Almost Galois ω-Stable classes

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Abstract

Theorem. Suppose that an $\aleph_0$-presentable Abstract Elementary Class (AEC), $\boldsymbol{K}$, is almost Galois ω-stable. If $\boldsymbol{K}$ has only countably many models in $\aleph_1$, then $\boldsymbol{K}$ is Galois ω-stable.

1 Introduction

This paper concerns two aspects of pseudo-elementary classes in $L_{\omega_1,\omega}$, the reducts to a vocabulary $\tau \subseteq \tau^+$ of models of an $L_{\omega_1,\omega}(\tau^+)$-sentence. In the first two sections we investigate the relationship between the number of countable models of such a class, Scott ranks, and the number of small (i.e., having a countable $L_{\omega_1,\omega}$-elementary submodel) models and large (not small) models of the class in $\aleph_1$; this yields some technical information about putative counterexamples to Vaught's conjecture. In the third section, we treat such classes as abstract elementary classes and investigate variations on Galois ω-stability.

We call an Abstract Elementary Classes (AEC) *almost Galois ω-stable* if for every countable model M, E_M (the equivalence relation of 'same Galois type over M') does not have a perfect set of equivalence classes. The immediate impetus for this paper was [3] which studied, what Baldwin and Larson called *analytically presented* Abstract Elementary Classes. These classes are called by many names: pseudo-elementary classes in $L_{\omega_1,\omega}$, $\aleph_0$-presentable classes, $\mathrm{PC}_{\aleph_0}$ [18], $\mathrm{PC}(\aleph_0,\aleph_0)$, $\mathrm{PC}\Gamma(\aleph_0,\aleph_0)$ [1] or, in the language of Keisler [11], PC_δ in $L_{\omega_1,\omega}$. In this paper we will most often refer to them as $\aleph_0$-presented. The term 'analytically presented' emphasizes that one can deduce from Burgess's theorem on analytic equivalence relations that if such a class is almost Galois ω-stable then each equivalence relation E_M has at most $\aleph_1$ equivalence classes. This topic first arose in [18]; for further background on the context see [3, 1, 17]. Our main goal is to prove that an almost Galois ω-stable $\aleph_0$-presentable Abstract Elementary Class with only countably many models in $\aleph_1$ is Galois ω-stable. This extends earlier work by Hyttinen-Kesala [10] and Kueker [13] proving the result for sentences of $L_{\omega_1,\omega}$ with no requirement on the number of uncountable models.

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Each class of models in this paper is $\aleph_0$-presented. A major tool for this investigation is to expand models of set theory by predicates encoding relevant properties of the models (for some vocabulary τ) being studied. This approach appears in Shelah's analysis in [15], Section VII, connecting the Hanf number for omitting families of types with the well-ordering number for classes defined by omitting types.

In [14], expanding the vocabulary to describe an analysis of the syntactic types allowed the construction of a 'small' (Definition 2.2) uncountable model in an $\aleph_0$-presentable class $\boldsymbol{K}$ from an uncountable model that is small with respect to every countable fragment of $L_{\omega_1,\omega}$. In Lemma 2.7, we use this method to show that if, in addition, there are only countably many models in $\aleph_1$, then they are each small. In Section 3, we combine these two techniques to show the main theorem as stated in the abstract.

2 Small Models

We refer the reader to [1] for the definition of Abstract Elementary Class (AEC).

{context}

Assumption 2.1. *$\boldsymbol{K} = (K, \leq_K)$ is an AEC which is $\aleph_0$-presented. Specifically, $\boldsymbol{K}$ is the class of reducts to τ of a class defined by a sentence $\phi \in L_{\omega_1,\omega}(\tau^+)$, where τ^+ is a countable vocabulary extending τ.*

This section deals with syntactic ($L_{\omega_1,\omega}$)-types in $\aleph_0$-presentable classes. As such the arguments are primarily syntactic and are minor variants on arguments Shelah used in [14, 16, 18]. In particular, no amalgamation assumptions are used in this section.

{small}

Definition 2.2. *1. A τ-structure M is L^*-*small *for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many L^*-types (i.e. only countably many L^*-n-types over $\emptyset$ for each $n < \omega$.)*

2. *A τ-structure M is called* locally τ-small *if for every countable fragment L^* of $L_{\omega_1,\omega}(\tau)$, M realizes only countably many L^*-types.*

3. *A τ-structure M is called* small *or* $L_{\omega_1,\omega}$-small *if M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.*

Note that 'small' is a much stronger requirement than 'locally small'. If $\tau \subseteq \tau'$ and $N \in \tau'$, we say that N is *locally τ-small* when $N \restriction \tau$ is. We emphasize τ when the ambient larger vocabulary plays a significant role. The following standard fact plays a key role below (see also pages 47-48 of [1]).

{Scottsent}

Fact 2.3. *Each small model satisfies a Scott-sentence, a complete sentence of $L_{\omega_1,\omega}$.*

We quickly review the proof of this fact, as the details will be important later. For any model M over a countable vocabulary τ, we can define for each finite tuple $\mathbf{a}$ (of size n) from M the n-ary formulas $\phi_{\mathbf{a},\alpha}(\overline{x})$ ($\alpha < \omega_1$) as follows.

- $\phi_{\mathbf{a},0}(\overline{x})$ is the conjunction of all atomic formulas satisfied by a,
- $\phi_{\mathbf{a},\alpha+1}(\overline{x})$ is the conjunction of the following three formulas:
  - $\phi_{\mathbf{a},\alpha}(\overline{x})$
  - $\bigwedge_{c\in M} \exists w \phi_{\mathbf{a}c,\alpha}(\overline{x}, w)$
  - $\forall w \bigvee_{c\in M} \phi_{\mathbf{a}c,\alpha}(\overline{x}, w)$
- for limit $\beta < \omega_1$, $\phi_{\mathbf{a},\beta}(\overline{x}) = \bigwedge_{\alpha<\beta} \phi_{\mathbf{a},\alpha}$.

The apparent uncountability of the conjunctions in the previous definition is obviated by identifying formulas $\phi_{\mathbf{a}c,\alpha}$ and $\phi_{\mathbf{a}'c,\alpha}$ when they are equivalent in M. Working by induction on α, one gets that if M is L^*-small for each countable fragment L^* of $L_{\omega_1,\omega}(\tau)$, then the set of formulas $\phi_{\mathbf{a},\alpha}$ is countable for each α, letting $\mathbf{a}$ range over all finite tuples from M. Finally, if M is small there exists an α such that

$$M \models \forall \overline{x}(\phi_{\mathbf{a},\alpha}(\overline{x}) \rightarrow \phi_{\mathbf{a},\alpha+1}(\overline{x}))$$

for all finite tuples $\mathbf{a}$. Then

$$\phi_{\langle\rangle,\alpha} \wedge \bigwedge_{\mathbf{a}\in M^{<\omega}} \forall \overline{x}(\phi_{\mathbf{a},\alpha}(\overline{x}) \rightarrow \phi_{\mathbf{a},\alpha+1}(\overline{x}))$$

is a Scott sentence for M. Fixing the least such α, we say that M has a Scott sentence of rank α.

We will also use the following fundamental result (see [11] or Theorem 5.2.5 of [1]; the notion of fragment is explained in both books). Roughly speaking, the fragment generated by a countable subset X of $L_{\omega_1,\omega}(\tau)$ is the closure of X under first order operations. We preserve Keisler's terminology to emphasize that the theorem deals only with the number of models and does not involve the choice of 'elementary embedding' on the class.

{keissmall}

Theorem 2.4 (Keisler)**.** *If a PC_δ over $L_{\omega_1,\omega}$ class $\boldsymbol{K}$ has an uncountable model but less than 2^{ω_1} models of power $\aleph_1$ then $\boldsymbol{K}$ is locally τ-small. That is, for any countable fragment L^* of $L_{\omega_1,\omega}(\tau)$, each $M \in \boldsymbol{K}$ realizes only countably many L^*-types over $\emptyset$.*

By just changing a few words in the proof of Theorem 6.3.1 of [1], (originally in [14]) one can obtain the following result, which was implicit in [18].

{getsmall}

Theorem 2.5. *If $\boldsymbol{K}$ is an $\aleph_0$-presentable AEC and some model $M \in \boldsymbol{K}$ of cardinality $\aleph_1$ is* locally τ-small*, then $\boldsymbol{K}$ has a $L_{\omega_1,\omega}(\tau)$-small model N of cardinality $\aleph_1$.*

Proof. Let ϕ be the τ^+-sentence whose reducts to τ are the members of $\boldsymbol{K}$. Without loss of generality we may assume the universe of M is ω_1. Add to τ^+ a binary relation $<$, interpreted as the usual order on ω_1. Using the fact that M realizes only countably many types in any τ-fragment, define a continuous increasing chain of countable fragments L_α for $\alpha < \aleph_1$ such that

- for each quantifier free (first order) n-type over the empty set realized in M, the conjunction of the type is in L_0, and
- the conjunction of each type in L_α that is realized in M is a formula in $L_{\alpha+1}$.

Extend the similarity type further to τ' by adding new $(2n+1)$-ary predicates $E_n(x, \mathbf{y}, \mathbf{z})$ and $(n+1)$-ary functions f_n for each $n \in \omega$. Let M satisfy $E_n(\alpha, \mathbf{a}, \mathbf{b})$ if and only if $\mathbf{a}$ and $\mathbf{b}$ realize the same L_α-type, and let the interpretation of f_n map M^{n+1} into ω in such a way that $E_n(\alpha, \mathbf{a}, \mathbf{b})$ if and only if $f_n(\alpha, \mathbf{a}) = f_n(\alpha, \mathbf{b})$ for all suitable α, $\mathbf{a}$, $\mathbf{b}$. Then the following hold.

1. The equivalence relations $E_n(\beta, \overline{x}, \overline{y})$ refines $E_n(\alpha, \overline{x}, \overline{y})$ if $\beta > \alpha$;
2. $E_n(0, \mathbf{a}, \mathbf{b})$ implies that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same quantifier free τ-formulas;
3. If $\beta > \alpha$ and $E_n(\beta, \mathbf{a}, \mathbf{b})$, then for every c_1 there exists c_2 such that $E_{n+1}(\alpha, c_1\mathbf{a}, c_2\mathbf{b})$, and
4. f_n witnesses that for any $a \in M$, each equivalence relation $E_n(a, \overline{x}, \overline{y})$ has only countably many classes.

All these assertions can be expressed by an $L_{\omega_1,\omega}(\tau')$ sentence χ. Let L^* be the smallest τ'-fragment containing $\chi \wedge \phi$. Now by the Lopez-Escobar bound on $L_{\omega_1,\omega}$ definable well-orderings, Theorem 5.3.8 of [1], there is a τ'-structure N of cardinality $\aleph_1$ satisfying $\chi \wedge \phi$ such that there is an infinite decreasing sequence $d_0 > d_1 > \ldots$ in N (alternately, one could use Lemma 2.5 of [3] for this step). For each n, let $E^+_n(\overline{x}, \overline{y})$ denote the assertion that for some i, $E_n(d_i, \overline{x}, \overline{y})$.

Using 1), 2) and 3) one can prove by induction on quantifier rank (for all $n \in \omega$ simultaneously) that for all n-ary $L_{\omega_1,\omega}(\tau)$ formulas μ, and all finite tuples $\mathbf{a}, \mathbf{b}$ from N, if $E^+_n(\mathbf{a}, \mathbf{b})$ holds then $N \models \mu(\mathbf{a})$ if and only if $N \models \mu(\mathbf{b})$. To see this, suppose that this assertion holds for all n and all θ with quantifier rank at most γ. Let $\mu(\overline{z})$ be an n-ary formula of the form $(\exists x)\theta(\overline{z}, x)$, where θ has quantifier rank γ. Let $\mathbf{a}, \mathbf{b}$ be n-tuples from N for which $E^+_n(\mathbf{a}, \mathbf{b})$ holds and $N \models \mu(\mathbf{a})$. Then for some i, $E_n(d_i, \mathbf{a}, \mathbf{b})$ and for some a, $N \models \theta(\mathbf{a}, a)$. By condition 3) above there is a b such that $E_{n+1}(d_{i+1}, \mathbf{a}, a, \mathbf{b}, b)$. By our induction hypothesis we have $N \models \theta(\mathbf{b}, b)$ and so $N \models \mu(\mathbf{b})$.

Now, for each n, $E_n(d_0, \overline{x}, \overline{y})$ refines $E^+_n(\overline{x}, \overline{y})$ and by 4) $E_n(d_0, \overline{x}, \overline{y})$ has only countably many classes, so $N\restriction\tau$ is small. $\square_{2.5}$

Definition 2.6. *We say a countable structure is* extendible *if it has an $L_{\omega_1,\omega}$-elementary extension to an uncountable model.*

{vfimpcomp}

Lemma 2.7. *Suppose that $\boldsymbol{K}$ is the class of reducts to τ of a class defined by a sentence $\phi \in L_{\omega_1,\omega}(\tau^+)$, where τ^+ is a countable vocabulary extending τ. If some uncountable $M \in \boldsymbol{K}$ is locally τ-small but is not $L_{\omega_1,\omega}(\tau)$-small then*

1. *There are at least $\aleph_1$ pairwise-inequivalent complete sentences of $L_{\omega_1,\omega}(\tau)$ which are satisfied by uncountable models in $\boldsymbol{K}$;*
2. *$\boldsymbol{K}$ has uncountably many small models in $\aleph_1$ that satisfy distinct complete sentences of $L_{\omega_1,\omega}(\tau)$;*
3. *$\boldsymbol{K}$ has uncountably many extendible models in $\aleph_0$.*

Proof. Suppose that M is a model in $\boldsymbol{K}$ with cardinality $\aleph_1$ that is is locally τ-small but is not $L_{\omega_1,\omega}(\tau)$-small. Let M^+ be an expansion of M to a τ^+-structure satisfying ϕ. We construct a sequence of τ^+-structures $\{N^+_\alpha : \alpha < \omega_1\}$ each with cardinality $\aleph_1$ and an increasing continuous family of countable fragments $\{L'_\alpha : \alpha < \omega_1\}$ of $L_{\omega_1,\omega}(\tau)$ and sentences χ_α such that:

1. $L'_0(\tau)$ is first order logic on τ;
2. all the models N^+_α satisfy ϕ;
3. for each $\alpha < \omega_1$, $N^+_\alpha\restriction\tau$ is $L_{\omega_1,\omega}(\tau)$-small;
4. χ_α is the $L_{\omega_1,\omega}(\tau)$-Scott sentence of N_α;
5. $L'_{\alpha+1}(\tau)$ is the smallest fragment of $L_{\omega_1,\omega}(\tau)$ containing $L'_\alpha(\tau) \cup \{\neg\chi_\alpha\}$;
6. For limit δ, $L'_\delta(\tau) = \bigcup_{\alpha<\delta} L'_\alpha(\tau)$;
7. For each α, $N_\alpha \equiv_{L'_\alpha(\tau)} M$.

Working by recursion, suppose that we have constructed N_α for all $\alpha < \beta$, for some countable ordinal β. This determines each χ_α ($\alpha < \beta$) as the Scott sentence of N_α and also determines $L'_\beta(\tau)$. Since M is not small, $M \models \neg\chi_\alpha$ for each $\alpha < \beta$. Apply Theorem 2.5 to M and the restriction of $\boldsymbol{K}$ to models $L'_\beta(\tau)$-elementarily equivalent to M to construct N_β.

Now the N_α are pairwise non-isomorphic since each satisfies a distinct complete sentence χ_α of $L_{\omega_1,\omega}(\tau)$, so conclusions 1) and 2) are satisfied. And each N_α has a countable elementary submodel with respect to $L'_{\alpha+1}(\tau)$, so there are at least $\aleph_1$ non-isomorphic extendible models in $\aleph_0$ as well. $\square_{2.7}$

Putting together Theorem 2.4 and Lemma 2.7, we have the following.

{allsmallcor}

Corollary 2.8. *If an $\aleph_0$-presented AEC $\boldsymbol{K}$ has only countably many models in $\aleph_1$, then every model in $\boldsymbol{K}$ is small.*

Lemma 2.7 leads to several corollaries connected to the Vaught conjecture. First we recall the following result of Harnik and Makkai [8].

{HM}

Theorem 2.9 (Harnik-Makkai). *If $\sigma \in L_{\omega_1,\omega}$ is a counterexample to Vaught's Conjecture then it has a model of cardinality $\aleph_1$ which is not small.*

{corvc}

Corollary 2.10. *If $\phi \in L_{\omega_1,\omega}$ is a counterexample to the Vaught conjecture then ϕ has $\aleph_1$* extendible *countable models.*

Proof. If $\phi \in L_{\omega_1,\omega}$ is a counterexample to Vaught's conjecture, then every uncountable model of ϕ is locally small. The result then follows from Theorem 2.9 and Lemma 2.7. $\square_{2.10}$

Su Gao pointed out another argument for this observation. By Becker [4], the Vaught conjecture holds for countable models such that $\mathrm{aut(M)}$ is a 'cli' group (admits a compatible left-invariant complete metric). By Gao [6], $\mathrm{aut(M)}$ is 'cli' if and only if M is not extendible. Thus if there is failure of Vaught's conjecture there must be $\aleph_1$ extendible countable models.

{iran1}

Remark 2.11. Clearly, if $\boldsymbol{K}$ has only countably many models in $\aleph_1$ then $\boldsymbol{K}$ has at most $\aleph_0$ non-isomorphic extendible countable models (since each uncountable model is $L_{\omega_1,\omega}$-equivalent to at most one model in $\aleph_0$). The three conclusions of Lemma 2.7 are easily seen to be equivalent; we separated them in the statement because both the countable and uncountable models arose naturally in the proof. The converse of Lemma 2.7 asserts if that $\boldsymbol{K}$ has uncountably many extendible countable models a locally small model in $\aleph_1$ then it has a non-small model in $\aleph_1$. Theorem 2.9 shows this is true if the hypothesis is changed to 'uncountably many countable models, but not a perfect set of countable models', without requiring extendibility. In general, the converse is false. The empty theory in a vocabulary with $\aleph_0$ constants has $2^{\aleph_0}$ models (depending on which constants are identified) in each of $\aleph_1$ and $\aleph_0$; all are small. But joint embedding and amalgamation fail even under first order elementary submodel. Example 2.1.1 of [2] is a sentence of $L_{\omega_1,\omega}$ giving rise to an AEC, with a particular notion of $\prec_{\boldsymbol{K}}$ (weaker than first order), which satisfies amalgamation and joint embedding and is $\aleph_1$-categorical, and for which the model in $\aleph_1$ is small. In this case there are $2^{\aleph_0}$ countable models, but only one of them is extendible.

Definition 2.12. *A sentence σ of $L_{\omega_1,\omega}$ is* large *if it has uncountably many countable models. A large sentence σ is* minimal *if for every sentence ϕ either $\sigma \wedge \phi$ or $\sigma \wedge \neg\phi$ is not large.*

As part of their proof of Theorem 2.9, Harnik and Makkai showed that any counterexample to Vaught's conjecture can strengthened to a minimal counterexample. We call a model of cardinality $\aleph_1$ *large* if it is not $L_{\omega_1,\omega}$-small in the sense of Definition 2.2. Lemma 2.7 implies that if ϕ has a large model in $\aleph_1$ then ϕ is large.

{cor2vc}

Corollary 2.13. *If ϕ is a minimal counterexample to Vaught's conjecture then ϕ has a large model in $\aleph_1$, and all large models of ϕ in $\aleph_1$ are $L_{\omega_1,\omega}$-elementarily equivalent.*

Proof. Theorem 2.9 says that ϕ has a large model N. Suppose that $\phi \in L_{\omega_1,\omega}$ holds in N. The fact that $\phi \wedge \psi$ has a large model implies by Lemma 2.7 that $\phi \wedge \psi$ has uncountably many models in $\aleph_0$. By minimality, $\phi \wedge \neg\psi$ has only countably many models in $\aleph_0$ and so by Lemma 2.7 again, all uncountable models of $\phi \wedge \neg\psi$ are small. $\square_{2.13}$

We pause to connect this analysis with a related but subtly distinct procedure.

{Manal}

Definition 2.14. *1.* **Morley's Analysis**

(a) *Let $L_0^{\mathbf{K}}$ be the set of first order τ-sentences.*

(b) *Let $L_{\alpha+1}^{\mathbf{K}}$ be the smallest fragment generated by $L_\alpha^{\mathbf{K}}$ and the sentences of the form $(\exists \mathbf{x}) \bigwedge p(\mathbf{x})$ where p is an $L_\alpha^{\mathbf{K}}$-type realized in a model in $\boldsymbol{K}$.*

(c) *For limit δ, $L_\delta^{\mathbf{K}} = \bigcup_{\alpha<\delta} L_\alpha^{\mathbf{K}}$.*

2. *$\boldsymbol{K}$ is scattered if and only if for each $\alpha < \omega_1$, $L_\alpha^{\mathbf{K}}$ is countable.*

Recall Morley's theorem, which is key to his approach to Vaught's conjecture.

{getscat}

Theorem 2.15 (Morley)**.** *If $\boldsymbol{K}$ is the class of models of a sentence in $L_{\omega_1,\omega}$ that has less than $2^{\aleph_0}$ models of power $\aleph_0$ then $\boldsymbol{K}$ is scattered.*

Remark 2.16. We cannot conclude that $\boldsymbol{K}$ is scattered from just counting models in $\aleph_1$, even from the hypothesis that $\boldsymbol{K}$ is $\aleph_1$-categorical. Again, Example 2.1.1 of [2] (Remark 2.11) is $\aleph_1$-categorical and has joint embedding for $\prec_{\boldsymbol{K}}$. But there are $2^{\aleph_0}$ first order types that give models that are not even first order mutually embeddible and the class $\boldsymbol{K}$ is not scattered.

Remark 2.17. *The sequence of languages in Theorem 2.5 might be labeled L_α^M. They come about by applying the Morley analysis solely to the types realized in M. So this gives a slower growing sequence of languages than the Morley analysis. If $\boldsymbol{K}$ has either less than $2^{\aleph_0}$ models in $\aleph_0$ or less than $2^{\aleph_1}$ models in $\aleph_1$, then every uncountable model of $\boldsymbol{K}$ is locally small.*

Remark 2.18. The arguments of Morley and Shelah have different goals. Being scattered is a condition on all models of an (in the interesting case for the Vaught conjecture) an *incomplete* sentence in $L_{\omega_1,\omega}$. The Shelah argument contracts $\boldsymbol{K}$ to a smaller class where every model is small and thus finds a $\boldsymbol{K}' \subset \boldsymbol{K}$ that is small and is axiomatized by a complete sentence. The hard part is to make sure $\boldsymbol{K}'$ has an uncountable model. In the most used case, $\boldsymbol{K}$ and *a fortiori* $\boldsymbol{K}'$ is $\aleph_1$-categorical.

2.1 Alternate proofs using Scott sentences

In this subsection we prove alternate versions of Theorem 2.5 and part of Lemma 2.7. Theorem 2.19 can be used in place of Theorem 2.5 in all of our applications of Theorem 2.5, and the basic idea of the proof is the same. For convenience we use the theory ZFC° from [3]. Any theory strong enough to carry out the construction of Scott sentences should be sufficient.

{getsmall2}

Theorem 2.19. *Let τ be a countable vocabulary, let M be a τ-structure, and let N be an ω-model of* ZFC$^\circ$ *with ω_1^N illfounded. Let β be the ordinal isomorphic to the longest wellfounded initial segment of ω_1^N. Suppose that, in N, M is locally τ-small and either large or small with Scott rank in the illfounded part of N. Then M is small, and the Scott rank of M is exactly β.*

Proof. Let t be the Scott rank of M in N if N thinks that M is small, and ω_1^N otherwise. Let

$$\langle \phi_{\mathbf{a},s} : \mathbf{a} \in M^{<\omega}, s < t\rangle$$

be the set of formulas defined in N in the first t many steps of the search for a Scott sentence for M. Then

$$\langle \phi_{\mathbf{a},\alpha} : \mathbf{a} \in M^{<\omega}, \alpha < \beta\rangle$$

is also the set of formulas defined in V in the first β many steps of the search for a Scott sentence for M. Since the Scott rank of M in N is in the illfounded part of N if it exists, the Scott rank of M in V is at least β.

We claim that for any $n \in \omega$ and any pair $\mathbf{a}$, $\mathbf{b}$ of n-tuples from M, if $\phi_{\mathbf{a},s} = \phi_{\mathbf{b},s}$ for any illfounded $s < t$, then $\mathbf{a}$ and $\mathbf{b}$ satisfy all the same $L_{\omega_1,\omega}(\tau)$-formulas in M (from the point of view of V). To see this, suppose that this assertion holds for all n and all formulas θ with quantifier rank at most γ. Let $\mu(\overline{z})$ be an n-ary formula of the form $(\exists x)\theta(\overline{z}, x)$, where θ has quantifier rank γ. Let $\mathbf{a}, \mathbf{b}$ be n-tuples from N, let $s < t$ be an illfounded ordinal of N such that $\phi_{\mathbf{a},s} = \phi_{\mathbf{b},s}$, and suppose that $M \models \mu(\mathbf{a})$. Then there is an illfounded $r < s$, and for any such r, $\phi_{\mathbf{a},r} = \phi_{\mathbf{b},r}$. Since $M \models \mu(\mathbf{a})$, there is a $c \in M$ such that $M \models \theta(\mathbf{a}, c)$. Since $r < s$ and $\phi_{\mathbf{a},s} = \phi_{\mathbf{b},s}$, $\phi_{\mathbf{a},r+1} = \phi_{\mathbf{b},r+1}$, which means that there is some $d \in M$ such that $\phi_{\mathbf{b}d,r} = \phi_{\mathbf{a}c,r}$. Thus by our induction hypothesis, $M \models \theta(\mathbf{b}, d)$ and thus $M \models \mu(\mathbf{b})$.

For each $n \in \omega$ and each pair $\mathbf{a}, \mathbf{b}$ of n-tuples from M, if $\phi_{\mathbf{a},\alpha} = \phi_{\mathbf{b},\alpha}$ for all $\alpha < \beta$, then $\phi_{\mathbf{a},s} = \phi_{\mathbf{b},s}$ for some illfounded $s < t$, since if $\phi_{\mathbf{a},r} \neq \phi_{\mathbf{b},r}$ for any $r < t$, then N thinks that there is a least such r, and there is no least illfounded ordinal of N. It follows then that the Scott rank of M (in V) is exactly β. □

Lemma 2.20 will make up part of the proof of our main theorem (Theorem 3.10). The proof is in fact a simplified part of the main argument in the proof of that theorem.

{vfimpcomp2}

Lemma 2.20. *Suppose that $\boldsymbol{K}$ is the class of reducts to τ of a class defined by a sentence $\phi \in L_{\omega_1,\omega}(\tau^+)$, where τ^+ is a countable vocabulary extending τ. If $\boldsymbol{K}$ has a model in $\boldsymbol{K}_{\aleph_1}$ that is locally τ-small, but is not $L_{\omega_1,\omega}(\tau)$-small then $\boldsymbol{K}$ has small models in $\aleph_1$ of uncountably many distinct Scott ranks.*

Proof. It suffices to show that for each $\gamma < \omega_1$ there exists a small model in $\boldsymbol{K}$ of cardinality $\aleph_1$ which has Scott rank at least γ. Fix $\gamma < \omega_1$. Suppose that M is a model in $\boldsymbol{K}$ with cardinality $\aleph_1$ that is is locally τ-small but is not $L_{\omega_1,\omega}(\tau)$-small.

Fix a regular cardinal $\theta > 2^{2^{\aleph_1}}$ and let X be a countable elementary submodel of $H(\theta)$ such that τ, ϕ, M and γ are in X. Let P be the transitive collapse of X, and let $\rho\colon X \to P$ be the corresponding collapsing map. Then $\rho(\omega_1) = \omega_1^P$ is the ordinal $X \cap \omega_1$, which is greater than γ.

By iterating the construction in [12], or by an iteration of ultrapowers of models of set theory as in Lemma 1.5 of [3], one can find an elementary extension P' of P with corresponding elementary embedding $\pi\colon P \to N$ such that ω_1^N is ill-founded and uncountable, and the critical point of π is ω_1^P. It follows that the ordinals of N are well-founded at least up to ω_1^P. (By the construction in [12], the wellfounded ordinals of N can be made to be exactly ω_1^P.)

Since ω_1^N is illfounded, Theorem 2.19 implies that $\pi(\rho(M))$ is $L_{\omega_1,\omega}(\tau)$-small, with Scott rank equal to the longest wellfounded initial segment of of ω_1^N, which is greater than γ. □

3 Almost Galois Stability

{ags}

Fixing a coding of hereditarily countable sets by real numbers, the notion of Galois types naturally induces an equivalence relation on the reals. For each countable $M \in \boldsymbol{K}$ we let E_M denote the corresponding equivalence relation for Galois types over M (this notation was used in [3]). If $\boldsymbol{K}$ is $\aleph_0$-presented, then each E_M is an analytic equivalence relation, and by Burgess's trichotomy for analytic equivalence relations, E_M has either countably many equivalence classes, $\aleph_1$ many, or a perfect set of inequivalent reals.[1] Because there are two notions of weak-stability in the literature of AEC ([10, 17], we call the following notion *almost Galois ω-stability*.

Definition 3.1. *$\boldsymbol{K}$ is* almost Galois ω-stable *if for no countable model M, E_M has a perfect set of equivalence classes.*

Remark 3.2 (Amalgamation, joint embedding, and maximal models)**.** In this remark, we collect a number of easy and well-known observations about the properties of the title. These observations should provide a background for understanding the choice of some 'background hypotheses'. If an AEC has no maximal models then it has arbitrarily large models. In general the converse fails; but the converse holds under joint embedding with one trivial exception: an AEC with a *unique* model[2] satisfies joint embedding. If we restrict to $\boldsymbol{K}_\lambda$, we see no maximal model of cardinality λ implies there is a model of λ^+ and joint embedding in λ implies no maximal model in λ.

Under amalgamation, the relation M and N have a common strong extension is an equivalence relation and each equivalence class is an AEC with joint embedding. Often we will assume amalgamation and joint embedding to avoid assuming only amalgamation and then having to restrict to one joint embedding class. Failure to make this assumption yields to trivial counterexamples. There are no universal models for the class of algbraically closed fields (because of characteristic) but fixing the characteristic (that is the joint embedding class) yields a smooth theory.

So assuming amalgamation in λ and no maximal model in λ is slightly more general that assuming amalgamation and joint embedding in λ but requires the restatement of standard arguments so it is more common to assume amalgamation and joint embedding in λ.

Amalgamation and some form of joint embedding easily allows one to show the following (see Corollary 8.23 of [1]); we give two variants.

{satmodel}

Lemma 3.3. *Suppose that $\boldsymbol{K}$ is almost Galois ω-stable and satisfies amalgamation in $\aleph_0$, then*

1. *If $\boldsymbol{K}$ has joint embedding in $\aleph_0$ then then there is a unique Galois-saturated model M in $\aleph_1$.*
2. *If $N \in \boldsymbol{K}$ is countable, and E_N has $\aleph_1$ many equivalence classes, then there is a Galois-saturated model M in $\aleph_1$ with $N \prec_{\boldsymbol{K}} M$.*

Proof. For the first, carefully construct an interweaving enumeration the Galois types over an increasing chain of countable models in order type ω_1 so that each Galois type over each model in the chain is realized.

For the second, let $\boldsymbol{K}_N$ be the equivalence class under joint embedding of the models that are jointly embeddable with N. Apply the first argument to this class. $\square_{3.3}$

Note that without joint embedding we cannot conclude uniqueness of Galois saturated models. As in this argument, there may be models which are Galois saturated with respect to a sub-AEC that are mutually non-embeddable. In particular, there may be countable models that are not extendible.

[1] Alternately, letting τ be the vocabulary associated to $\boldsymbol{K}$, the set of τ-structures with domain ω can be viewed as a Polish space, with the set of countable models in $\boldsymbol{K}$ as an analytic subset. See [7].

[2] Unique means there is no extension, even one isomorphic to itself.

{notsofast}

Example 3.4. Let $\boldsymbol{K}$ be the set of structures in the language with a single equivalence relation E that have infinitely many elements in each class and exactly $\aleph_0$ many classes. Let $\prec_{\boldsymbol{K}}$ be the relation of first order elementarity. This gives an $\aleph_0$-presentable class. There are $\aleph_0$ models in $\aleph_1$ (given by the number of classes that have cardinality $\aleph_0$ and $\aleph_1$ respectively). Let M be the $\aleph_1$-saturated model in $\boldsymbol{K}$ of cardinality $\aleph_1$, which has uncountably many elements in each class. Let M_0 be a countable submodel of M.

Given an element a of a model $N \in \boldsymbol{K}$, let $[a]_N$ denote the E-equivalence class of a as interpreted in N. Let $\boldsymbol{K}'$ be the class of countable models $N \in \boldsymbol{K}$ containing M_0 for which

$$\{[a]_N \mid a \in M_0 \wedge |[a]_N \setminus [a]_{M_0}| = n\}$$

is finite, for each $m \in \omega$. For each $N \in \boldsymbol{K}'$, let $c_N : \omega \to \omega$ be defined by setting

$$c_N(m) = |\{[a]_N \mid a \in M_0 \wedge |[a]_N \setminus [a]_{M_0}| = n\}|.$$

Then for each $c\colon \omega \to \omega$ there is an $N \in \boldsymbol{K}'$ for which $c_N = c$, and for any two $N, N' \in \boldsymbol{K}'$, N and N' are isomorphic via an isomorphism fixing M_0 setwise if and only if $c_N = c_{N'}$. Furthermore, even among the elements N of $\boldsymbol{K}'$ for which c_N is the constant function 1, there are $2^{\aleph_0}$ many models (corresponding naturally to the permutations of ω) which are not isomorphic via isomorphisms fixing M_0 pointwise.

Finally, if N and N' are elements of $\boldsymbol{K}'$ with $N \subseteq N'$, then $c_N = c_{N'}$ if and only if $[a]_N = [a]_{N'}$ for every $a \in M_0$ with $[a]_N$ finite (in which case there is an isomorphism between N and N' fixing M_0 pointwise). To see this, suppose that the latter statement is false and consider the least m for which there is an $a \in M_0$ with $m = [a]_N \setminus [a]_{M_0} \neq [a]_{N'} \setminus [a]_{M_0}$. Then $c_N(m) > c_{N'}(m)$.

This example illustrates the importance of studying truth in a particular model when we do not have a monster model that is homogenous over sets.

For any AEC $\boldsymbol{K}$, if $M, N \in \boldsymbol{K}$ and $M \prec_K N$, then M is a substructure of N, but the definition of AEC does not require even that M be a first-order elementary submodel of N. Before proving the main result of this section, Theorem 3.10, we prove a lemma which reduces the proof to the case where $M \prec_K N$ implies $L_{\omega_1,\omega}(\tau)$-elementarity. A similar reduction appears in Theorem 3.6 E) of [18] and Lemma 2.5 of [14].

Definition 3.5. *Let $\boldsymbol{K}$ be an AEC in a countable similarity type τ, with Löwenheim-Skolem number $\aleph_0$, such that $\boldsymbol{K}$ has a unique Galois-saturated model M in $\aleph_1$ that is small.*

Let $\boldsymbol{K}^* = \{N \in \boldsymbol{K} : |N| = \aleph_0 \wedge N \prec_{\boldsymbol{K}'} M\}$, where $N_0 \prec_{\boldsymbol{K}'} N_1$ if $N_0 \prec_{\boldsymbol{K}} N_1$ and $N_0 \prec_{\infty,\omega} N_1$.

Let $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ be the closure of $(\boldsymbol{K}^*, \prec_{\boldsymbol{K}'})$ under isomorphism and direct limits of arbitrary length.

To discuss the relationship between (almost) Galois stability of $\boldsymbol{K}$ and $\boldsymbol{K}'$, we introduce some notation. We first give a standard equivalent for the definition of Galois type, but parameterized for the comparisons we need here. The class $\boldsymbol{K}_0$ below will be $\boldsymbol{K}$ or $\boldsymbol{K}'$ in our applications.

Notation 3.6. *Let $\boldsymbol{K}_0$ be an AEC with a Galois saturated (equivalently, $\boldsymbol{K}_0$-$\aleph_1$-homogenous-universal) model M in $\aleph_1$.*

1. *If $M_0 \prec_{\boldsymbol{K}_0} M$, $\mathbb{S}_{\boldsymbol{K}_0}(M_0)$ is the collection of orbits of elements of M under $\mathrm{aut}_{\mathrm{M}_0}(\mathrm{M})$ (the automorphisms of M fixing M_0 pointwise).*
2. $\alpha(\boldsymbol{K}_0) = \sup\{|\mathbb{S}_{\boldsymbol{K}_0}(M_0)| : M_0 \in \boldsymbol{K}_0, |M_0| = \aleph_0\}$.

{addelema}

Lemma 3.7. *Let $\boldsymbol{K}$ be an AEC in a countable similarity type τ, with Löwenheim-Skolem number $\aleph_0$, such that $\boldsymbol{K}$ has a unique Galois-saturated model M in $\aleph_1$ that is small. Then the following hold.*

1. $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ *is an AEC with Löwenheim-Skolem number* $\aleph_0$.

2. M *is* $\boldsymbol{K}'$-$\aleph_1$*-homogenous-universal.*

3. $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ *satisfies amalgamation in* $\aleph_0$.

4. *For every* $M_0 \in \boldsymbol{K}'_{\aleph_0}$, $\mathbb{S}_{\boldsymbol{K}}(M_0) = \mathbb{S}_{\boldsymbol{K}'}(M_0)$.

5. $\alpha(\boldsymbol{K}) = \alpha(\boldsymbol{K}')$.

6. *If* $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *is* $\aleph_0$*-presented then so is* $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$.

Proof. 1) The coherence and unions of chains axioms are immediate on $\boldsymbol{K}^*$. For Löwenheim-Skolem, note that M can be written as an increasing chain of $\boldsymbol{K}'$-submodels. Thus, $\boldsymbol{K}^*$ is a weak AEC in the sense of Definition 16.10 of [1] and so $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ is an AEC by Exercise 16.12.

2) Let $M_0 \prec_{\boldsymbol{K}'} M_1$ be countable. Then there are $\boldsymbol{K}'$-maps f and g such that $f(M_0) \prec_{\boldsymbol{K}'} M$ and $g(M_1) \prec_{\boldsymbol{K}'} M$ by the definition of $\boldsymbol{K}'$. But since M is $\boldsymbol{K}$-$\aleph_1$-homogenous-universal, there is an h in $\mathrm{aut(M)}$ mapping $f(M_0)$ onto $g(M_0)$. Since both $\prec_{\boldsymbol{K}}$ and $\prec_{\boldsymbol{K}'}$ are preserved by automorphisms, h is a $\boldsymbol{K}'$-map. So $h \circ g$ is a $\boldsymbol{K}'$ embedding of M_1 into M extending f.

3) Suppose $M_0 \prec_{\boldsymbol{K}'} M_1, M_2$. Then there are $\boldsymbol{K}'$-embeddings of M_1 and M_2 over M_0 into M. So amalgamation holds.

4) The Galois types are determined by $\mathrm{aut_{M_0}M}$ which does not depend on the choice of AEC.

5) We have that $\alpha(\boldsymbol{K}) \geq \alpha(\boldsymbol{K}')$ since the supremum is taken over a smaller set. But for each $M_0 \in \boldsymbol{K}_{\aleph_0}$, there is an $M_1 \in \boldsymbol{K}'_{\aleph_0}$ with $M_0 \prec_{\boldsymbol{K}} M_1 \prec_{\boldsymbol{K}'} M$ and by the extendability of $\boldsymbol{K}$-Galois types, and part 4, $|\mathbb{S}_{\boldsymbol{K}}(M_0)| \leq |\mathbb{S}_{\boldsymbol{K}}(M_1)| = |\mathbb{S}_{\boldsymbol{K}'}(M_0)|$.

6) Let τ' and $\tau'' = \tau' \cup \{P\}$ be the vocabularies which witness that $\boldsymbol{K}$ is $\aleph_0$-presented; let ψ_1 be the τ' sentence whose reducts are the models in $\boldsymbol{K}$; Let ψ_2 be the τ'' sentence whose reducts are pairs (N, M) with $N \prec_{\boldsymbol{K}} M$. Further suppose ϕ is the Scott sentence of M. The following sentences witness that $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ is $\aleph_0$-presented: $\hat{\psi}_1 = \psi_1 \wedge \phi$ and $\hat{\psi}_2 = \psi_2 \wedge \chi$ where $(M, N) \models \chi$ if $M \prec_{L^*} N$ where L^* is least countable fragment containing ϕ. $\square_{3.7}$

Conclusion 5 immediately yields.

Corollary 3.8. *Under the conditions of Lemma 3.7,*

- $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *is Galois* ω*-stable if and only if* $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ *is;*
- $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *is almost Galois* ω*-stable if and only if* $(\boldsymbol{K}', \prec_{\boldsymbol{K}'})$ *is.*

By Lemma 2.7, we get the small saturated model from assuming countably many models in $\aleph_1$.

Corollary 3.9. *If* $\boldsymbol{K}$ *is an almost Galois stable class with amalgamation and the joint embedding property, and only countably many models in* $\aleph_1$ *then the hypotheses of Lemma 3.7 hold.*

Moreover, the hypothesis of joint embedding is only a convenience; if $\boldsymbol{K}$ has the amalgamation property then joint embedding is an equivalence relation and each of the equivalence classes is an AEC with joint embedding preserving all other properties. At least one class fails Galois ω-stability if $\boldsymbol{K}$ does.

If we dropped the assumption of joint embedding in the following Theorem with slightly more complication we could handle the case where there were countable non-extendible models that cannot be jointly embedded with the others.

{mthm}

Theorem 3.10. *Suppose that $\boldsymbol{K}$ is an $\aleph_0$-presented AEC which satisfies amalgamation, and JEP for countable models, and that $\boldsymbol{K}$ is almost Galois ω-stable. If $\boldsymbol{K}$ has only countably many models in $\aleph_1$, then* {stillsmall1}

1. *For every countable $N \in \boldsymbol{K}$ there is a countable fragment $L^*(\tau')$ of the expanded language τ' where constants are added for each member of N such that for every $M \in \boldsymbol{K}$ with $N \prec_{\boldsymbol{K}} M$ and $N \prec_{\infty,\omega} M$, M is L_1-atomic. This implies* {getstab}
2. *$\boldsymbol{K}$ is Galois ω-stable.*

Proof. The proof of Theorem 3.10 takes up the rest of this section. By Corollary 2.8, all models in $\boldsymbol{K}$ are small. Applying Lemma 3.3, let M be a small Galois-saturated model of cardinality $\aleph_1$, and let ϕ be the Scott sentence of M. By Lemma 3.7, we may assume that the relation $\prec_{\boldsymbol{K}}$ implies full $L_{\omega_1,\omega}(\tau)$-elementarity.

Notation 3.11. *Fix any countable model $M_0 \in \boldsymbol{K}$. Let τ' be the extension of τ formed by adding new constant symbols c_i ($i \in \omega$), and let M' be a τ'-structure expanding M, where the interpretation of the c_i's enumerates M_0.*

Now there are three cases; we will show cases 1) and 2a) contradict the hypothesis of almost Galois ω-stability while case 2b) implies statement 1) of Theorem 3.10 which yields statement 2) Galois ω-stability.

1. For some countable fragment $L^*(\tau')$ of $L_{\omega_1,\omega}(\tau')$, there are uncountably many $L^*(\tau')$-types realized in M' .
2. For every countable fragment $L_0(\tau')$ of $L_{\omega_1,\omega}(\tau')$, only countably many $L_0(\tau')$-types are realized in M'. Then one of the following holds.
   (a) The model M' is not $L_{\omega_1,\omega}(\tau')$-small.
   (b) The model M' is $L_{\omega_1,\omega}(\tau')$-small, so for some countable fragment $L^*(\tau')$, M' has a Scott sentence in $L^*(\tau')$;

In Case 1, there exists a perfect set of syntactic types in $L^*(\tau')$. Since $\prec_{\boldsymbol{K}}$ implies $L_{\omega_1,\omega}(\tau)$-elementarity, this implies the existence of a perfect set of Galois types over M_0, contradicting the almost Galois ω-stability of $\mathbf{K}$.

The bulk of the proof derives a contradiction from Case 2a. Since $\boldsymbol{K}$ is almost Galois ω-stable but not Galois ω-stable, there exists a countable $M_0 \in \mathbf{K}$ such that E_{M_0} has exactly $\aleph_1$ many equivalence classes and satisfies 2a). We use this model to show case 2a) is impossible. Note that only this case requires that there are only countably many models in $\aleph_1$.

{nice2dec}

Definition 3.12. *Suppose that a model M with cardinality $\aleph_1$ is the union of an uncountable chain of countable models $\langle M_\alpha : \alpha < \omega_1\rangle$. For each $\alpha < \omega_1$, let F_α be an automorphism of M mapping M_0 onto M_α. Then we say that $\langle M_\alpha, F_\alpha : \alpha < \omega_1\rangle$ is a* nice decomposition *of M, and we let $F_{\alpha,\beta}$ denote $F_\beta \circ F_\alpha^{-1}$.*

Recall that ϕ is the Scott sentence for M. Let $\overline{M} = \langle M_\alpha : \alpha < \omega_1\rangle$ be such that (as above) M_0 is the model enumerated by $\mathbf{c}$, $M = \bigcup_{\alpha<\omega_1} M_\alpha$ and the following hold for each $\alpha < \omega_1$:

- M_α is a countable element of $\boldsymbol{K}$;
- $M_\alpha \prec_{\boldsymbol{K}} M$;

- $M_\alpha \models \phi$;
- M_α is a proper subset of $M_{\alpha+1}$;
- if α is a limit ordinal, then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.

The models M_α are all isomorphic, as they satisfy the same Scott sentence. As M is Galois saturated, there is a set $\overline{F} = \{F_\alpha : \alpha < \omega_1\}$ such that $\langle M_\alpha, F_\alpha : \alpha < \omega_1\rangle$ is a nice decomposition of M.

Let τ^+ be the expansion of our vocabulary τ' to the τ' of Theorem 2.5 (i.e., add the symbols E_n, f_n $(n \in \omega)$, and a binary relation ordering the domain of M in order type ω_1; alternately, using Theorem 2.19 below we could skip this step). Fix a regular cardinal θ large enough so that M', τ^+, $\overline{M}$ and $\overline{F}$ are elements of $H(\theta)$ (to apply the methods of [3], we need θ to be larger than $2^{2^{\aleph_1}}$).

Let $\langle X_\alpha : \alpha < \omega_1\rangle$ be a properly $\subseteq$-increasing continuous chain of countable elementary submodels of $\mathcal{A}$. In particular $\omega_1^{\mathcal{A}} \in X_0$ and for every $\alpha < \omega_1$ there is a countable ordinal $\beta \in X_{\alpha+1} - X_\alpha$. For each $\alpha < \omega_1$, let P_α be the transitive collapse of X_α, and let $\rho_\alpha \colon X_\alpha \to P_\alpha$ be the corresponding collapsing map. Then $\rho_\alpha(\omega_1) = \omega_1^{P_\alpha}$ is the ordinal $X_\alpha \cap \omega_1$.

The following is a paraphrase of Theorem 2.1 of [9] (Hutchinson built on work of Keisler and Morley [12]; Enayat provides a useful source on this work in [5]). The following argument can also be carried out via iterated ultrapowers as in [3]. Section 4 of [9] describes the fragment of ZFC needed for Fact 3.13; this fragment is easily seen to follow from the theory ZFC° of [3].

{Hutch}

Fact 3.13. *Let $\mathcal{B}$ be a countable model of ZFC and c a regular cardinal in $\mathcal{B}$. Then there is a countable elementary extension $\mathcal{C}$ of $\mathcal{B}$ such that each a such that $\mathcal{B} \models a \in c$ is fixed (i.e. has no new elements in $\mathcal{C}$) but c is enlarged and there is a least new element of $\mathcal{C}$.*

Either iterating the construction in Fact 3.13, or by an iteration of ultrapowers of models of set theory as in Lemma 1.5 of [3], construct a family $\{P'_\alpha : \alpha < \omega_1\}$ of countable models of set theory so that, for each $\alpha < \omega_1$, there is an elementary extension of P_α to P'_α (with corresponding elementary embedding $\chi_\alpha \colon P_\alpha \to P'_\alpha$) such that

1. the critical point of χ_α is $\omega_1^{P_\alpha}$, so $\omega_1^{P_\alpha}$ is an initial segment of $\omega_1^{P'_\alpha}$;
2. $\omega_1^{P'_\alpha}$ is ill-founded, {goodclub}
3. in V, there is a continuous increasing ω_1-sequence $\langle t^\alpha_\gamma : \gamma < \omega_1\rangle$ consisting of elements of $\omega_1^{P'_\alpha}$

Item 3 above implies in particular that each $\omega_1^{P'_\alpha}$ is uncountable. Each P'_α can be realized as the union of a increasing elementary chain of models $\langle P^\alpha_\gamma : \gamma < \omega_1\rangle$, where $P^\alpha_0 = P_\alpha$,

$$P'_\alpha = \bigcup_{\gamma<\omega_1} P^\alpha_\gamma$$

for limit α, and each $P^\alpha_{\gamma+1}$ can be obtained by applying Fact 3.13 (or a generic ultrapower) to P^α_γ. Then each t^α_γ (the c of Fact 3.13) can be taken to be $\omega_1^{P^\alpha_\gamma}$.

Recall that M is the union of the continuous $\subseteq$-increasing chain $\langle M_\alpha : \alpha < \omega_1\rangle$. It follows then for each $\alpha < \omega_1$, that $M_{\omega_1^{P_\alpha}} = \rho_\alpha(M) \subset P_\alpha$, and that $M_{\omega_1^{P_\alpha}}$ has cardinality $\aleph_1$ in P_α. For each $\alpha < \omega_1$, let $N_\alpha = \chi_\alpha(M_{\omega_1^{P_\alpha}})$ and let $N'_\alpha = \chi_\alpha(\rho_\alpha(M'))$. Then each N'_α is an expansion of N_α via the given enumeration of M_0 by the constants c_i, and it has cardinality $\aleph_1$ in P'_α.

In the argument for Theorem 2.5 replace the appeal to Lopez-Escobar (Theorem 5.3.8 of [1]) with the observation that the induced ordering on N'_α is not well-founded by construction. The rest of the argument for Theorem 2.5 (or Theorem 2.19) shows that, in V, each N'_α is small for $L_{\omega_1,\omega}(\tau')$. Nevertheless, by the elementarity of $\chi_\alpha \circ \rho_\alpha$, each P'_α thinks that N'_α is not $L_{\omega_1,\omega}(\tau')$-small.

Since $\overline{M}$ is a sequence indexed by ω_1 in V (or in X_α), $\chi_\alpha(\rho_\alpha(\overline{M}))$ is a sequence indexed by $\omega_1^{P'_\alpha}$ in P'_α. So, in P'_α, for each element t of its ω_1, there is a t-th element of the sequence, which we denote by M^α_t. Furthermore, in P'_α, $\chi_\alpha(\rho_\alpha(\overline{F}))$ is a set$\{F^\alpha_t : t \in \omega_1^{P'_\alpha}\}$ consisting of automorphisms of N_α, such that each $F^\alpha_t \in P'_\alpha$ is an automorphism of N_α sending M_0 to M^α_t. Each F^α_t is then an automorphism of N_α in V also.

Since each N'_α is small, each N_α is as well. Since we are assuming that there are *only countably many models* in $\mathbf{K}$ of cardinality $\aleph_1$, there exists an uncountable set $S \subseteq \omega_1$ such that N_{α_0} and N_{α_1} are isomorphic (in V) for all α_0, α_1 in S. Fix for a moment a pair of elements α_0, α_1 of S and an isomorphism $\pi\colon N_{\alpha_0} \to N_{\alpha_1}$. Applying item 3 above and the continuity (in the sense of P'_{α_j}, for $j = 0, 1$) of the sequences $\langle M^{\alpha_0}_t : t \in \omega_1^{P'_{\alpha_0}}\rangle$ and $\langle M^{\alpha_1}_t : t \in \omega_1^{P'_{\alpha_1}}\rangle$, there must be $t_0 \in \omega_1^{P'_{\alpha_0}}$ and $t_1 \in \omega_1^{P'_{\alpha_1}}$ such that π maps $M^{\alpha_0}_{t_0}$ onto $M^{\alpha_1}_{t_1}$. To see this, start with $\gamma_0 = 0$ and, for each $n \in \omega$, let γ_{n+1} be large enough so that

$$\pi[M^{\alpha_0}_{t^{\alpha_0}_{\gamma_n}}] \subseteq M^{\alpha_1}_{t^{\alpha_1}_{\gamma_{n+1}}}$$

and

$$\pi^{-1}[M^{\alpha_1}_{t^{\alpha_1}_{\gamma_n}}] \subseteq M^{\alpha_0}_{t^{\alpha_0}_{\gamma_{n+1}}}.$$

Then let $s_0 = t^{\alpha_0}_{\sup_{n\in\omega}\gamma_n}$ and let $s_1 = t^{\alpha_1}_{\sup_{n\in\omega}\gamma_n}$. By the continuity in item 3, the s_j's are in the respective P'_{α_j}, for $j \in \{0,1\}$. So, for each j, by the continuity in P'_{α_j} of $M^{\alpha_j}_t$,

$$M^{\alpha_j}_{s_j} = \bigcup_{n<\omega} M^{\alpha_j}_{t^{\alpha_j}_{\gamma_n}}.$$

Then $(F^{\alpha_1}_{s_1})^{-1} \circ \pi \circ F^{\alpha_0}_{s_0}$ is an isomorphism of N_{α_0} and N_{α_1} fixing M_0 setwise, though not necessarily pointwise.

Finally, we show that for each $\alpha_0 < \omega_1$ such an isomorphism is impossible for sufficiently large $\alpha_1 < \omega_1$.

Each model P'_α thinks that N'_α is small for every countable fragment of $L_{\omega_1,\omega}(\tau')$ but not $L_{\omega_1,\omega}(\tau')$-small. Thus, from the point of view of P'_α, there is no ordinal t such that $\phi_{\overline{a},t}(\overline{x}) \equiv \phi_{\overline{a},t+1}(\overline{x})$ (in the terms of the Scott construction) for all finite tuples $\overline{a}$ of N'_α. For each well-founded ordinal γ of P'_α (this includes the members of $\omega_1^{P'_\alpha} = \omega_1 \cap X_\alpha$, by item 1 above), and each finite tuple $\overline{a}$ of N'_α, P'_α sees the same formula $\phi_{\overline{a},\gamma}(\overline{x})$ that the true universe V does, which means that the Scott sentence for N'_α has rank at least $\omega_1 \cap X_\alpha$ (and slightly more than this, in fact, in the approach from [3]). Alternately, Lemma 2.20 implies that the Scott rank of N'_α is exactly the wellfounded part of $\omega_1^{P'_\alpha}$.

Now choose $\alpha_0, \alpha_1 \in S$ such that $\omega_1 \cap X_{\alpha_1}$ is greater than the Scott rank (in V) of N'_{α_0}. Since permuting the constants c_i in terms of their enumeration of M_0 has no effect on the rank of the Scott sentence for N'_{α_1}, there cannot be then an isomorphism of N_{α_0} and N_{α_1} fixing M_0 setwise, since this would imply that N'_{α_0} and N'_{α_1} have the same Scott rank (indeed, their Scott sentences would differ only by a permutation of the c_i's). Thus we have a contradiction in case 2a.

We have ruled out cases 1) and 2a) and are left with case 2b). We show that Case 2b gives the conclusions of Theorem 3.10. We use the following remark.

{fixfrag}

Remark 3.14. Suppose that a structure M is small. Then there is a countable fragment L_1 of $L_{\omega_1,\omega}$ such that M is L_1-atomic. That is, for any $\mathbf{a} \in M$, there is $\chi_{\mathbf{a}}(\mathbf{x}) \in L_1$ such that for any $\lambda(\mathbf{x}) \in L_{\omega_1,\omega}$ if $M \models \lambda(\mathbf{a})$, then

$$M \models (\forall \mathbf{x})[\chi_{\mathbf{a}}(\mathbf{x}) \to \lambda(\mathbf{x})].$$

Namely, let L_1 be the least countable fragment containing the canonical Scott sentence of M.

Since we have reduced to case 2b) we have the hypotheses of the following Lemma for any countable $M_0 \in \boldsymbol{K}$. Note that although we have a Galois-saturated model in power $\aleph_1$, *a priori*, it might realize countably many Galois types over some countable submodel.

{2b}

Lemma 3.15. *Assume that M' is $L_{\omega_1,\omega}(\tau')$-small and Galois saturated for $\boldsymbol{K}$. Then for some $L^*(\tau')$, M' is $L^*(\tau')$-atomic. This implies that M realizes only countably many Galois types over M_0.*

Proof. By Remark 3.14, M' is atomic in $L^*(\tau')$, the countable fragment in which M' has a Scott sentence; this is Theorem 3.10.1. We will show that for any $a \in M$ the $L^*(\tau')$-type of a determines the Galois type (in $\boldsymbol{K}$) of a over M_0. Since M' is $L_{\omega_1,\omega}(\tau')$-small, it follows that only countably many Galois types over M_0 are realized in M. Suppose that some $a, b \in M$ realize the same $L^*(\tau')$-type in M'. Then this type is given by a formula in $L^*(\tau')$, by $L^*(\tau')$-atomicity. There exists a countable $\hat{M} \in \mathbf{K}$ such that $M_0ab \subset \hat{M} \prec_{L^*(\tau')} M$, and, as $\hat{M}$ is $L^*(\tau')$-atomic, there exists an automorphism g of $\hat{M}$, fixing M_0 pointwise with $g(a) = b$. Thus, a and b have the same the Galois type over M_0. So M realizes only countably many Galois types over M_0. Since M_0 was an arbitrary countable model, we have Theorem 3.10.2. $\square_{3.15}$

This completes the proof of the main theorem. $\square_{3.10}$

Remark 3.16. Note that argument ruling out case 2a) uses the set theoretic argument to find $\aleph_1$ τ'-small models in $\aleph_1$ with distinct τ'-Scott rank. By the automorphism argument, this contradicts the assumption that there are only $\aleph_0$ τ-models in $\aleph_1$.

4 Categoricity

It is shown in [3] (see Theorems 2.1 and 6.2) that, given an $\aleph_0$-presented AEC $\boldsymbol{K}$, the statement that $\boldsymbol{K}$ has an uncountable model is Σ^1_1 in a real coding $\boldsymbol{K}$, and the statement that $\boldsymbol{K}$ is almost Galois ω-stable is Π^1_1 in a real coding $\boldsymbol{K}$. These statements are therefore absolute. Amalgamation for such a $\boldsymbol{K}$ is easily seen to be Π^1_2 in a code for $\boldsymbol{K}$. In this section we apply Theorem 3.10 to prove the following theorem.

Theorem 4.1. *Suppose that $\boldsymbol{K}$ is an $\aleph_0$-presented almost Galois ω-stable AEC with Löwenheim-Skolem number $\aleph_0$, satisfying amalgamation and joint embedding and having an uncountable model. Then the $\aleph_1$-categoricity of $\boldsymbol{K}$ is equivalent to a $\utilde{\Pi}^1_2$ statement, and therefore absolute.*

Proof. In this situation, $\aleph_1$-categoricity is equivalent to the conjunction of the following statements.

{allsameScott}

1. All uncountable models in $\boldsymbol{K}$ satisfy the same Scott sentence.

{stillsmall}

2. There exist a countable $N \in \boldsymbol{K}$ and a countable fragment L_1 of the expanded language where constants are added for each member of N such that for every $M \in \boldsymbol{K}$ with $N \prec_{\boldsymbol{K}} M$ and $N \prec_{\infty,\omega} M$, M is L_1-atomic.

{nomiss}

3. There do not exist $N \prec_{\boldsymbol{K}} M$ with N countable and M uncountable, such that only countably many Galois types over N are realized in M, and some Galois type over N is not realized in M.

Corollary 2.8 implies that all uncountable models of any $\aleph_1$-categorical $\aleph_0$-presented AEC satisfy the same Scott sentence giving clause (1). By Lemma 3.10.1, we have clause (2) for all countable $N \in \boldsymbol{K}$. Lemma 3.10.2 implies Galois ω-stability for $\boldsymbol{K}$. Thus there is a Galois saturated model in $\aleph_1$ that satisfies clause (3).

For the other direction, item (1) and Lemma 3.3 imply that $\boldsymbol{K}$ has a small Galois saturated model. By Lemma 3.15 Item (2) implies that $\boldsymbol{K}$ is Galois ω-stable. Then item (3) implies that the Galois saturated model is the only model in $\boldsymbol{K}$ of cardinality $\aleph_1$.

Item (1) is easily seen to be Π^1_1 in a real coding $\boldsymbol{K}$. Item (2) is easily seen to be Π^1_2 in a real coding $\boldsymbol{K}$. Theorem 6.1 of [3] shows that item (3) above is Π^1_1 in a real coding $\boldsymbol{K}$. Since each conjunct is Π^1_2, the assertion is absolute.

$\square$

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