Variants on the Morley Analysis

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Today's Topics

Acknowledgements

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and On a theorem on simultaneous omitting and realizing types and its applications by B.S. Baizhanov, T.S. Zambarnaya
and WEAKLY AND ALMOST ORTHOGONALITY OF TYPES B.S.Baizhanov, A.D.Yershigeshova
with further analysis and extensions by myself and Aida Alibek.

Strategies

Analyze the countable models

1. Specific examples
2. the stability theory analysis: analyze the structure of models

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1. Specific examples
2. the stability theory analysis: analyze the structure of models
3. Analyze countable isomorphism - e.g. by descriptive set theory

Do uncountable models count?

Two ways to approximate the countable by the uncountable

1. Obtain more definable sets by expanding the logic: Morley
2. Use our better understanding of models in $\aleph_1$ to understand countable models.

Morley Analysis

Morley's Analysis

Let $\boldsymbol{K}$ be the class of models of a sentence of $L_{\omega_1,\omega}$.

1. Let $L_0^{\mathbf{K}}$ be the set of first order τ-sentences.
2. Let $L_{\alpha+1}^{\mathbf{K}}$ be the smallest fragment generated by $L_\alpha^{\mathbf{K}}$ and the sentences of the form $(\exists \mathbf{x}) \bigwedge p(\mathbf{x})$ where p is an $L_\alpha^{\mathbf{K}}$-type realized in a model in $\boldsymbol{K}$.
3. For limit δ, $L_\delta^{\mathbf{K}} = \bigcup_{\alpha<\delta} L_\alpha^{\mathbf{K}}$.

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Scattered

$\boldsymbol{K}$ (or ϕ if $\boldsymbol{K} = \mathrm{mod}\phi$) is scattered if and only if for each $\alpha < \omega_1$, $L_\alpha^{\mathbf{K}}$ is countable.

Locally Small Models

Definition: Locally Small

1. A τ-structure M is L^*-small for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many L^*-types (i.e. only countably many L^*-n-types over $\emptyset$ for each $n < \omega$).
2. A τ-structure M is called locally τ-small if for every countable fragment L^* of $L_{\omega_1,\omega}(\tau)$, M realizes only countably many L^*-types.

Of course, every model of a scattered sentence is locally small.

Large and Small models

Definition: small

1. A τ-structure M is called small or $L_{\omega_1,\omega}$-small if M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.
2. Otherwise it is large (i.e. $L_{\omega_1,\omega}$-large).

Scott's Theorem in the uncountable

A model of arbitrary cardinality has a Scott sentence if and only if it small.

ϕ-scattered implies every model is locally small; converse fails (B-Hytinnen-Kesala).

Large Sentences

Definition

A sentence σ of $L_{\omega_1,\omega}$ is large if it has uncountably many countable models.

A large sentence σ is minimal if for every sentence ϕ either $\sigma \wedge \phi$ or $\sigma \wedge \neg\phi$ is not large.

Minimal counterexamples and large models

Lemma [Harnik-Makkai]

For every counterexample σ to Vaught's conjecture, there is a minimal counterexample ϕ such that $\phi \models \sigma$.

Lemma [Harnik-Makkai]

Every (minimal) counterexample σ to Vaught's conjecture has a large uncountable model.

Shelah Analysis of Model

Fix a scattered $L_{\omega_1,\omega}$ sentence σ and $M \models \sigma$ with universe ω_1.
Add a binary relation $<$, interpreted as the usual order on ω_1.
Define a continuous increasing chain of countable fragments L_α for $\alpha < \aleph_1$ such that

- for each (first order) *n*-type over the empty set realized in *M*, the conjunction of the type is in L_0, and
- the conjunction of each type in L_α that is realized in *M* is a formula in $L_{\alpha+1}$.

Shelah Analysis cont.

Extend the similarity type to τ' by adding new $(2n+1)$-ary predicates $E_n(x, \mathbf{y}, \mathbf{z})$.
Let M satisfy $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ if and only if $\boldsymbol{a}$ and $\mathbf{b}$ realize the same L_α-type.
By Lopez-Escobar, there is an infinite $<$-decreasing sequence d_i
Define:

$$E_n^+(\mathbf{x}, \mathbf{y}) \text{iff for some i } E_n(d_i, \mathbf{x}, \mathbf{y})$$

Locally small implies There Exists small

With this apparatus using only Lopez-Escobar Shelah showed:
Induction on qf ranks shows:
$E_n^+(\mathbf{x}, \mathbf{y})$ implies $\boldsymbol{a} \equiv_{\omega_1,\omega} \mathbf{b}$.
For any n, $E_n(d_0, \mathbf{x}, \mathbf{y})$ refines $E_n^+(\mathbf{x}, \mathbf{y})$ and has only countably many classes is a single $L_{\omega_1,\omega}$-formula that implies $L_{\omega_1,\omega}$-equivalence.

Theorem

Suppose $M \models \phi$ has cardinality $\aleph_1$ and is locally τ-small, then ϕ has a $L_{\omega_1,\omega}(\tau)$-small model N of cardinality $\aleph_1$.

This shows every ce to VC has a small model in $\aleph_1$.

Nice countable models of a ce to VC

Definition

We say a countable structure is extendible if it has an $L_{\omega_1,\omega}$-elementary extension to an uncountable model.

Refining the Shelah Analysis

Theorem (B-Larson-Shelah)

Suppose that $\boldsymbol{K}$ is the class of models of $\phi \in L_{\omega_1,\omega}(\tau)$. If some uncountable $M \in \boldsymbol{K}$ is locally τ-small but is not $L_{\omega_1,\omega}(\tau)$-small then

1. There are at least $\aleph_1$ pairwise-inequivalent complete sentences of $L_{\omega_1,\omega}(\tau)$ which are satisfied by uncountable models in $\boldsymbol{K}$;
2. $\boldsymbol{K}$ has uncountably many small models in $\aleph_1$ that satisfy distinct complete sentences of $L_{\omega_1,\omega}(\tau)$;
3. $\boldsymbol{K}$ has uncountably many extendible models in $\aleph_0$.

Proof I

Let $|M| = \aleph_1$ be locally τ-small but not $L_{\omega_1,\omega}(\tau)$-small.

Construct models of ϕ, $\{N_\alpha : \alpha < \omega_1\}$, $|N_\alpha| = \aleph_1$, countable fragments $\{L'_\alpha : \alpha < \omega_1\}$ of $L_{\omega_1,\omega}(\tau)$ and sentences χ_α:

1. $L'_0(\tau)$ is first order logic on τ;
2. $\forall\ \alpha < \omega_1$, N_α is $L_{\omega_1,\omega}$-small; but $N_\alpha \equiv_{L'_\alpha(\tau)} M$.
3. χ_α is the Scott sentence of N_α;
4. $L'_{\alpha+1}(\tau)$ is the smallest fragment of $L_{\omega_1,\omega}$ containing $L'_\alpha \cup \{\neg\chi_\alpha\}$;
5. For limit δ, $L'_\delta(\tau) = \bigcup_{\alpha<\delta} L'_\alpha(\tau)$;

Proof II

Working by recursion, suppose that we have constructed N_α for all $\alpha < \beta$, for some countable ordinal β.

This determines each χ_α $(\alpha < \beta)$ as the Scott sentence of N_α and also determines $L'_\beta(\tau)$.

Since M is not small, $M \models \neg\chi_\alpha$ for each $\alpha < \beta$.

Proof III

By the previous theorem there is an L'_β-elementarily equivalent to M that is small; call it N_β.

The N_α are pairwise non-isomorphic since each satisfies a distinct complete sentence χ_α of $L_{\omega_1,\omega}(\tau)$, so conclusions 1) and 2) are satisfied. And each N_α has a countable elementary submodel with respect to $L'_{\alpha+1}(\tau)$, so there are at least $\aleph_1$ non-isomorphic extendible models in $\aleph_0$ as well.

Large models are $L_{\omega_1,\omega}$-equivalent.

Corollary

If ϕ is a minimal counterexample to Vaught's conjecture then ϕ has a large model in $\aleph_1$,
and all large models of ϕ in $\aleph_1$ are $L_{\omega_1,\omega}$-elementarily equivalent.

Proof. We saw that ϕ has a large model N. Suppose that $\psi \in L_{\omega_1,\omega}$ holds in N.
By 3) $\phi \wedge \psi$ has uncountably many models in $\aleph_0$.
By minimality, $\phi \wedge \neg\psi$ has only countably many models in $\aleph_0$.

By the contrapositive of hyp implies 3) all uncountable models of $\phi \wedge \neg\psi$ are small.

Dowries

Finite Diagrams renamed

Fix a countable vocabulary τ and a small τ-theory T in a countable fragment $\mathcal{F}$ of $L_{\omega_1,\omega}$.
$S(T) = S_{\mathcal{F}}(T)$ is the collection of types of any arity over $\emptyset$.

1. The dowry of M. $\mathcal{D}(M)$ is the collection of $p \in S(T)$ that are realized in M.
2. A dowry Δ is big if uncountably many non-isomorphic M satisfy $\mathcal{D}(M) = \Delta$.
   A big dowry gives rise to a new ce to VC.
3. Let E_D denoted $\sim_D$ be the equivalence relation on the logic space.

$$M \sim_D N \text{ if and only if } \mathcal{D}(M) = \mathcal{D}(N)$$

Shelah 1973 calls this notion the finite diagram. We wanted to keep the D but not confuse with the diagram of a model

Big Dowries exist

Observation

E_D is a Borel equivalence relation.

By Silver's theorem:

Theorem: Baizhanov-Zambarnaya

If T is counterexample to Vaught's conjecture there exist an family of $\aleph_1$ models of T with the same dowry.

Note then that a minimal counterexample ϕ will specify that all models of ϕ realize the same F-diagram for some fragment F slightly bigger than the one generated by ϕ.

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Marker uses E_D and Silver to prove Morley's theorem: ϕ is not scattered implies ϕ has perfect set of models.

‘Siberian model theory’

The ‘elementary’ study of small theories

Sudoplatov, Baizhanov et al

1. study types over finite sets
2. $p(x) \perp^w q(y)$ if $\{p(x)\} \cup \{q(y)\}$ is complete
3. $p \leq_{RK} q$ if p is realized in the prime model over a realization of q.
4. Unfortunately, they use the notation $p \not\perp^a q$ for $q \leq_{RK} p$

Some theorems

Theorem: (Sudoplatov

Every model of a compete theory with only finitely many models is either prime over a finite set or
$M = \bigcup_{i<\omega} M_i$
where each M_i is prime over a realization of the same type p.

Constructing more big Dowries

Theorem: Baizhanov-Yershegeshova (B-Alibek)

Let M be a countable model of a small theory T and suppose $M \prec N$, an ω_1-saturated model. Then there exists a countable elementary extension $M_{\overline{c}}$ with $M \prec M_{\overline{c}} \prec N$ and a sequence of finite tuples $\overline{a}_i$ from M such that

$$M_{\overline{c}} = \bigcup_{i<\omega} M[\overline{a}_i, \overline{c}]$$

where $M[\overline{a}_i, \overline{c}]$ denotes the prime model over $\overline{a}_i, \overline{c}$.

Properties of M_c

Theorem

Let M be a countable, non-homogeneous model of a small theory T, and $p \in S(T)$ be a non-isolated type such that

1. for any non-isolated $r \in D(M)$, $r \not\leq_{RK} p$ and
2. for any $q(x, \overline{y}) \in D(M)$, such that there exists $\overline{\alpha} \in M,\ q(N, \overline{\alpha}) \cap M = \emptyset$ we have for any $p' \in S(\overline{\alpha})$, such that $p \subset p'$, $q(x, \alpha) \not\leq_{RK} p'$.

If M_c is constructed as above then if $\alpha \in M$ and $q(x, \overline{\alpha})$ is not realized in M, then $q(x, \overline{\alpha})$ is not realized in $M_{\mathbf{c}}$; Thus, $M_{\mathbf{c}}$ is not homogeneous.

Conjectures

Write $p \perp^w M$ if $p \perp^w r$ for each r realized in M.
$M \in \mathrm{mod}(\Delta)$ if $\mathcal{D}(M) = \Delta$.

Baizhanov's conjectures

Suppose $\mathcal{D}(M) = \mathcal{D}(N) = \Delta$, $p \perp^w N$ and c, d realize p. Then

1. $\mathcal{D}(M_c) = \mathcal{D}(N_d) = \Delta'$
2. The map $M \to M_c$ is $1-1$ from $\mathrm{mod}(\Delta)$ to $\mathrm{mod}(\Delta')$.

$\mathrm{mod}(\Delta) = \{N : \mathcal{D}(N) = \Delta\}$

Questions/Speculations

1. in first order or any reasonable fragment
   1. Are there $\aleph_1$ weakly saturated models of a ce?
   2. in first order: Are there $\aleph_1$ universal models of a ce?
2. Are there a pair of models in $\aleph_1$ with $M \prec N$?
3. What about amalgamation? 3 amalgamation

The importance of Vaught's Conjecturef

”Some people think this [Vaught's conjecture] is the most important question in model theory as its solution will give us an understanding of countable models which is the most important kind of models. We disagree with all those three statements.”

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VC may be a good test question for our understanding of countable models.

Why VC is an important question

Model theory, recursion theory, set theory

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LOGIC

A criteria for recognizing the subject of VC

If yes, then the methods used in the solution will indicate the area.

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If yes, then the methods used in the solution will indicate the area.
if no, model theory will provide an analysis of which theories satisfy the conjecture.

THANKS

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TO THE ORGANIZERS

FOR AN IMMENSELY
STIMULATING
AND ENJOYABLE
CONFERENCE