpha})| > |M_{\alpha}|^{2^{|T|}} \). Then for every \( \lambda \geq |T| \), there is an \( M \) with \( |S^n(T)(M)| > |M| = \lambda \).

**Remark 2.4 (Proof Sketch)** Before the formal proof we outline the argument. We start with a sequence of models \( M_{\alpha} \) and many distinct types over each of them. By an argument which is completely uniform in \( \alpha \), we construct triples \( \langle a_{\alpha,i}, b_{\alpha,i}, d_{\alpha,i} \rangle \) for \( i < \mu^+_{\alpha} \) with the required properties. Finally, this set of indiscernibles easily yields models of all cardinalities with the required properties.

**Remark 2.5** The idea of the proof can be seen by ignoring the \( \alpha \) and proving a slightly weaker result from one model of size \( \beth \).

**Notation 2.6** In this section, \( \lambda_{\alpha} = |M_{\alpha}|^{2^{|T|}} \); \( \mu_{\alpha} = \beth_{\alpha}(|T|) \); \( \kappa_{\alpha} = \beth_{\alpha+2}(|T|) \).

**Lemma 2.7** There is \( \Phi \), proper for linear orders, in a vocabulary \( \tau_{\Phi} \) extending \( \tau \) with \( |\tau_{\Phi}| = |\tau| \), with fixed additional unary predicates \( P, P_1 \) and binary \( R \) such that:

1. For every linear ordering \( I \), \( N_I = EM_{\tau}(I, \Phi) \models T \) and \( M_I = EM_{\tau}(I, \Phi) \models T \). Naturally, \( J \subseteq I \) implies \( N_J \subset N_I \) and \( M_J \subset M_I \).

2. The skeleton of \( N_I \) is \( \langle a_i, b_i, c_i : i \in I \rangle \) and \( lg(c_i) = m \).

3. For some first order \( \phi \):

\[
N_I \models (\phi(c_t, a_s) \equiv \phi(c_t, b_s)) \text{ iff } s <_I t.
\]

4. \( M_I \cup c_i \subset N_I \) and is atomic.

5. For \( S_{mod}(M) \), we add the requirement that for each \( s \in I \),

\[
M_{I,s} = N_I \models \{ d : N_I \models R(d, c_s) \}
\]

is an atomic elementary submodel of \( N_I \) containing \( M_I c_s \).

Proof. The proof of Lemma 2.7 requires a number of steps. Fix for each \( \alpha < \delta(T) \), \( M_{\alpha} \in K \) with \( |M_{\alpha}| = \lambda_{\alpha} \) such that \( |S^n(T)(M_{\alpha})| > \lambda_{\alpha} = |M_{\alpha}|^{2^{|T|}} \). Fix \( p_{\alpha,i} \) for \( i < \lambda^+_{\alpha} \), a list of distinct types in \( S^n(T)(M_{\alpha}) \). We work throughout in a monster model \( M \) of \( T \).

**Notation 2.8** In the following construction, we choose by induction triples \( \langle a_{\alpha,i}, b_{\alpha,i}, d_{\alpha,i} \rangle \) for \( i < \mu^+_{\alpha} \). We use the following notation for initial segments of the sequences.

1. \( D_{\alpha,i} = \{ d_{\alpha,j} : j < i \} \).
2. $X_{\alpha,i} = \{a_{\alpha,j}, b_{\alpha,j} : j < i\}$.

3. $q_{\alpha,i}$ is the type of $d_{\alpha,i}$ over $X_{\alpha,i}$.

The following variant on splitting is crucial to carry out the construction. We call it ex-splitting (for external) because the elements which exemplify splitting are required to satisfy the same type over a set $D$ which is not in (so external to) the model $M$ and, in particular, is not required to be realized in an atomic set.

**Definition 2.9** Let $M$ be a model, $X \subseteq M$ and $D \subseteq \mathbb{M}$. We say that $p \in S^m_\alpha(M)$ ex-splits over $(D, X)$ if there exist $a, b \in M$, $f \in \mathbb{M}$ so that $f$ realizes $p \rest X$, $a \equiv_D b$ but $(a, f)$ and $(b, f)$ realize different types over $\emptyset$.

We will apply the next claim to $M_\alpha$, $X_{\alpha,i}$, and $D_{\alpha,i}$ when carrying out the construction in paragraph 2.12. Note that this computation does not depend on $|M|$.

**Claim 2.10** For any model $M$, the number of types in $S^m_\alpha(M)$ that do not ex-split over a pair $(D, X)$ with $|X| = |D| \leq \mu_\alpha$ is at most $\mu_{\alpha+2}$.

Proof. Let $P$ denote the collection of $tp(e/M)$ with $\lg(e) = m$ that do not ex-split over a pair $(D, X)$. Each type $r$ in $P$ is determined by knowing $r \rest X$ and for each formula $\phi_i(x_1, \ldots, x_{k_i})$ for $i < |T|$ the restriction of $r$ to one $k_i$-tuple from each equivalence class of the equivalence relation $E_k$ on $M$ defined by $aE_k b$ if $a$ and $b$ realize the same $k_i$-type over $D$. So, since $|D| = \mu_\alpha$, there are at most

$$2^{\mu_\alpha} \times (2^{2^{\mu_\alpha}})^{|T|} = (2^{2^{\mu_\alpha}})^{|T|} = \mu_{\alpha+2}$$

possible such $r$. \qed

As noted, for each $M_\alpha$ we will be constructing by induction on $i < \mu^+_\alpha$, sets $X_{\alpha,i}, D_{\alpha,i}$ of cardinality $\mu_\alpha$. We need to choose in advance a type $p_\alpha$ which does not ex-split over any $(X_{\alpha,i}, D_{\alpha,i})$ that arises. In order to do that we restrict the source of $D_{\alpha,i}$: clearly $X_{\alpha,i} \subseteq M_\alpha$. That is, we will fix $M'_\alpha$ with $M_\alpha < M'_\alpha$, $|M'_\alpha| = \lambda_\alpha$ and $M'_\alpha$ is $\mu_\alpha^+$-saturated and choose $D_{\alpha,i} \subseteq M'_\alpha$. (Note then that $M'_\alpha$ is not in general atomic.)

The number of types in $S^m_\alpha(M_\alpha)$ that do not ex-split over any pair $(D, X)$ with $|X| = |D| = \beth_\alpha$ is bounded by the number of such sets, $|M'_\alpha|^{|X|}$, times the number of types in $S^m_\alpha(M_\alpha)$ that do not ex-split over a particular choice of $(D, X)$, which is $\mu_{\alpha+2}$ by Claim 2.10. That is, the bound is $|M'_\alpha|^{|X|} \times \mu_{\alpha+2}$. Since this number is less than $\lambda^{\beta(T)}_\alpha$, we can fix a type $p_\alpha \in S^m_\beta(M_\alpha)$ which does not ex-split over any of the relevant $(D, X)$.

**Definition 2.11** For each $\alpha < \delta(T)$, fix $M'_\alpha$ with $M_\alpha < M'_\alpha$, $|M'_\alpha| = \lambda_\alpha$, and $M'_\alpha$ is $\mu^+_\alpha$ saturated. Choose, by induction on $i < \mu^+_\alpha$, triples $e_{\alpha,i} = (a_{\alpha,i}, b_{\alpha,i}, d_{\alpha,i})$ where

a) $d_{\alpha,i} \in M'_\alpha$.

b) $a_{\alpha,i}, b_{\alpha,i}$ are sequences of the same length from $M_\alpha$ that realize the same type over $D_{\alpha,i} = \{d_{\alpha,j} : j < i\}$. 5
c) The types over the empty set of \((a_{\alpha,i}, d_{\alpha,i})\) and \((b_{\alpha,i}, d_{\alpha,i})\) differ.

d) \(q_{\alpha,i} = p_{\alpha} \restriction X_{\alpha,i} = \text{tp}(d_{\alpha,i}/X_{\alpha,i})\) so if \(j < i\), \(q_{\alpha,j} \subseteq q_{\alpha,i}\).

e) \(M_{\alpha}d_{\alpha,i}\) is an atomic set for each \(i\). (In the mod-version \(N_{\alpha,i}\) is an atomic model containing \(M_{\alpha}d_{\alpha,i}\).)

**Construction 2.12** Choose \(d_{\alpha,i}\) to realize \(p_{\alpha} \restriction X_{\alpha,i}\). By Claim 2.10 and since \(|S_{i}(M_{\alpha})| > \lambda_{\alpha}\) we can choose \(a_{\alpha,i}\) and \(b_{\alpha,i}\) to satisfy conditions b) and c). So we have

\[
\text{tp}(d_{\alpha,i}, a_{\alpha,j}) = \text{tp}(d_{\alpha,i}, b_{\alpha,j}) \text{ if and only if } i < j.
\]

We want this order condition for a single formula. For each \(i \in \mu_{\alpha}^{+}\), the types of \((a_{\alpha,i}, d_{\alpha,i})\) and \((b_{\alpha,i}, d_{\alpha,i})\) differ. That is, \(\phi_{\alpha,i}(a_{\alpha,i}, d_{\alpha,i})\) and \(\neg\phi_{\alpha,i}(b_{\alpha,i}, d_{\alpha,i})\) for some \(\phi_{\alpha,i}\). By the pigeon-hole principal we may assume the \(\phi_{\alpha,i}\) is always the same \(\phi_{\alpha}\). (Further, since \(|T|\) is not cofinal in \(\delta(T)\), we can assume the \(\phi_{\alpha}\) is the same \(\phi\) for all \(\alpha\).)

Now the construction is completed. We expand \(\tau\) to a language \(\tau_{\Phi} \supseteq \tau\) by adding predicates \(P, <, R\) and Skolem functions. We add Skolem axioms to \(T\) to get a theory \(T_{1}\) that admits quantifier elimination, requiring that these Skolem functions applied to elements of \(P\) (\(P_{n}\)) give an element of \(P\) (\(P_{n}\)) so that \(P\) (\(P_{n}\)) will pick out an elementary submodel. (We make a similar requirement for \(R(x, y)\) in the mod-case.) Let \(M_{\alpha}^{\tau}\) be a model of \(T_{1}\) (submodel of \(M_{\alpha}^{\nu}\)) with cardinality \(\mu_{\alpha}^{+}\) containing \(M_{\alpha}\) and all the \(d_{\alpha,i}\) (\(N_{\alpha,i}\) in the mod-case). Interpret \(P\) as the model \(M_{\alpha}\), \(P_{n}\) as \(M_{\alpha}^{\nu}\), and the relation < as the ordering on the triples \(\langle e_{\alpha,i} : i < \mu_{\alpha}^{+}\rangle\) imposed by \(\phi_{\alpha}\).

Assign the Skolem functions so that the \(e_{\alpha,i}\) generate \(M_{\alpha}^{\nu}\) and interpret \(R\) by

\[
R = \{ e \cdot d_{\alpha,i} : e \in M_{\alpha}, i < \mu_{\alpha}^{+}\}.
\]

(In the \(S_{\text{mod}}(M)\) case, interpret \(R\) as \(\{ e \cdot d_{\alpha,i} : i < \mu_{\alpha}^{+}, e \in N_{\alpha,i}\}\).)

**Notation 2.13** Let \(\Gamma\) be the collection of types \(P_{n} \cup Q_{n}\). Each non-principal \(n\)-type \(q\) over the empty set determines one element of \(P_{n}\) and each non-principal \(n + m\)-type \(q\) determines one element of \(Q_{n}\):

1. \(P_{n} = \{ \bigwedge_{i < n} P(x_{i}) \} \cup \{ q(x) : q\text{ is a non-principal } n\text{-type} \}
2. \(Q_{n} = \{ \bigwedge_{i < n} R(x_{i}, y) \} \cup \{ q(x,y) : q\text{ is a non-principal } n + m\text{-type}, m < \omega \}

Now apply Morley’s omitting types theorem\(^1\) to the \(\tau_{\Phi}\)-theory \(T_{1}\) and the collection of \(M_{\alpha}^{\tau}\) to get a countable sequence \(\Gamma\) of order indiscernibles and an extension \(\Phi\) of \(T_{1}\), (the EM-template) such that \(\Phi\) is realized in each \(M_{\alpha}^{\tau}\) and such that for every linear order \(J, EM_{J}(\Phi) \models T_{1}\) and omits \(\Gamma\).

\(^1\)See Appendix A.3.1 of [Bal09] for a precisely tailored version. See [She78a] or [Hod93], page 587 for a version with the role of the ordering more explicit. The latter two sources make the connection with the well-ordering number clear.
Remark 2.14 (Morley’s Method) The next observation requires a little care in proving Morley’s theorem rather than just quoting it. The $M'_\alpha$ are generated by the $e_{\alpha,i}$ and we have interpreted $<$ so that these are exactly the domain of $<$. So in proving the omitting types theorem, all witnesses for the consistency of the template $\Phi(c)$ can be chosen from the domain of $<$. We use this fact below. It is this extra care that in the mind of the first author distinguishes “Morley’s Method” from Morley’s theorem. But this may be an idiosyncratic interpretation. The earliest mention of the phrase I have found is in [She74] and that refers to a standard application of the two cardinal theorem for cardinals far apart.

Note that any $\tau_\phi$ formula $\phi(x)$ is in $\Phi$ if it is true of every tuple $\langle e_{\alpha,i_1}, \ldots, e_{\alpha,i_n} \rangle$ with $i_1 < i_2 < \ldots < i_n$. We describe a crucial such sentence.

Let $x^1x^2x^3$ be a triple of sequences with the first two having the same length as $\lg(a) = \lg(b)$ and the third has length $m$. Let $\psi(x, y)$ denote:

$$\phi(y^3, x^1) \equiv \phi(y^3, x^2).$$

Let $\psi_1$ be the assertion that $\phi$ defines a linear order on its domain; this directly translates precisely Lemma 2.7.3 and is true by the displayed statement 1. These structures clearly satisfy all the conditions of the requirements in Lemma 2.7 and we complete the proof.

Proof of Theorem 2.3: To show instability in $\lambda$, let $I$ be a dense linear ordering with cardinality $\lambda$ which has more than $\lambda$ cuts and choose $J \supset I$, that realizes more than $|I|$ cuts over $I$. Then $EM_{\tau}(J, \Phi)$ realizes more than $\lambda$ types in $S^m_i(P(EM_{\tau}(I, \Phi)))$. To see this, consider for any cut in $I$ realized by an element $j \in J$ the type:

$$\{ \psi(\langle a_i, b_i, c_i \rangle, x) : i < j \} \cup \{ \psi(\langle a_i, b_i, c_i \rangle, x) : i \geq j \}.$$ 

Then $\langle a_j, b_j, c_j \rangle$ realizes the type in $EM_{\tau}(J, \Phi)$ and $P(EM_{\tau}(J, \Phi))c_j$ is an atomic set since $Q$ was omitted. For the mod-case, use the interpretation of $R$ to define $N_{\alpha,i}$. \hfill $\square_{2.7}$

3 Strictly stable case

In Theorem III.5.15 of [She78b], Shelah determines the form of the least stability cardinal for a stable first order theory. Patterned on these arguments we take some steps towards such a characterization for atomic classes.

As the following examples show, it is easy to have superstable (i.e. eventually stable) (incomplete) sentences of $L_{\omega_1, \omega}$ that are not stable in arbitrarily large cardinals below the Hanf number $H$. Theorem 3.3 has two easily stated corollaries: If $K$ is not eventually stable then it is not stable in every $\lambda$ with $\lambda^\omega > \lambda$. If $K$ is eventually stable then it is stable in some $\lambda < H$.

The results here are related to those in [GS86] but the combinatorics here is considerably simpler than in [GS86] for two related reasons. First, we construct tree indiscernibles indexed by $\leq \lambda$ while they are concerned with $\leq \lambda$; the limit node is much
more difficult to handle. Second, they are constructing many non-isomorphic models, we only construct many different types. To obtain these stronger results, they assume the existence of large cardinals while this paper is in ZFC.

Example 3.1 For $\alpha < \omega_1$, let $\phi_\alpha$ be Morley’s sentence [Mor65b] that has a model in $\mathfrak{d}_\alpha$ but no larger model. It is easy to see that the sentences are not stable in the cardinalities where they have models. Let $\psi$ be the Scott sentence of an infinite set with only equality. Now let $\psi_\alpha$ assert that either a structure has a nontrivial relation and obeys $\phi_\alpha$ or just $\psi$. Then $\phi_\alpha$ is $\mathfrak{d}_\alpha$-unstable but stable (indeed categorical) in all cardinals beyond $\mathfrak{d}_{\omega_1}$.

If one adds even joint embedding such trivial examples are no longer apparent.

Question 3.2 Is there a complete sentence of $L_{\omega_1\omega}$ which is stable beyond $H$ (for either mod or at) but fails stability for some cardinals less than $H$?

We retain the value of $\mu_\alpha = \mathfrak{d}_\alpha(|T|)$ from the first section but $\lambda_\alpha$ is redefined in the hypothesis of the next theorem to be a sufficiently large cardinal in which stability fails.

Theorem 3.3 Suppose that for every $\alpha < \delta(T)$, there is $M^\alpha \in K$ such that $\lambda_\alpha = |M^\alpha| \geq \mu_\alpha$ and $S_i^m(M^\alpha) > \lambda_\alpha$. Then for any $\mu$ with $\mu^{\aleph_0} > \mu$, $K$ is not stable in $\mu$.

Proof. Fix for each $\alpha < \delta(T)$, $M^\alpha \in K$ such that $|S_i^m(M^\alpha)| > \lambda_\alpha$. Fix $p_{\alpha,i}$ for $i < \lambda_\alpha^+$, a list of distinct types in $S_i^m(M^\alpha)$. We work throughout in a monster model $M$ of $T$.

To prepare for the application of an appropriate version of Morley’s omitting types theorem we construct a sequence of models and certain types. For this, we construct trees of types that arise from failure of stability. The combinatorics slightly extends the classical arguments and avoids compactness. Note that this stage of the construction takes place in the original language. We will apply the following general result uniformly to each $M^\alpha$.

Fact 3.4 Suppose $|M| \geq \mu_{\alpha+1}$ and $\mathcal{P}$ is a collection of $\lambda_\alpha = |M|$ members of $S_i^m(M)$. Then there exists a sequence $\langle b_j : j < \mu_\alpha \rangle$ with each $b_j \in M$ and a formula $\phi(x,y) = \phi_\mathcal{P}$ such that for each $j < \mu_\alpha$,

$$|\{p \in \mathcal{P} : i < j \Rightarrow \phi(x, b_i) \in p \text{ and } \neg \phi(x, b_j) \in p\}| > \lambda_\alpha.$$ (2)

Proof. We consider many possibilities for $\phi$ and prove one works. We choose $\{\phi_{\eta} : \eta \in T_i\}$ by induction on $i < \mu_\alpha$ where each $T_i$ is a subset of $\{2\}$ and each $b_{\eta} \in M^\alpha$ so that

1. $j < i$ and $\eta \in T_i$ implies $\eta \upharpoonright j \in T_j$.
2. If $\eta \in T_i$ then $p_{\eta} = \{\phi_{\eta \upharpoonright j}(x, b_{\eta \upharpoonright j}) : j < i\} \subseteq \lambda_\alpha$ members of $\mathcal{P}$.
3. For limit $i$,

$$T_i = \{\eta \in \{2\} : (\forall j < i) \eta \upharpoonright j \in T_j \text{ and } p_{\eta} \text{ is included in } \lambda_\alpha \text{ members of } \mathcal{P}\}$$
4. If \( i = j + 1 \) then \( T_i = \{ \eta \} 0, \eta 1 : \eta \in T_j \}. 

For the successor step in the induction recall the following crucial observation of Morley. Suppose there are more than \(|M|\) types over \( M \) extending a partial type \( p \). Then there exists a formula \( \phi(x, a) \) with \( a \in M \) such that both \( p \cup \{ \phi(x, a) \} \) and \( p \cup \{ \neg \phi(x, a) \} \) have more than \(|M|\) extensions to complete types over \( M \). (We are extending Morley’s analysis to types in \( S_i^m(M) \) but the argument is just counting; there is a unique type which has more than \( \lambda_\alpha \) extensions.)

The interesting point in the induction is the limit stage. We cannot guarantee that individual paths survive. But at each stage in the induction, we have defined types over a set of cardinality \( \mu_\alpha \). So there are at most \( \mu_{\alpha + 1} \) types over \( \{ b_\eta : \lg(\eta) < \delta \} \). So one of the paths must have more than \( \lambda_\alpha \) extensions to \( S_i^m(M) \).

So \( T_{\mu_\alpha} \neq \emptyset \). Choose \( \eta \in T_{\mu_\alpha} \). Let \( \phi_j(x, b_j) = \phi_{\eta, j}(x, b_{\eta, j}) \eta(j) \) for \( j < \mu_\alpha \). Since the path has length \( \mu_\alpha = 2^{2^\delta} \), by the pigeonhole principle we may assume there is a single formula \( \phi \). (The pigeonhole argument fails for \( \alpha = 0 \) but we need the result only for large \( \alpha \) so we may assume \( \alpha \neq 0 \).) This completes the construction of the \( \phi \) and the \( b_j \). We have the result by condition 4.

□3.4

Now we apply this fact to construct from the original \( M^\alpha \) given in the hypothesis of Theorem 3.3 a sequence of models \( M^\alpha \) and associated sequences \( b_{\alpha, \rho} \) and \( c_{\alpha, \rho} \) for \( \rho \in ^{<\omega} \mu_\alpha \).

**Definition 3.5** Let \( \tilde{M}^\alpha \) be a \( \mu_\alpha^+ \) saturated elementary extension of \( M^\alpha \). We construct for each \( \alpha \) by induction on \( n < \omega \), submodels \( M_n^\alpha \) of \( M^\alpha \) and types \( \{ q_\nu^\alpha : \nu \in ^{<\omega} \mu_\alpha \} \) with \( q_\nu^\alpha \in S_i^m(M_{\lg(\nu)}^\alpha) \) and realizations \( c_{\alpha, \nu} \in M^\alpha \) of \( q_\nu^\alpha \) satisfying the following conditions.

1. \( \langle M_n^\alpha : n < \omega \rangle \) is an increasing chain of submodels of \( M^\alpha \), each with cardinality \( \mu_\alpha \).
2. If \( k \leq n \) and \( \nu \in ^k \mu_\alpha \), then \( q_\nu^\alpha \in S_i^m(M_k^\alpha) \).
3. Each \( q_\nu^\alpha \in S_i^m(M_n^\alpha) \) has \( \lambda_\alpha \) extensions to \( S_i^m(M^\alpha) \)
4. Suppose \( k < r \leq n, \nu \in ^k \mu_\alpha, \rho \in ^r \mu_\alpha \) and \( \rho \) extends \( \nu \):

\[
q_\nu^\alpha \subseteq q_\rho^\alpha.
\]

5. If \( \nu \in ^k \lambda_\alpha, k < n, i \neq j \), then

\[
q_\nu^\alpha i \neq q_\nu^\alpha j.
\]

They are distinguished by the \( b_{\alpha, \rho} \), as specified in statement 3 below.

6. \( c_{\alpha, \nu} \in M^\alpha \) realizes \( q_\nu^\alpha \). (In the mod-case, \( N_{\alpha, \rho} \) is the universe of an atomic model containing \( M_{c_{\alpha, \rho}} \).)
Construction 3.6 We use Fact 3.4 to construct objects meeting this definition. Let the subscript \( \tau \) denote at or mod. By induction, for each \( \rho \in \kappa \mu \), the type \( q^\rho_\mu \in S^\mu \alpha (M^n) \) has \( \lambda_\alpha \) extensions to \( S^\mu (M^n) \). Let \( P_\rho = \{ r \in S_\mu (M_\alpha) : q^\rho_\mu \subseteq r \} \) so \( |P_\rho| > \lambda_\alpha \).

By Fact 3.4, we find \( \langle b_{\alpha,\rho}, j < \mu_\alpha \rangle \) and \( \phi_\rho \) satisfying displayed statement 2.

Let \( M^n_{n+1} \) be a submodel of \( M^n \) with \( M^n \cup \{ b_{\alpha, \rho} : \rho \in \kappa (\mu_\alpha) \} \subseteq M^n_{n+1} \) and with cardinality \( \mu_\alpha \). \( M^n_{n+1} \subset M_\alpha \) so is an atomic model and each \( q^\rho_\mu \) extends to an atomic type over \( M^n \).

For \( \rho \in \kappa (\mu_\alpha) \) and \( i < \mu_\alpha \) first define
\[
p^\rho_{\rho, i} = q^\rho_\mu \cup \{ \phi_\rho(x, b_{\alpha, \rho}, j) : j < i \} \cup \{ \neg \phi_\rho(x, b_{\alpha, \rho}, i) \}.
\]

Since \( \lambda_\alpha < |\{ r \in S_\mu (M^n) : p^\rho_\mu \subseteq r \}| \), we can find \( p^\rho_{\rho, i} \in S_\mu (M^n) \) extending \( p^\rho_{\rho, i} \) such that \( P^\rho_{\rho, i} = \{ r \in S_\mu (M^n) : p^\rho_{\rho, i} \subseteq r \} \) has cardinality \( > \lambda_\alpha \). Note that
\[
p^\rho_{\rho, i} \supseteq q^\rho_\mu \cup \{ \phi_\rho(x, b_{\alpha, \rho}, j) : j < i \} \cup \{ \neg \phi_\rho(x, b_{\alpha, \rho}, i) \}.
\]

This completes the \( n + 1 \)st stage of the construction. So we can construct the \( M^n_\alpha \) and \( \{ q_\nu, \nu \in \kappa (\mu_\alpha) \} \), \( M^n \) and by \( \mu_\alpha \)-saturation choose \( c_{\alpha, \rho} \in M^n \). In the mod-case choose an atomic model \( N_{\alpha, \rho} \) with \( M^n c_{\alpha, \rho} \subseteq N_{\alpha, \rho} < M^n \). Note
\[
\{ \phi_\rho(c_{\alpha, \rho}, i, b_{\alpha, \rho}, j) : j < i \} \cup \{ \neg \phi_\rho(c_{\alpha, \rho}, i, b_{\alpha, \rho}, i) \}. \tag{3}
\]

With the construction complete, we expand \( \tau \) to a language \( \tau_\Phi \supseteq \tau \) in two stages. Form \( \tau' \) by adding predicates \( P, P_\alpha, <, <^*, R \) and Skolem functions. We add Skolem axioms to \( T \) to get a theory \( T' \) that admits quantifier elimination, requiring that these Skolem functions applied to elements of \( P \) give an element of \( P \) so that \( P \) will pick out an elementary submodel.

Let \( M^n_\alpha \) be a model of \( T \) (submodel of \( M^n \)) with cardinality \( \mu_\alpha \) containing \( M^n_\alpha \) for \( n < \omega \) and all the \( c_{\alpha, \rho} \). Assign the \( \tau' \)-Skolem functions so that \( P(M^n) = M^n = \bigcup_{n< \omega} M^n_\alpha \) is generated by the \( b_{\alpha, \rho} \) for \( \rho \in \kappa (\mu_\alpha) \). Let \( X_\alpha \) be the tree with domain \( \langle b_{\alpha, \rho} : \rho \in \kappa (\mu_\alpha) \rangle \) and the following relations. Interpret \( < \) as the partial order on the \( \langle b_{\alpha, \rho} : \rho \in \kappa (\mu_\alpha) \rangle \) given by inclusion on the \( \rho \)-indices. Let \( <^* \) be a linear order of the \( \langle b_{\alpha, \rho} : \rho \in \kappa (\mu_\alpha) \rangle \) given by lexicographic order on the \( \rho \)-indices. Interpret \( R \) as
\[
\{ e(c_{\alpha, \rho}, \rho) : \rho \in \kappa (\mu_\alpha), e \in \bigcup_{n< \omega} M^n_\alpha \}.
\]

Form \( \tau_\Phi \) by adding function symbols \( F_n \). Define \( F_n(b_{\alpha, \rho}) = c_{\alpha, \rho} \). Now let \( T_1 \) be the collection of all \( L(\tau_\Phi) \)-sentences that are true in each \( M^n_\alpha \).

In the \( S_{mod}(M) \) case, we must do a bit more. Interpret \( R \) as
\[
\{ e(c_{\alpha, \rho}, \rho) : \rho \in \kappa (\mu_\alpha), and \ e \in N_{\alpha, \rho} \}.
\]

Define the \( \tau' \)-Skolem functions so that the Skolem closure of \( Mc_{\alpha, \rho} \) is \( N_{\alpha, \rho} \). This implies that if \( R(e, c_{\alpha, \rho}) \) holds then \( e \) is a sequence given by \( \tau' \)-Skolem functions with arguments a finite number of members of \( P(M^n_\alpha) \) and \( c_{\alpha, \rho} \).

By conditions 4-6 of Definition 3.5,
Claim 3.7 For any finite linearly ordered initial \(<\)-segment of the tree with length \(n + 1\), enumerated by \(x_0, \ldots, x_n\), (so \(P_i(x_i)\)):

1. \(\bigwedge_{i \leq n} [P_i(z) \land z <^* x_i \rightarrow \phi_i(F_n(x_n), z)]\)
2. \(\bigwedge_{i \leq n} \lnot \phi_i(F_n(x_n), x_i)\).

The universal quantification of each such sentence is true in each \(\tilde{M}_n\) and so is in \(T_1\).

As in Notation 2.13 let \(\Gamma\) be the collection of types:

1. \(P_n = \{\bigwedge_{i < n} P(x_i)\} \cup \{q(x) : q \text{ is a non-principal } n\text{-type}\}\)
2. \(Q_n = \{\bigwedge_{i < n} R(x_i, y)\} \cup \{q(x, y) : q \text{ is a non-principal } n + m\text{-type, } m < \omega\}\)

Now apply the omitting types theorem (as stated in Section 4) to the \(\tau_b\)-theory \(T_1\) and the collection of \(M^+_n\) to get a countable set of tree-indiscernibles in order type \(<^\omega \omega\) and an extension \(\Phi\) of \(T_1\) (the EM-template) such that for every tree of \(J\) of order \(<^\omega \lambda\), \(EM_{\tau}(J, \Phi) \models T_1\) and omits \(\Gamma\).

Finally we must show there are many types; we separate the mod and at cases.

Claim 3.8 If \(\lambda^\omega > \lambda\) then there is an \(I\) with \(|I| = \lambda\) such that \(S_{\text{at}}^m(M_I) > \lambda\), where \(M_I = EM(I, \Phi) \upharpoonright P\).

Proof. Note that by displayed statement 3 and Claim 3.7 we have:

1. If \(\rho, \rho \tilde{i} \in I\), \(\text{tp}(F_n(\rho) / P_n(M)) \subseteq \text{tp}(F_{n+1}(\rho \tilde{i}) / P_{n+1}(M)).\)
2. If \(\rho \in I\) and \(i \neq j\), \(\text{tp}(F_{n+1}(\rho \tilde{j}) / P_{n+1}(M)) \neq \text{tp}(F_{n+1}(\rho \tilde{i}) / P_{n+1}(M)).\)

Now in any \(M_I = EM(I, \Phi)\) for any \(\rho \in J\) define \(p_\rho \in S_{\text{at}}^m(P_{\text{at}}(M_I)) = \text{tp}(F_n(\rho), P_n(M)).\) Now letting \(p_\eta \in S_{\text{at}}^m(P(M_I)) = \bigcup_{i < \omega} p_{\eta i}^n\), we find \(\lambda^\omega\) members of \(S_{\text{at}}^m(P(M_I)).\) The definition of \(S_{\text{at}}^m\) guarantees the union is in \(S_{\text{at}}^m\). \(\square_{3.8}\)

Now we extend this result to mod.

Claim 3.9 If \(\lambda^\omega > \lambda\) then there is an \(I\) with \(|I| = \lambda\) such that \(S_{\text{mod}}^m(M_I) > \lambda\), where \(M_I = EM(I, \Phi) \upharpoonright P\).

Proof. We need to construct an atomic model \(N_\eta\) containing \(M_I c_\eta\) (from the proof of Claim 3.8). The natural choice is the \(\tau^\omega\)-Skolem closure of \(M_I c_\eta\). The reason the reduct of this structure to \(\tau\) is atomic is that any finite sequence is of the form \(a, b\) where the \(a\) come from \(P_n(M_I)\) (for a fixed \(n\)) and each of the \(b\) has the form \(G(a, c_\eta)\) where \(G\) is a \(\tau^\omega\)-Skolem function. But then the \(\tau\) type of \(a b\) is the same as the \(\tau\)-type of a sequence \(a' b'\) where \(a' \in P_n(\tilde{M}^\alpha)\) and each \(b' \in N_{\alpha, \rho}\) is of the form \(G(a', c_{\eta i n}).\) \(\square_{3.9}\)
Remark 3.10 We investigate the difference in hypotheses between Theorem 2.3 and Theorem 3.3. We first study Theorem 2.3.

Let $\kappa = |M_\alpha|$. 

Case 1. $\kappa \leq \beth_\alpha$: then $\kappa \beth_\alpha = 2^{\beth_\alpha} = \beth_{\alpha + 1}$. The assumption of the theorem is that $|S^m_\alpha(M_\alpha)| > \kappa \beth_\alpha = 2^{\beth_\alpha}$. This case is not possible since $|S^m_\alpha(M_\alpha)| \leq 2^{2^{\beth_\alpha}}$.

Case 2. $\kappa > \beth_\alpha$. On one hand we have $\kappa \leq \kappa \beth_\alpha$; on the other $\beth_{\alpha + 1} = \beth_\alpha \leq \kappa \beth_\alpha$. Thus, $\kappa \beth_\alpha \geq \max(\beth_{\alpha + 1}, \kappa)$. The hypothesis in the theorem says that $|S^m_\alpha(M_\alpha)| > \kappa \beth_\alpha$, so $|S^m_\alpha(M_\alpha)| > \max(\beth_{\alpha + 1}, \kappa)$.

This leads to two cases:

Case 2a. $\kappa \geq \beth_{\alpha + 1}$: then $|S^m_\alpha(M_\alpha)| > \max(\beth_{\alpha + 1}, \kappa \beth_\alpha) \geq \kappa$. So the requirement is at least instability in $\kappa$.

Case 2b. $\beth_\alpha < \kappa < \beth_{\alpha + 1}$: then $|S^m_\alpha(M_\alpha)| > \max(\beth_{\alpha + 1}, \kappa \beth_\alpha) = \beth_{\alpha + 1} > \kappa$.

This yields instability in $\kappa$. (Under GCH, of course, this case is empty.)

In general, the hypothesis in case 2b) requires more than instability in $\kappa$: if $\kappa$ has cofinality less than or equal to the cofinality of $\beth_\alpha$, then $\kappa \beth_\alpha > \kappa$, and the number of types needs to be (possibly) much greater than $\kappa$.

Theorem 3.3 asserts that $K$ is unstable in some cardinal then it is unstable in any $\lambda$ with $\lambda^\omega > \lambda$ so it is analogous to the first order case. Further it asserts that the first stability cardinal for a superstable class is less than $H$.

Thus, in Theorem 2.3 we assume 'serious' instability and get instability everywhere and in Theorem 3.3 we assume "just" instability, and get instability for cardinals of countable cofinality only.

We further analyze case 2a under GCH. The possible values of $\kappa \beth_\alpha$, given that $\kappa > \beth_{\alpha + 1}$, become $\kappa$ and $2^\kappa = \kappa^+$ (the first is the case when the cofinality of $\kappa$ is greater than the cofinality of $\beth_\alpha$; otherwise, the second alternative holds).

Under the GCH the difference between 'serious' and 'just' instability disappears. Moreover, we can expect to find $M_\alpha$ satisfying the hypothesis only for $|M_\alpha|$ of cofinality greater than the cofinality of $\beth_\alpha$. So under the GCH, the difference between the hypotheses in 2.1 and 3.2 disappears, but the conclusion of 3.2 is weaker.

4 Tree Indiscernibility

The main result of this section is the existence of tree indiscernibles as needed in the previous section. But we take the occasion to discuss the role of various types of index sets for indiscernible collections and to make explicit the role of expanding the vocabulary when finding indiscernibles in various contexts.

The theorem reported here is implicit in the literature (e.g. [She78a, GS86]) but we could not find an explicit statement. Theorem VII.3.6 of [She78a] finds an indiscernible tree in the first order case on $\omega \omega$ but we want to omit types as well. The basic plan of the proof dates to Morley [Mor65b]. We indicate the modifications needed for the more complicated combinatorics to build models to omit types that are over indiscernible trees instead of over linear orders.

This analysis was worked out by the first author and Alexei Kolesnikov.
Many variants of tree indiscernibles are used in various parts of model theory; we describe several of these applications to provide a context for the current version and to emphasize some important distinctions. Indiscernibles may be ordered by linear orders, or trees of the form \( \leq \omega \), \( \leq \omega \lambda \) or even \( \leq \omega \), \( \leq \omega \lambda \). We may want to find the ordering in the basic vocabulary \( \tau \) (to witness unstability at some level) or not (to avoid introducing instability) but only in an expanded vocabulary \( \tau^* \).

We first consider situations where the (partial) ordering is explicitly added to the language. Ehrenfeucht and Mostowski (finding automorphisms) index the indiscernible by a linear order which is not in the base language. Morley’s proof that \( \aleph_1 \)-categoricity implies \( \omega \)-stability pursues the same strategy. He is counting the number of \( \tau \) types and there is certainly no ordering in the vocabulary \( \tau \). There are further applications using the extended vocabulary to two-cardinal models [She75, She76] and to Peano arithmetic [MP84]. Tree indiscernibles on \( \leq \omega \) rely on Halpern-Lauchli; tree indiscernibles on \( \leq \omega \lambda \) rely on Erdos-Rado.

To show an unstable first order theory has the maximal number of models, it is essential that Shelah is building order indiscernibles with respect to an ordering that is definable in the base vocabulary \( \tau \). To investigate the difference in stability spectrum for stable but not superstable theories, the (tree)-order must be expressible in \( \tau \) and the tree is \( \lambda \leq \omega \). But to count the number of models of superstable theory involves trees of height \( \omega + 1 \). The proof in this paper differs from [She78a], where the number of models of an unsuperstable theory is computed, because in working with \( L_{\omega_1, \omega} \), we must omit types. In VII.3.6 of [She78a], Erdos-Rado is applied to show the existence of a ‘uniform’ \( \beta \)-tree implies the existence of a tree of indiscernibles indexed by \( \leq \omega \). Thus indiscernibles indexed by linear orders as well as the trees \( \lambda \leq \) and \( \lambda \leq \omega \) have a long history. More recently, the tree orders occur in [Dö4, KKS, LS03, Sco]. The use of trees indexed by \( 2 \leq \omega \) to construct many models in \( \aleph_1 \) if a countable theory is not \( \omega \)-stable appears in [She78a]. (The tree is found in VI.3.7; it is used to construct many models in VIII.1.2.) An exposition of this result and some extensions to uncountable languages occur in [Bal89]. The construction of many models from infinitary order properties in [GS86], with trees of the form \( \lambda \leq \omega \) requires large cardinal axioms for the combinatorics.

We see three steps in this kind of construction. The references in parentheses are to the application of this method to the proof of the strictly stable case in this paper.

1. Model theoretic construction of specific syntactic-combinatoric configurations on models. (Construction 3.6.)

2. Application of Erdos-Rado or Halpern-Lauchli and compactness to extract a countable family of indiscernibles. (Theorem 4.7.)

3. Application of Ehrenfeucht-Mostowski models to obtain models of arbitrary cardinality. (Claim 3.8.) This is sometimes called ‘stretching’.

We first establish some background notation. The exact vocabulary for describing the partial order is significant; we follow [She78b] rather.

**Notation 4.1**

1. A tree \( T \) is a subset of \( \leq \omega \lambda \) that is closed under initial segment.
2. atp means atomic (quantifier-free) type.

3. The vocabulary $\tau^*$ will denote the vocabulary for trees we use. It contains the partial order on the tree, $<$, the lexicographic order on the tree $<^*$, $\land$ (meet) and the levels $P_n$. $\tau^*_n$ omits the $P_i$ with $i > n$.

4. When elements $a_\eta$ and $a_\tau$ in a structure $M$ are indexed by $\eta, \tau \in T$ that realize the same quantifier free $\tau^*$-type in the tree then $a_\tau$ and $a_\eta$ have the same length.

5. If $\nu$ is an $n$-element sequence from $T$, $a_\nu$ denotes $\langle a_\nu(0), \ldots, a_\nu(n-1) \rangle$.

**Definition 4.2** For any vocabulary $\tau$, let $M$ be a $\tau$-structure and $\Sigma$ a set of $\tau$-formulas. If $\text{atp}_{\tau^*}(\eta/\emptyset) = \text{atp}_{\tau^*}(\nu/\emptyset)$ implies $\text{tp}_\Sigma(a_\eta/\emptyset) = \text{tp}_\Sigma(a_\nu/\emptyset)$ in $M$ then we call $\langle a_\eta : \eta \in T \rangle \subseteq M$ a set of $\Sigma$-tree indiscernibles:

We just say tree indiscernibles if $\Sigma$ contains all formulas in $L(\tau)$.

We rely on a combinatorial lemma that follows from Erdos-Rado. The result is proved as Theorem 2.6 in the appendix to [She78a]. A stronger result (the bound on $k(m, n)$ is smaller) with a shorter proof is sketched in the appendix of [GS86]. Kim, Kim, and Scow have recently included a full argument in [KKS]; their paper includes a careful distinction of several notions of ‘tree-indiscernibility’.

**Lemma 4.3 ([She78a])** For every $n, m < \omega$, there is a $k = k(n, m) < \omega$ such that if $\lambda = \beth_k(\chi)^+$ the following is true. For any function $f : [\leq n] m \rightarrow \chi$, there exists a $T \subseteq [\leq n] \lambda$ such that

1. Each $\eta \in T$ has $\chi^+$ immediate successors in $T$.

2. If $\nu$ and $\tau$ are $m$-tuples from $T$ with $\text{atp}_{\tau^*}(\eta/\emptyset) = \text{atp}_{\tau^*}(\nu/\emptyset)$, then $f(\tau) = f(\eta)$.

We now prove the theorem on the existence of tree-indiscernibles. In order to be clear about the definability of the tree in the original vocabulary we extend Notation 4.1 and are quite pedantic about the vocabularies involved.

**Notation 4.4**

1. $\tau_\Phi$ includes both $\tau$ and $\tau^*$ and includes Skolem functions for $\tau_\Phi$, where the Skolem axioms and relations with crucial $\tau$-formulas are axiomatized in a $\tau_\Phi$-theory $T_1$.

2. The set of constants $C$ which guarantee the consistency of the order are added to $\tau_\Phi$.

3. $\Sigma_i$ denotes the set of formulas $\phi \in \tau_\Phi - \{P_j : j > i\}$ with at most $i$ free variables.

Tree-indiscernibles are a special case of generalized indiscernibility as defined in VII.2 of [She78a]. Indiscernibles indexed by other types of structure appear for example in [LS03, Dv04, Sco]. The following notion of modeling property, based on one introduced by Scow[Sco] in a slightly different context is helpful for stating the results
The point is that although the type of an infinite collection of indiscernibles may not be realized in any of the input models, each (even complete) type of a finite subsequence is. Thus properties of finite character (such as realizing a finite type) follow immediately if the indiscernibles have the modeling property. We use \( \approx \) for isomorphic.

**Definition 4.5** Let \( \Sigma \) be a collection of \( \tau_\Phi \)-formulas. A collection of \( \Sigma \)-tree-indiscernibles \( B = \{ b_\eta : \eta \in T \} \) has the modeling property if it is derived from a sequence \( (M_\alpha, X_\alpha) \) (where \( M_\alpha \supset X_\alpha = \{ a_\eta : \eta \in T_\alpha \} \), and \( T_\alpha \approx T \) for \( \alpha < H \)) such that for every finite sequence \( \nu \) from \( T \) and every sequence \( b_\nu \) from \( B \) and some \( \alpha \) there is a sequence \( a_\nu' \in X_\alpha \) with \( \nu' \) having the same \( \tau^* \)-type as \( \nu \) and such that \( a_\nu' \) and \( b_\nu \) have the same \( \Sigma \)-type.

Note that in the argument below when the \( X_\alpha \) are refined using Lemma 4.3 a tuple \( a_\nu \in X_{i,n}^\alpha \) was originally named \( a_\nu' \in X_{0+m^*+n}^\alpha \) (where \( m^* < \omega \) can be easily computed). But, \( \nu \) and \( \nu' \) realize the same \( \tau^* \)-type.

**Remark 4.6** There are at least four approaches to the proof of Morley’s omitting types theorem that differ subtly. 1) In [CK73, Mar02] the language is countable and there are separate steps to guarantee indiscernibility and omission of the types (meeting indiscernibility type omission requirements in turn for each formula and for each type). 2) The extension to uncountable languages is announced without proof in the original paper [Mor65a]. In the argument here, we use the Skolemization of the models \( M_\alpha \) to deduce the omission of types from the indiscernibility. This argument strategy is forced because in dealing with uncountable languages, working with one formula at each step (as in [Mar02]) makes the induction too long to be actually carried out. We replace this long induction by working with all formulas with \( n \)-free variables at step \( n \). 3) The arguments in [She78a, GL02, GS86] employ nonstandard-models of set theory. 4) Finally, the arguments in [Hod87, Kei71], work directly in infinitary logic using Hintikka sets or consistency properties. The arguments of [She78a, GS86, Hod87] make the connection with well-ordering numbers explicit.

Tsuboi [Tsu08] shows that a family of \( < 2^{\aleph_0} \) complete types that is omitted up to \( \beth_1 \) can be omitted in arbitrarily large models; this argument introduces some new combinatorial ideas.

Recall that \( \mu_\alpha = \beth_\alpha(|T|) \). Writing \( \mu_\alpha \) rather than \( \beth_\alpha \) and considering \( M_\alpha \) for \( \alpha < \delta(T) = (2^{|T|})^+ \) is part of the price for dealing with uncountable \( T \).

Note that when applying this theorem in Section 3, the \( M_\alpha \) here are the \( M_\alpha^+ \) (as Skolemized) there.

**Theorem 4.7** Let \( T \) be a theory with Skolem functions in a vocabulary \( \tau_\Phi \). Suppose for \( \alpha < \delta(T) \), there exists a model \( M_\alpha \) of \( T \) with \( |M_\alpha| \geq \mu_\alpha \) such that \( M_\alpha \) omits a family \( \Gamma \) of \( \tau \)-types. \( \tau_\Phi \) contains the vocabulary \( \tau^* \) and \( X_\alpha \) is a set of elements in \( M_\alpha \) that form a tree of type \( \langle \omega \rangle^\omega \mu_\alpha \) in \( M_\alpha \) defined by the interpretations of \( <, <^*, \land, P_n \), for
The sequence \( \langle M_i, X_{\alpha,n}^i \rangle \) has the property that for each \( \alpha \):

If \( \eta, \nu \in T_{\alpha,n}^i \) both realize the same quantifier-free \( \tau^* \)-type \( r \), and \( n \leq i \) then for each \( \phi \in \Sigma_n \)

\[
\phi(c_\eta) \equiv \phi(c_\nu). \tag{4}
\]

Moreover, \( (X_{\alpha,n}^i, <, <^*, \land) \approx (\leq_n \mu_\alpha, <, <^*, \land) \).

\(^4\)An application of this observation is in the paragraph after Remark 2.14.
Proof. Consider \((M_{i+k}, X_{i+k,i})\) where \(k = k(m,i)\). Let \(f : [X_{i+k,i}]^m \to S_n\), where \(f(\nu) = s\) if \(\tp_{\tau^*}(a_{\nu}) = s\). Now by Lemma 4.3, there is a \(Y_{\alpha,i}\) (contained in \(X_{i+k,i} \subseteq \bigcup_{j \leq i} P_j(M_{\alpha})\)) and with \((Y_{\alpha,i},<,<^*) \approx \leq \mu_{\alpha}\) and (4) is true on \(Y_{\alpha,i}\).

Denote \(Y_{\alpha,i}\) as \(X_{i+1,\alpha,i}\) and \(M_{\alpha+k}\) as \(M_{\alpha+1}\). For \(j \geq i\), let \(X_{i+1,\alpha,j}\) be the elements of \(X_{\alpha+i,j}\) that extend members of \(Y_{\alpha,i} = X_{i+1,\alpha,i}\). \(\square\)

To complete the proof of Theorem 4.7, we refine (and rename for convenience) the index set of ordinals to guarantee that for all \(\alpha\), each \(\tau^*\)-type in \(S_n\) is given the same truth value for all tuples from \(X_{i+1,\alpha,i}\) realizing \(r\). This assignment gives us \(\Phi_{n+1}\). We can do this because at any stage, the number of \(\Sigma\)-theories is at most \(2^{2\theta_1}\) which is not cofinal in \((2^{\theta_1})^+\). Note that as \(i\) increases in this induction, the indiscernibility is being insured for larger \(\Sigma_i\). Since the \(\Sigma_i\) are increasing this results in a consistent theory \(\Phi\) giving tree-indiscernibility in \(L(\tau \Phi)\).

At stage \(i\), we have assigned to each \(\tau^*_i\) type \(r\), a complete \(\Sigma_i\)-diagram in \(\tau \Phi\); each formula \(\phi(v) \in \Sigma_i\) has a fixed truth value for all \(v_{\eta}\) where \(\eta\) realizes \(r\). In particular, since all \(M_{\alpha}\) omit each \(\ell\)-type \(p \in \Gamma\) for any finite \(\ell\), for each sequence of \(\ell\)-Skolem functions \(t\) in a most \(m\)-variable, and each \(\eta\) realizing a \(\tau^*\)-type in \(m\)-variables there is a \(\phi_p \in \Sigma_{\ell,m}\) with \(\phi_p \in p\) and \(\neg \phi_p(t(c_{\eta})\).

\(\square\)

This completes the general proof for obtaining tree indiscernibles and so the proof of Theorem 3.3 is complete as well.

References


