Almost Galois ω-Stable classes

John T. Baldwin*
University of Illinois at Chicago

Paul B. Larson†
Miami University
Oxford, Ohio USA

Saharon Shelah‡
Hebrew University of Jerusalem
Rutgers University

November 20, 2013

Abstract

Theorem. Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ is an $\aleph_0$-presentable Abstract Elementary Class with Löwenheim-Skolem number $\aleph_0$, satisfying the joint embedding and amalgamation properties in $\aleph_0$. If $\boldsymbol{K}$ has only countably many models in $\aleph_1$, then all are small. If, in addition, $\boldsymbol{k}$ is almost Galois ω-stable then $\boldsymbol{k}$ is Galois ω-stable. Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ is an $\aleph_0$-presented almost Galois ω-stable AEC with Löwenheim-Skolem number $\aleph_0$, satisfying amalgamation for countable models, and having a model of cardinality $\aleph_1$. The assertion that $\boldsymbol{K}$ is $\aleph_1$-categorical is then absolute.

1 Introduction

This paper concerns two aspects of pseudo-elementary classes in $L_{\omega_1,\omega}$, the reducts to a vocabulary $\tau \subseteq \tau^+$ of models of an $L_{\omega_1,\omega}(\tau^+)$-sentence. In the first two sections we investigate the relationship among the number of countable models of such a class, Scott ranks, and the number of small (i.e., having a countable $L_{\omega_1,\omega}$-elementary submodel) models and large (not small) models of the class in $\aleph_1$; this yields some technical information about putative counterexamples to Vaught's conjecture. Building on this material, in the third section, we treat such classes as abstract elementary classes and investigate variations on Galois ω-stability. In the final section we use the results presented here and in [5] to prove a theorem on the absoluteness of $\aleph_1$-categoricity for pseudo-elementary classes in $L_{\omega_1,\omega}$ that are also abstract elementary classes.

We call an Abstract Elementary Classes (AEC) *almost Galois ω-stable* if for every countable model M, E_M (the equivalence relation of 'same Galois type over M' see Definition 3.1) does not have a perfect set of inequivalent members. An AEC is *strictly almost Galois ω-stable* if in addition it is not Galois ω-stable. The immediate impetus for this paper was [5], which studied what Baldwin and Larson called *analytically presented* Abstract Elementary Classes. These classes are called by many names: pseudo-elementary classes

*Research partially supported by Simons travel grant G5402
†Supported in part by NSF Grants DMS-0801009 and DMS-1201494.
‡Research partially supported by NSF grant no: DMS 1101597, and German-Israeli Foundation for Scientific Research & Development grant no. 963-98.6/2007. Publication number 1003. Part of f1179 on the Shelah Archive. All authors thank Rutgers University. Diagram drawn using Paul Taylor's diagrams.sty.

in $L_{\omega_1,\omega}$, $\aleph_0$-presentable classes, $\mathrm{PC}(\aleph_0,\aleph_0)$ or $\mathrm{PC}_{\aleph_0}$ [26], $\mathrm{PC}\Gamma(\aleph_0,\aleph_0)$ [1] or, in the language of Keisler [13], PC_δ in $L_{\omega_1,\omega}$. In this paper we will most often refer to them as $\aleph_0$-presented. The term 'analytically presented' emphasizes that one can deduce from Burgess's theorem on analytic equivalence relations (see [8], Theorem 9.1.5, for instance) that if such a class is almost Galois ω-stable then each equivalence relation E_M has at most $\aleph_1$ equivalence classes. This topic first arose in [26] and several of the arguments here just expand ideas Shelah mentioned there; for further background on the context see [5, 1, 24]. Our main goal, Theorem 3.18, is to prove that an almost Galois ω-stable $\aleph_0$-presentable Abstract Elementary Class with only countably many models in $\aleph_1$ is Galois ω-stable. This extends earlier work by Hyttinen-Kesala [11] and Kueker [15] proving the result for sentences of $L_{\omega_1,\omega}$ with no requirement on the number of uncountable models.

Each class of models in this paper is $\aleph_0$-presented. A major tool for this investigation is to expand models of set theory by predicates encoding relevant properties of the models (for some vocabulary τ) being studied. This approach appears in Shelah's analysis in [21], Section VII, connecting the Hanf number for omitting families of types with the well-ordering number for classes defined by omitting types. In [20], expanding the vocabulary to describe an analysis of the syntactic types allowed the construction of a 'small' (Definition 2.2) uncountable model in an $\aleph_0$-presentable class $\boldsymbol{K}$ from an uncountable model that is small with respect to every countable fragment of $L_{\omega_1,\omega}$. In Lemma 2.7, we use this method to show that if, in addition, there are only countably many models in $\aleph_1$, then each is small. In Section 3, we combine this technique with constructions using iterated models of set theory to prove Theorem 3.18. In Section 4, we give sufficient conditions of categoricity in $\aleph_1$ of an $\aleph_0$-presented AEC to be absolute.

2 Small Models

We refer the reader to [1, 24] for the definition of Abstract Elementary Class (AEC).

Assumption 2.1. *$\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ is an AEC which is $\aleph_0$-presented. Specifically, $\boldsymbol{K}$ is the class of reducts to τ of a class defined by a sentence $\phi \in L_{\omega_1,\omega}(\tau^+)$, where τ^+ is a countable vocabulary extending τ.*

If λ is a cardinal, we let $\boldsymbol{K}_\lambda$ be the class of models in $\boldsymbol{K}$ of cardinality λ.

This section deals with syntactic $L_{\omega_1,\omega}$-types in $\aleph_0$-presentable classes. As such the arguments are primarily syntactic and are minor variants on arguments Shelah used in [20, 22, 26]. In particular, no amalgamation assumptions are used in this section.

Definition 2.2.

1. *A τ-structure M is L^*-*small *for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many L^*-types (i.e. only countably many L^*-n-types over $\emptyset$ for each $n < \omega$).*
2. *A τ-structure M is called* locally τ-small *if for every countable fragment L^* of $L_{\omega_1,\omega}(\tau)$, M realizes only countably many L^*-types.*
3. *A τ-structure M is called* small *or* $L_{\omega_1,\omega}$-small *if M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.*

Note that 'small' is a much stronger requirement than 'locally small'. If $\tau \subseteq \tau'$ and $N \in \tau'$, we say that N is *locally τ-small* when $N \restriction \tau$ is. We emphasize τ when the ambient larger vocabulary plays a significant role. The following standard fact plays a key role below (see also pages 47-48 of [1]).

Fact 2.3. *Each small model satisfies a Scott sentence, a complete sentence of $L_{\omega_1,\omega}$.*

We quickly review the proof of this fact, as the details will be important later. For any model M over a countable vocabulary τ, we can define for each finite tuple $\mathbf{a}$ (of size n) from M the n-ary formulas $\phi_{\mathbf{a},\alpha}(\overline{x})$ ($\alpha < |M|^+$) as follows.

- $\phi_{\mathbf{a},0}(\overline{x})$ is the conjunction of all atomic and negated atomic formulas satisfied by $\mathbf{a}$,
- $\phi_{\mathbf{a},\alpha+1}(\overline{x})$ is the conjunction of the following three formulas:
  - $\phi_{\mathbf{a},\alpha}(\overline{x})$
  - $\bigwedge_{c\in M} \exists w \phi_{\mathbf{a}c,\alpha}(\overline{x}, w)$
  - $\forall w \bigvee_{c\in M} \phi_{\mathbf{a}c,\alpha}(\overline{x}, w)$
- for limit $\beta < |M|^+$, $\phi_{\mathbf{a},\beta}(\overline{x}) = \bigwedge_{\alpha<\beta} \phi_{\mathbf{a},\alpha}$.

The apparent uncountability of the conjunctions in the previous definition is obviated by identifying formulas $\phi_{\mathbf{a}c,\alpha}$ and $\phi_{\mathbf{a}'c,\alpha}$ when they are equivalent in M. Working by induction on α, one gets that if M is L^*-small for each countable fragment L^* of $L_{\omega_1,\omega}(\tau)$, then the set of formulas $\phi_{\mathbf{a},\alpha}$ is countable for each α, letting $\mathbf{a}$ range over all finite tuples from M. Finally, if M is small there exists an $\alpha < \omega_1$ such that

$$M \models \forall \overline{x}(\phi_{\mathbf{a},\alpha}(\overline{x}) \rightarrow \phi_{\mathbf{a},\alpha+1}(\overline{x}))$$

for all finite tuples $\mathbf{a}$. Then

$$\phi_{\langle\rangle,\alpha} \wedge \bigwedge_{\mathbf{a}\in M^{<\omega}} \forall \overline{x}(\phi_{\mathbf{a},\alpha}(\overline{x}) \rightarrow \phi_{\mathbf{a},\alpha+1}(\overline{x}))$$

is a Scott sentence for M. Fixing the least such α, we say that M has Scott rank α.

We will also use the following fundamental result (see [13] or Theorem 5.2.5 of [1]; the notion of fragment is explained in both books). Roughly speaking, the fragment generated by a countable subset X of $L_{\omega_1,\omega}(\tau)$ is the closure of X under first order operations. We preserve Keisler's terminology to emphasize that the theorem deals only with the number of models and does not involve the choice of 'elementary embedding' on the class.

Theorem 2.4 (Keisler). *If a PC_δ over $L_{\omega_1,\omega}$ class $\boldsymbol{K}$ has an uncountable model but less than 2^{ω_1} models of power $\aleph_1$ then $\boldsymbol{K}$ is locally τ-small. That is, for any countable fragment L^* of $L_{\omega_1,\omega}(\tau)$, each $M \in \boldsymbol{K}$ realizes only countably many L^*-types over $\emptyset$.*

By just changing a few words in the proof of Theorem 6.3.1 of [1], (originally in [20]) one can obtain the following result, which was implicit in [26].

Theorem 2.5. *If $\boldsymbol{K}$ is an $\aleph_0$-presentable AEC and some model $M \in \boldsymbol{K}$ of cardinality $\aleph_1$ is* locally τ-small*, then $\boldsymbol{K}$ has a $L_{\omega_1,\omega}(\tau)$-small model N of cardinality $\aleph_1$.*

Proof. Let ϕ be the τ^+-sentence whose reducts to τ are the members of $\boldsymbol{K}$. Without loss of generality we may assume the universe of M is ω_1. Add to τ^+ a binary relation $<$, interpreted as the usual order on ω_1. Using the fact that M realizes only countably many types in any τ-fragment, define a continuous increasing chain of countable fragments L_α for $\alpha < \aleph_1$ such that

- for each quantifier free (first order) n-type over the empty set realized in M, the conjunction of the type is in L_0, and

- the conjunction of each type in L_α that is realized in M is a formula in $L_{\alpha+1}$.

Extend the similarity type further to τ' by adding new $(2n+1)$-ary predicates $E_n(x, \mathbf{y}, \mathbf{z})$ and $(n+1)$-ary functions f_n for each $n \in \omega$. Let M satisfy $E_n(\alpha, \mathbf{a}, \mathbf{b})$ if and only if $\mathbf{a}$ and $\mathbf{b}$ realize the same L_α-type, and let the interpretation of f_n map M^{n+1} into ω in such a way that $E_n(\alpha, \mathbf{a}, \mathbf{b})$ if and only if $f_n(\alpha, \mathbf{a}) = f_n(\alpha, \mathbf{b})$ for all suitable α, $\mathbf{a}$, $\mathbf{b}$. Then the following hold.

1. The equivalence relations $E_n(\beta, \overline{x}, \overline{y})$ refines $E_n(\alpha, \overline{x}, \overline{y})$ if $\beta > \alpha$;
2. $E_n(0, \mathbf{a}, \mathbf{b})$ implies that $\mathbf{a}$ and $\mathbf{b}$ satisfy the same quantifier free τ-formulas;
3. If $\beta > \alpha$ and $E_n(\beta, \mathbf{a}, \mathbf{b})$, then for every c_1 there exists c_2 such that $E_{n+1}(\alpha, c_1\mathbf{a}, c_2\mathbf{b})$, and
4. f_n witnesses that for any $a \in M$, each equivalence relation $E_n(a, \overline{x}, \overline{y})$ has only countably many classes.

All these assertions can be expressed by an $L_{\omega_1,\omega}(\tau')$ sentence χ. Let L^* be the smallest τ'-fragment containing $\chi \wedge \phi$. Now by the Lopez-Escobar bound on $L_{\omega_1,\omega}$ definable well-orderings, Theorem 5.3.8 of [1], there is a τ'-structure N of cardinality $\aleph_1$ satisfying $\chi \wedge \phi$ such that there is an infinite decreasing sequence $d_0 > d_1 > \ldots$ in N (alternately, one could use Lemma 2.5 of [5] for this step). For each n, let $E_n^+(\overline{x}, \overline{y})$ denote the assertion that for some i, $E_n(d_i, \overline{x}, \overline{y})$.

Using 1), 2) and 3) one can prove by induction on quantifier rank (for all $n \in \omega$ simultaneously) that for all n-ary $L_{\omega_1,\omega}(\tau)$ formulas μ, and all finite tuples $\mathbf{a}$, $\mathbf{b}$ from N, if $E_n^+(\mathbf{a}, \mathbf{b})$ holds then $N \models \mu(\mathbf{a})$ if and only if $N \models \mu(\mathbf{b})$. To see this, suppose that this assertion holds for all n and all θ with quantifier rank at most γ. Let $\mu(\overline{z})$ be an n-ary formula of the form $(\exists x)\theta(\overline{z}, x)$, where θ has quantifier rank γ. Let $\mathbf{a}, \mathbf{b}$ be n-tuples from N for which $E_n^+(\mathbf{a}, \mathbf{b})$ holds and $N \models \mu(\mathbf{a})$. Then for some i, $E_n(d_i, \mathbf{a}, \mathbf{b})$ and for some a, $N \models \theta(\mathbf{a}, a)$. By condition 3) above there is a b such that $E_{n+1}(d_{i+1}, \mathbf{a}, a, \mathbf{b}, b)$. By our induction hypothesis we have $N \models \theta(\mathbf{b}, b)$ and so $N \models \mu(\mathbf{b})$.

Now, for each n, $E_n(d_0, \overline{x}, \overline{y})$ refines $E_n^+(\overline{x}, \overline{y})$ and by 4) $E_n(d_0, \overline{x}, \overline{y})$ has only countably many classes, so $N \restriction \tau$ is small. $\square_{2.5}$

Definition 2.6. *We say a countable structure is* extendible *if it has an $L_{\omega_1,\omega}$-elementary extension to an uncountable model.*

Lemma 2.7. *Suppose that $\boldsymbol{K}$ is the class of reducts to τ of a class defined by a sentence $\phi \in L_{\omega_1,\omega}(\tau^+)$, where τ^+ is a countable vocabulary extending τ. If some uncountable $M \in \boldsymbol{K}$ is locally τ-small but is not $L_{\omega_1,\omega}(\tau)$-small then*

1. *There are at least $\aleph_1$ pairwise-inequivalent complete sentences of $L_{\omega_1,\omega}(\tau)$ which are satisfied by uncountable models in $\boldsymbol{K}$;*
2. *$\boldsymbol{K}$ has uncountably many small models in $\aleph_1$ that satisfy distinct complete sentences of $L_{\omega_1,\omega}(\tau)$;*
3. *$\boldsymbol{K}$ has uncountably many extendible models in $\aleph_0$.*

Proof. Suppose that M is a model in $\boldsymbol{K}$ with cardinality $\aleph_1$ that is is locally τ-small but is not $L_{\omega_1,\omega}(\tau)$-small. Let M^+ be an expansion of M to a τ^+-structure satisfying ϕ. We construct a sequence of τ^+-structures $\{N_\alpha^+ : \alpha < \omega_1\}$ each with cardinality $\aleph_1$ and an increasing continuous family of countable fragments $\{L'_\alpha : \alpha < \omega_1\}$ of $L_{\omega_1,\omega}(\tau)$ and sentences χ_α such that:

1. $L'_0(\tau)$ is first order logic on τ;

2. all the models N_α^+ satisfy ϕ;
3. for each $\alpha < \omega_1$, $N_\alpha^+ \restriction \tau$ is $L_{\omega_1,\omega}(\tau)$-small;
4. χ_α is the $L_{\omega_1,\omega}(\tau)$-Scott sentence of N_α;
5. $L'_{\alpha+1}(\tau)$ is the smallest fragment of $L_{\omega_1,\omega}(\tau)$ containing $L'_\alpha(\tau) \cup \{\neg\chi_\alpha\}$;
6. For limit δ, $L'_\delta(\tau) = \bigcup_{\alpha<\delta} L'_\alpha(\tau)$;
7. For each α, $N_\alpha \equiv_{L'_\alpha(\tau)} M$.

Working by recursion, suppose that we have constructed N_α for all $\alpha < \beta$, for some countable ordinal β. This determines each χ_α $(\alpha < \beta)$ as the Scott sentence of N_α and also determines $L'_\beta(\tau)$. Since M is not small, $M \models \neg\chi_\alpha$ for each $\alpha < \beta$. Apply Theorem 2.5 to M and the restriction of $\boldsymbol{K}$ to models $L'_\beta(\tau)$-elementarily equivalent to M to construct N_β.

Now the N_α are pairwise non-isomorphic since each satisfies a distinct complete sentence χ_α of $L_{\omega_1,\omega}(\tau)$, so conclusions 1) and 2) are satisfied. And each N_α has a countable elementary submodel with respect to $L'_{\alpha+1}(\tau)$, so there are at least $\aleph_1$ non-isomorphic extendible models in $\aleph_0$ as well. $\square_{2.7}$

Putting together Theorem 2.4 and Lemma 2.7, we have the following.

Corollary 2.8. *If an $\aleph_0$-presented AEC $\boldsymbol{K}$ has only countably many models in $\aleph_1$, then every model in $\boldsymbol{K}$ is small.*

Lemma 2.7 leads to several corollaries connected to the Vaught conjecture. First we recall the following result of Harnik and Makkai [9].

Theorem 2.9 (Harnik-Makkai). *If $\sigma \in L_{\omega_1,\omega}$ is a counterexample to Vaught's Conjecture then it has a model of cardinality $\aleph_1$ which is not small.*

Corollary 2.10. *If $\phi \in L_{\omega_1,\omega}$ is a counterexample to the Vaught conjecture then ϕ has $\aleph_1$* extendible *countable models.*

Proof. If $\phi \in L_{\omega_1,\omega}$ is a counterexample to Vaught's conjecture, then every uncountable model of ϕ is locally small. The result then follows from Theorem 2.9 and Lemma 2.7. $\square_{2.10}$

Remark 2.11. Clearly, if $\boldsymbol{K}$ has only countably many models in $\aleph_1$ then $\boldsymbol{K}$ has at most $\aleph_0$ non-isomorphic extendible countable models (since each uncountable model is $L_{\omega_1,\omega}$-equivalent to at most one model in $\aleph_0$). The three conclusions of Lemma 2.7 are easily seen to be equivalent; we separated them in the statement because both the countable and uncountable models arose naturally in the proof. The converse of Lemma 2.7 asserts that if $\boldsymbol{K}$ has uncountably many extendible countable models and a locally small model in $\aleph_1$ then it has a non-small model in $\aleph_1$. Theorem 2.9 shows this is true if the hypothesis is changed to 'uncountably many countable models, but not a perfect set of countable models', without requiring extendibility, and the class of countable models is Borel, as opposed to analytic. In general, the converse is false. The empty theory in a vocabulary with $\aleph_0$ constants has $2^{\aleph_0}$ models (depending on which constants are identified) in each of $\aleph_1$ and $\aleph_0$; all are small. But joint embedding and amalgamation fail even under first order elementarity. Example 2.1.1 of [4] is a sentence of $L_{\omega_1,\omega}$ giving rise to an AEC, with a particular notion of $\prec_{\boldsymbol{K}}$ (weaker than first order), which satisfies amalgamation and joint embedding and is $\aleph_1$-categorical, and for which the model in $\aleph_1$ is small. In this case there are $2^{\aleph_0}$ countable models, but only one of them is extendible.

Definition 2.12. *A sentence σ of $L_{\omega_1,\omega}$ is* large *if it has uncountably many countable models. A large sentence σ is* minimal *if for every sentence ϕ either $\sigma \wedge \phi$ or $\sigma \wedge \neg\phi$ is not large.*

As part of their proof of Theorem 2.9, Harnik and Makkai showed that any counterexample to Vaught's conjecture can strengthened to a minimal counterexample. We call a model of cardinality $\aleph_1$ *large* if it is not $L_{\omega_1,\omega}$-small in the sense of Definition 2.2. Lemma 2.7 implies that if ϕ has a large model in $\aleph_1$ then ϕ is large.

Corollary 2.13. *If ϕ is a minimal counterexample to Vaught's conjecture then ϕ has a large model in $\aleph_1$, and all large models of ϕ in $\aleph_1$ are $L_{\omega_1,\omega}$-elementarily equivalent.*

Proof. Theorem 2.9 says that ϕ has a large model N. Suppose that $\psi \in L_{\omega_1,\omega}$ holds in N. The fact that $\phi \wedge \psi$ has a large model implies by Lemma 2.7 that $\phi \wedge \psi$ has uncountably many models in $\aleph_0$. By minimality, $\phi \wedge \neg\psi$ has only countably many models in $\aleph_0$ and so by Lemma 2.7 again, all uncountable models of $\phi \wedge \neg\psi$ are small. $\square_{2.13}$

Harrington[1] showed that any counterexample to Vaught's conjecture has models in $\aleph_1$ with Scott ranks (using sentences in $L_{\omega_2,\omega}$) cofinal in $\aleph_2$.

Question 2.14. *Can one say anything about the embedability relation on the large models of a counterexample to Vaught's conjecture?*

2.1 Connections with the Morley Analysis

We pause to connect this analysis in Section 2.2 with a related but subtly distinct procedure.

Definition 2.15. *1.* **Morley's Analysis** *Let $\boldsymbol{K}$ be the class of models of a sentence of $L_{\omega_1,\omega}$.*

(a) Let $L_0^{\mathbf{K}}$ be the set of first order τ-sentences.

(b) Let $L_{\alpha+1}^{\mathbf{K}}$ be the smallest fragment generated by $L_\alpha^{\mathbf{K}}$ and the sentences of the form $(\exists \mathbf{x}) \bigwedge p(\mathbf{x})$ where p is an $L_\alpha^{\mathbf{K}}$-type realized in a model in $\boldsymbol{K}$.

(c) For limit δ, $L_\delta^{\mathbf{K}} = \bigcup_{\alpha<\delta} L_\alpha^{\mathbf{K}}$.

2. $\boldsymbol{K}$ is scattered if and only if for each $\alpha < \omega_1$, $L_\alpha^{\mathbf{K}}$ is countable.

Recall Morley's theorem, which is key to his approach to Vaught's conjecture.

Theorem 2.16 (Morley). *If $\boldsymbol{K}$ is the class of models of a sentence in $L_{\omega_1,\omega}$ that has less than $2^{\aleph_0}$ models of power $\aleph_0$ then $\boldsymbol{K}$ is scattered.*

Remark 2.17. We cannot conclude that $\boldsymbol{K}$ is scattered from just counting models in $\aleph_1$, even from the hypothesis that $\boldsymbol{K}$ is $\aleph_1$-categorical. Again, Example 2.1.1 of [4] (Remark 2.11) is $\aleph_1$-categorical and has joint embedding for $\prec_{\boldsymbol{k}}$. But there are $2^{\aleph_0}$ first order types that give models that are not even first order mutually embeddible and the class $\boldsymbol{K}$ is not scattered.

Remark 2.18. The sequence of languages in Theorem 2.5 might be labeled L_α^M. They come about by applying the Morley analysis solely to the types realized in M. So this gives a slower growing sequence of languages than the Morley analysis. Clearly if $\boldsymbol{K}$ is scattered, every model of $\boldsymbol{K}$ is locally small. So from Theorem 2.16 and Theorem 2.4, we conclude. If $\boldsymbol{K}$ has either less than $2^{\aleph_0}$ models in $\aleph_0$ or less than $2^{\aleph_1}$ models in $\aleph_1$, then every uncountable model of $\boldsymbol{K}$ is locally small.

[1] See [18] for an account of Harrington's proof, Larson [16] for his proof using Scott processes and [3, 17] for a proof that encompasses the construction of an uncountable atomic model of a first order theory in a vocabulary of size $\aleph_1$.

Remark 2.19. The arguments of Morley and Shelah have different goals. Being scattered is a condition on all models of (in the interesting case for the Vaught conjecture) an *incomplete* sentence in $L_{\omega_1,\omega}$. The Shelah argument contracts $\boldsymbol{K}$ to a smaller class where every model is small and thus finds a $\boldsymbol{K}' \subset \boldsymbol{K}$ that is small and is axiomatized by a complete sentence. The hard part is to make sure $\boldsymbol{K}'$ has an uncountable model. In the most used case, $\boldsymbol{K}$ and *a fortiori* $\boldsymbol{K}'$ is $\aleph_1$-categorical.

2.2 Alternate proofs using Scott sentences

In this subsection we prove alternate versions of Theorem 2.5 and part of Lemma 2.7. Theorem 2.20 can be used in place of Theorem 2.5 in all of our applications of Theorem 2.5, and the basic use of ill-foundedness is the same. Ill-foundedness can be obtained either by Lopez-Escobar or by iterated ultrapowers of models of set theory. For convenience we use the theory ZFC° from [5]. Any theory strong enough to carry out the construction of Scott sentences should be sufficient.

Theorem 2.20. *Let τ be a countable vocabulary, let M be a τ-structure, and let N be an ω-model of* ZFC° *with ω_1^N ill-founded. Let β be the ordinal isomorphic to the longest well-founded initial segment of ω_1^N. Suppose that, in N, M is locally τ-small and either large or small with Scott rank in the ill-founded part of N. Then M is small, and the Scott rank of M is exactly β.*

Proof. Let t be the Scott rank of M in N if N thinks that M is small, and ω_1^N otherwise. Let

$$\langle \phi_{\mathbf{a},s} : \mathbf{a} \in M^{<\omega}, s < t\rangle$$

be the set of formulas defined in N in the first t many steps of the search for a Scott sentence for M. Then

$$\langle \phi_{\mathbf{a},\alpha} : \mathbf{a} \in M^{<\omega}, \alpha < \beta\rangle$$

is also the set of formulas defined in V in the first β many steps of the search for a Scott sentence for M. Since the Scott rank of M in N is in the ill-founded part of N if it exists, the Scott rank of M in V is at least β.

We claim that for any $n \in \omega$ and any pair $\mathbf{a}, \mathbf{b}$ of n-tuples from M, if $\phi_{\mathbf{a},s} = \phi_{\mathbf{b},s}$ for any ill-founded $s < t$, then $\mathbf{a}$ and $\mathbf{b}$ satisfy all the same $L_{\omega_1,\omega}(\tau)$-formulas in M (from the point of view of V). To see this, suppose that this assertion holds for all n and all formulas θ with quantifier rank at most γ. Let $\mu(\overline{z})$ be an n-ary formula of the form $(\exists x)\theta(\overline{z}, x)$, where θ has quantifier rank γ. Let $\mathbf{a}, \mathbf{b}$ be n-tuples from N, let $s < t$ be an ill-founded ordinal of N such that $\phi_{\mathbf{a},s} = \phi_{\mathbf{b},s}$, and suppose that $M \models \mu(\mathbf{a})$. Then there is an ill-founded $r < s$, and for any such r, $\phi_{\mathbf{a},r} = \phi_{\mathbf{b},r}$. Since $M \models \mu(\mathbf{a})$, there is a $c \in M$ such that $M \models \theta(\mathbf{a}, c)$. Since $r < s$ and $\phi_{\mathbf{a},s} = \phi_{\mathbf{b},s}$, $\phi_{\mathbf{a},r+1} = \phi_{\mathbf{b},r+1}$, which means that there is some $d \in M$ such that $\phi_{\mathbf{b}d,r} = \phi_{\mathbf{a}c,r}$. Thus by our induction hypothesis, $M \models \theta(\mathbf{b}, d)$ and thus $M \models \mu(\mathbf{b})$.

For each $n \in \omega$ and each pair $\mathbf{a}, \mathbf{b}$ of n-tuples from M, if $\phi_{\mathbf{a},\alpha} = \phi_{\mathbf{b},\alpha}$ for all $\alpha < \beta$, then $\phi_{\mathbf{a},s} = \phi_{\mathbf{b},s}$ for some ill-founded $s < t$, since if $\phi_{\mathbf{a},r} \neq \phi_{\mathbf{b},r}$ for any $r < t$, then N thinks that there is a least such r, and there is no least ill-founded ordinal of N. It follows then that the Scott rank of M (in V) is exactly β. $\square_{2.20}$

Lemma 2.21 will make up part of the proof of our main theorem (Theorem 3.18). The proof is in fact a simplified part of the main argument in the proof of that theorem.

Lemma 2.21. *Suppose that $\boldsymbol{K}$ is the class of reducts to τ of a class defined by a sentence $\phi \in L_{\omega_1,\omega}(\tau^+)$, where τ^+ is a countable vocabulary extending τ. If $\boldsymbol{K}$ has a model in $\boldsymbol{K}_{\aleph_1}$ that is locally τ-small, but is not $L_{\omega_1,\omega}(\tau)$-small then $\boldsymbol{K}$ has small models in $\aleph_1$ of club many distinct Scott ranks.*

Proof. Suppose that M is a model in $\boldsymbol{K}$ with cardinality $\aleph_1$ that is is locally τ-small but is not $L_{\omega_1,\omega}(\tau)$-small. Fix a regular cardinal $\theta > 2^{2^{\aleph_1}}$. It suffices to show that for every countable elementary submodel X of $H(\theta)$ with τ, ϕ and M in X, there exists a small model in $\boldsymbol{K}$ of cardinality $\aleph_1$ whose Scott rank is $X \cap \omega_1$. Fix such an X. Let P be the transitive collapse of X, and let $\rho\colon X \to P$ be the corresponding collapsing map. Then $\rho(\omega_1) = \omega_1^P$ is the ordinal $X \cap \omega_1$.

By iterating the construction in [14], one can find an elementary extension P' of P with corresponding elementary embedding $\pi\colon P \to N$, with critical point ω_1^P, such that ω_1^N is ill-founded and uncountable, and such that the well-founded ordinals of N are exactly the members of ω_1^P. Since ω_1^N is ill-founded, Theorem 2.20 implies that $\pi(\rho(M))$ is $L_{\omega_1,\omega}(\tau)$-small, with Scott rank equal to the longest well-founded initial segment of of ω_1^N, which is $X \cap \omega_1$. $\square_{2.21}$

Theorem 2.20 gives part of the proof of the following fact, which is used in the proof of Theorem 4.1.

Lemma 2.22. *Let $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ be an $\aleph_0$-presented AEC with Löwenheim-Skolem number $\aleph_0$, having an uncountable model. The statement that every uncountable model in $\boldsymbol{K}$ satisfies the same Scott sentence in $L_{\omega_1,\omega}$ can be expressed as both a Σ^1_2 sentence and a Π^1_2 sentence, each in a given real parameter for $\boldsymbol{K}$.*

Proof. First consider the statement that there is a complete sentence θ in $L_{\omega_1,\omega}$ such that whenever M is a countable model in $\boldsymbol{K}$ and N is a countable ω-model of ZFC$^\circ$ with $\theta \in N$ and M uncountable in N, $M \models \theta$. By Theorem 8.9 in Marker's appendix to [2], being a complete sentence in $L_{\omega_1,\omega}$ is Π^1_1, so this sentence is Σ^1_2. If there is a nonsmall uncountable model in $\boldsymbol{K}$, or if there are uncountable small models with distinct Scott sentences in $L_{\omega_1,\omega}$, then this Σ^1_2 statement can be shown to be false by taking the transitive collapses of a suitable countable elementary submodels (note that a nonsmall model satisfies the negation of each complete sentence in $L_{\omega_1,\omega}$). On the other hand, for any sentence θ of $L_{\omega_1,\omega}$, if there exist a countable model M in $\boldsymbol{K}$ and a countable ω-model N of ZFC$^\circ$ with $\theta \in N$, M uncountable in N and $M \models \neg\theta$, then one can find an uncountable model in $\boldsymbol{K}$ satisfying $\neg\theta$, by taking an iterated generic elementary embedding of length ω_1 (as in the proof of Theorem 2.1 of [5]).

Now consider the statement that whenever M and N are countable models in $\boldsymbol{K}$ and P and Q are countable ω-models of ZFC$^\circ$ with M an uncountable model in P and N an uncountable model in Q, then M and N are isomorphic. This statement is easily seen to be Π^1_2 in a code for $\boldsymbol{K}$. As above, if there is a nonsmall uncountable model in $\boldsymbol{K}$, or if there are uncountable small models with distinct Scott sentences in $L_{\omega_1,\omega}$, then this Π^1_2 statement can be shown to be false by taking the transitive collapses of suitable countable elementary submodels of $H(\kappa)$, for any regular κ greater than $2^{2^{\aleph_1}}$.

In the other direction, suppose first that there exist a countable model P of ZFC$^\circ$ with ω_1^P wellfounded, and a model $M \in \boldsymbol{K}$ in P which P thinks has uncountable Scott rank. Then we can produce two uncountable models in $\boldsymbol{K}$ of distinct Scott ranks by taking elementary extensions of P. We start by finding two elementary embeddings, $k_1\colon P \to R_1$ and $k_2\colon P \to R_2$, each with critical point ω_1^P, where the well-founded part of R_1 is exactly ω_1^P, and the well-founded part of R_2 is at least ω_2^P (for the first of these, use the construction in [14]; for the second use [10] or [5]). We can then iteratively extend R_1 and R_2 each ω_1 many times (iterating either the construction in [14] or the one in [5]), producing elementary embeddings $j_1\colon R_1 \to R_1^*$ and $j_2\colon R_2 \to R_2^*$, where $\omega_1^{R_1^*}$ and $\omega_1^{R_2^*}$ are uncountable. By Theorem 2.20, $j_1(k_1(M))$ will have Scott rank ω_1^P. By the elementarity of $j_2 \circ k_2$, the Scott rank of $j_2(k_2(M))$ will be at least ω_2^P (and uncountable if $\omega_1^{R_2^*} = \omega_1$).

Supposing now that there exists no pair (M, P) as in the previous paragraph, suppose that we have two ω-models P and Q of ZFC$^\circ$, containing countable models M and N (respectively) in $\boldsymbol{K}$ which they think are uncountable, and suppose that M and N satisfy different Scott sentences in V. Then either P thinks that M is small, or ω_1^P is ill-founded, and the same holds for Q and N. Using the constructions from either [14] or [5], iterate P and Q each ω_1 times, producing models M' and N' of cardinality $\aleph_1$. Then M and M' have the same Scott sentence, as do N and N'. To see this, note that the Scott rank of M as computed in P is either in the well-founded part of P (in which case it must be countable in P, since we are not in the case of the previous paragraph) or not. In the first case, M and M' have the same Scott sentence by elementarity. In the second case, they have the same Scott sentence by Theorem 2.20. Since this applies to Q and N also, M' and N' are uncountable models in $\boldsymbol{K}$ satisfying distinct Scott sentences in $L_{\omega_1,\omega}$, and we are done. $\square_{2.22}$

3 Almost Galois Stability

The section is concerned about stability and almost stability with respect to Galois types.

Definition 3.1. *Given an AEC $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$, we define a reflexive and symmetric relation $\sim_0$ on the set of triples of the form (M, a, N), where $M, N \in \boldsymbol{K}_{\aleph_0}$, $M \prec_{\mathbf{K}} N$, and $a \in N \setminus M$. We say that $(M_0, a_0, N_0) \sim_0 (M_1, a_1, N_1)$ if $M_0 = M_1$ and there exist a structure $N \in \mathbf{K}$ and $\prec_{\boldsymbol{k}}$-embeddings $f_0 \colon N_0 \to N$ and $f_1 \colon N_1 \to N$ such that $f_0|M_0 = f_1|M_1$ and $f_0(a_0) = f_1(a_1)$. We let $\sim$ be the transitive closure of $\sim_0$. The equivalence classes of $\sim$ are called* Galois types.

Note that if $\boldsymbol{K}$ satisfies the amalgamation property then $\sim=\sim_0$. This identity is used crucially in proving the equivalence of model-homogeneity and Galois-saturation. When we use this equivalence we will assume amalgamation.

Fixing a coding of hereditarily countable sets by subsets of ω, the notion of Galois types naturally induces an equivalence relation on $\mathcal{P}(\omega)$.[2] For each $M \in \boldsymbol{K}_{\aleph_0}$ we let E_M denote the corresponding equivalence relation for Galois types over M (This notation was used in [5][3].). The domain of E_M is then the set of subsets of ω coding triples of the form (M, a, N), where $N \in \boldsymbol{K}_{\aleph_0}$, $M \prec_K N$ and $a \in N \setminus M$. If $\boldsymbol{k}$ is $\aleph_0$-presented, then each E_M is an analytic equivalence relation, and by Burgess's trichotomy for analytic equivalence relations, E_M has either countably many equivalence classes, $\aleph_1$ many, or a perfect set of inequivalent reals. Because there are two notions of weak-stability in the literature of AEC ([11, 24], we call the following notion *almost Galois ω-stability*.

Definition 3.2. *$\boldsymbol{K}$ is* almost Galois ω-stable *if there do not exist a countable model M in $\boldsymbol{K}$ and a perfect subset P of the domain of E_M whose members are E_M-inequivalent.*

Galois types are very much a property of the monster model. That is, given $M \prec_{\boldsymbol{k}} N$ and $a \in N \setminus M$, the Galois type of a over M cannot be determined by just looking at automorphism of N fixing M in isolation; one must consider an embedding of N into the monster model.

Remark 3.3. *[Amalgamation, joint embedding, and maximal models]* This remark collects a number of easy and well-known observations about the properties in its title. These observations should provide a background for understanding the choice of some 'background hypotheses' below. If an AEC has no maximal models[4] then it has arbitrarily large models. In general the converse fails; but the converse holds under joint embedding with one exception: an AEC with a *unique* maximal model may satisfy joint embedding (See part (1) of Corollary 3.6). The class of well-orders of order type $\leq \omega_1$, with $\prec_{\boldsymbol{k}}$ as end extension is a

[2]Alternately, letting τ be the vocabulary associated to $\boldsymbol{K}$, the set of τ-structures with domain ω can be viewed as a Polish space, with the set of codes for members of $\boldsymbol{K}_{\aleph_0}$ as an analytic subset. See [8].

[3]It might be natural to write E^1_M instead, as we are referring to the Galois 1-types. Recent work of Boney [6] shows that for an AEC satisfying amalgamation for countable models, the set of Galois n-types over M has the same cardinality for each $n \in \omega$.

[4]*Maximal* means there is no proper $\prec_{\boldsymbol{k}}$-extension, even one isomorphic to itself.

standard example of part 1 of Theorem 3.7: an AEC with a unique maximal model in $\aleph_1$ but amalgamation in $\aleph_0$.

Assuming amalgamation, the relation 'M and N can be $\prec_{\boldsymbol{k}}$-embedded into a common model' is an equivalence relation and each equivalence class is an AEC with joint embedding. Often we will assume amalgamation and joint embedding to avoid assuming only amalgamation and then having to restrict to one joint embedding class. Failure to make this assumption yields trivial counterexamples. There are no universal models for the class of algebraically closed fields (because of characteristic) but fixing the characteristic (that is the joint embedding class) yields a family of classes each with the joint embedding property. The technique of restricting to an equivalence class is illustrated by the generalization of Theorem 3.18 to Corollary 3.22.

We record some additional observations about amalgamation, joint embedding and maximal models. The next theorem shows that joint embedding and amalgamation in κ implies members of $\boldsymbol{K}$ with cardinality κ^+ can be amalgamated over submodels of cardinality κ. We restrict to the case $\kappa = \aleph_0$ as this is the relevant case for this paper. It is an open question (usually conjectured to be false) whether amalgamation in κ implies amalgamation in κ^+. Note that (1) of Theorem 3.4 is an easy consequence of part (2).

Theorem 3.4. *Let $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ be an AEC with Löwenheim-Skolem number $\aleph_0$ which satisfies amalgamation for countable models.*

1. *If $M, N \in \boldsymbol{K}_{\aleph_0}$ and $P \in \boldsymbol{K}_{\aleph_1}$, with $M \prec_{\boldsymbol{k}} P$ and $M \prec_{\boldsymbol{k}} N$, then there exist $Q \in \boldsymbol{K}_{\aleph_1}$ with $N \prec_{\boldsymbol{k}} Q$, and a $\prec_{\boldsymbol{k}}$-embedding $f\colon P \to Q$ such that f is the identity function on M.*
2. *If $M \in \boldsymbol{K}_{\aleph_0}$, $P, Q \in \boldsymbol{K}_{\aleph_1}$ with $M \prec_{\boldsymbol{k}} P$ and $M \prec_{\boldsymbol{k}} Q$, then there exist $R \in \boldsymbol{K}_{\aleph_1}$ and $\prec_{\boldsymbol{k}}$-embeddings $f\colon P \to R$ and $g\colon Q \to R$ with $f \restriction M = g \restriction M$.*

Proof. Since $\boldsymbol{K}$ has Löwenheim-Skolem number $\aleph_0$, each model in $\boldsymbol{K}_{\aleph_1}$ is the union of a continuous $\prec_{\boldsymbol{k}}$-increasing sequence of length ω_1, consisting elements of $\boldsymbol{K}_{\aleph_0}$.

For the first part of the lemma, fix a continuous $\prec_{\boldsymbol{k}}$-increasing sequence $\langle P_\alpha : \alpha < \omega_1\rangle$, consisting of elements of $\boldsymbol{K}_{\aleph_0}$, with union P, and $P_0 = M$. Recursively build a continuous $\prec_{\boldsymbol{k}}$-increasing chain of elements of $\boldsymbol{K}_{\aleph_0}$, $\langle N_\alpha : \alpha < \omega_1\rangle$, with $N_0 = N$, by choosing $N_{\alpha+1}$ to amalgamate N_α and $P_{\alpha+1}$ over P_α, for each $\alpha < \omega_1$. Then for each limit ordinal $\beta < \omega_1$, $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ is in $\boldsymbol{K}$, and for each $\alpha < \omega_1$, $P_\alpha \prec_{\boldsymbol{k}} N_\alpha$. Finally, $Q = \bigcup_{\alpha<\omega_1} N_\alpha$ is as desired.

For the second part of the lemma, fix continuous $\prec_{\boldsymbol{k}}$-increasing sequences

$$\langle P_\alpha : \alpha < \omega_1\rangle, \langle Q_\alpha : \alpha < \omega_1\rangle$$

consisting of members of $\boldsymbol{K}_{\aleph_0}$, with unions P and Q respectively, and $P_0 = Q_0 = M$. Recursively choose

1. $R_\alpha \in \boldsymbol{K}_{\aleph_0}$, for $\alpha \in \omega_1 \setminus \{0\}$;
2. $P'_\alpha, Q'_\alpha \in \boldsymbol{K}_{\aleph_0}$, for nonlimit $\alpha \in \omega_1 \setminus 2$;
3. $\prec_{\boldsymbol{k}}$-embeddings
   (a) $f_\alpha\colon P_\alpha \to R_\alpha$ and $g_\alpha\colon Q_\alpha \to R_\alpha$, for $\alpha \in (\omega_1 + 1) \setminus \{0\}$;
   (b) $j_\alpha\colon R_\alpha \to P'_{\alpha+1}$ and $k_\alpha\colon R_\alpha \to Q'_{\alpha+1}$, for$\alpha \in \omega_1 \setminus \{0\}$;
   (c) $j'_\alpha\colon P'_\alpha \to R_\alpha$ and $k'_\alpha\colon Q'_\alpha \to R_\alpha$, for each nonlimit $\alpha \in \omega_1 \setminus 2$;

such that

i) $f_1 \restriction M = g_1 \restriction M$;

ii) when $\alpha = 1$ or $\alpha < \omega_1$ is a limit ordinal, $j_\alpha \circ f_\alpha = f'_{\alpha+1} \restriction P_\alpha$ and $k_\alpha \circ g_\alpha = g'_{\alpha+1} \restriction Q_\alpha$;

iii) for all non-limit $\alpha \in \omega_1 \setminus 2$, $j_\alpha \circ j'_\alpha \circ f'_\alpha = f'_{\alpha+1} \restriction P_\alpha$ and $k_\alpha \circ k'_\alpha \circ g'_\alpha = g'_{\alpha+1} \restriction Q_\alpha$;

iv) for all $\alpha \in \omega_1 \setminus 1$, $j'_{\alpha+1} \circ j_\alpha = k'_{\alpha+1} \circ k_\alpha$;

v) for all non-limit $\alpha \in \omega_1 \setminus 2$, $f_\alpha = j'_\alpha \circ f'_\alpha$ and $g_\alpha = k'_\alpha \circ g'_\alpha$;

vi) for each limit ordinal $\beta < \omega_1$,

   (a) R_β is the direct limit of R_α ($\alpha < \beta$) via the maps $j'_{\alpha+1} \circ j_\alpha$;
   (b) f_β and g_β are induced by this direct limit and the maps f_α, g_α ($\alpha < \beta$).

These objects are all chosen by amalgamation, as follows:

- f_1, g_1 and R_1 amalgamate over M, P_1 and Q_1;
- for all $\alpha \in \omega_1 \setminus 1$, $R_{\alpha+1}$, $j'_{\alpha+1}$ and $k'_{\alpha+1}$ amalgamate over R_α, j_α and k_α;
- for $\alpha \in \omega_1 \setminus \{0\}$,
  - $P'_{\alpha+1}$, $f'_{\alpha+1}$ and j_α amalgamate over P_α, f_α, R_α and $P_{\alpha+1}$;
  - $Q'_{\alpha+1}$, $g'_{\alpha+1}$ and k_α amalgamate over Q_α, g_α, R_α and $Q_{\alpha+1}$.

$$
\begin{tikzcd}
P_5 \arrow[rrrr, "f'_5"] & & & & P'_5 \arrow[r, "j'_5"] & R_5 \\
P_4 \arrow[u] \arrow[rrr, "f'_4"] & & & P'_4 \arrow[r, "j'_4"] & R_4 \arrow[u, "j_4"] \arrow[r, "k_4"] & Q'_5 \arrow[u, "k'_5"] \\
P_3 \arrow[u] \arrow[rr, "f'_3"] & & P'_3 \arrow[r, "j'_3"] & R_3 \arrow[u, "j_3"] \arrow[r, "k_3"] & Q'_4 \arrow[u, "k'_4"] & \\
P_2 \arrow[u] \arrow[r, "f'_2"] & P'_2 \arrow[r, "j'_2"] & R_2 \arrow[u, "j_2"] \arrow[r, "k_2"] & Q'_3 \arrow[u, "k'_3"] & & \\
P_1 \arrow[u] \arrow[r, "f_1"] & R_1 \arrow[u, "j_1"] \arrow[r, "k_1"] & Q'_2 \arrow[u, "k'_2"] & & & \\
M \arrow[u] \arrow[r] & Q_1 \arrow[u, "g_1"] \arrow[r] & Q_2 \arrow[u, "g'_2"] \arrow[r] & Q_3 \arrow[uu, "g'_3"] \arrow[r] & Q_4 \arrow[uuu, "g'_4"] \arrow[r] & Q_5 \arrow[uuuu, "g'_5"]
\end{tikzcd}
$$

The first five stages of the construction.

Finally, we can let R be the direct limit of R_α ($\alpha < \omega_1$) via the maps $j'_{\alpha+1} \circ j_\alpha$. Then the desired f and g are induced by this direct limit. $\square_{3.4}$

In Part (1) of Lemma 3.4, we have not asserted that Q is a *proper* extension of P; the first example in Remark 3.3 shows that is too strong.

Definition 3.5. 1. *M is* μ-model homogeneous *if for every $N \prec_{\boldsymbol{k}} M$ and every $N' \in \boldsymbol{K}$ with $|N'| < \mu$ and $N \prec_{\boldsymbol{k}} N'$ there is a $\boldsymbol{K}$-embedding of N' into M over N.*

2. *M is* strongly μ-model homogeneous *if it is μ-model homogeneous and for any $N, N' \prec_{\boldsymbol{k}} M$ and $|N|, |N'| < \mu$, every isomorphism f from N to N' extends to an automorphism of M.*

3. *M is* strongly model homogeneous *if it is strongly $|M|$-model homogeneous.*

Theorem 3.4 implies that for AEC's with Löwenheim-Skolem number $\aleph_0$ satisfying amalgamation for countable models, maximal models of cardinality $\aleph_1$ are strongly $\aleph_1$-model homogeneous. By Theorem 8.3 of [1], $\aleph_1$-model homogeneous models are isomorphic for AEC's with Löwenheim-Skolem number $\aleph_0$ satisfying joint embedding for countable models. We note two additional consequences of Theorem 3.4 for maximal models.

Corollary 3.6. *Let $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ be an AEC with Löwenheim-Skolem number $\aleph_0$ which satisfies amalgamation for countable models.*

1. *If $\boldsymbol{k}$ satisfies joint embedding for countable models, and M and P are elements of $\boldsymbol{K}$, with M countable and P maximal and of cardinality $\aleph_1$, then M $\prec_{\boldsymbol{k}}$-embeds into P.*

2. *If M, P and Q are elements of $\boldsymbol{K}$, with M countable, P and Q maximal of cardinality $\aleph_1$, $M \prec_{\boldsymbol{k}} P$ and $M \prec_{\boldsymbol{k}} Q$, then there is an isomorphism of P and Q fixing M.*

Amalgamation and some form of joint embedding easily allows one to show the following (see Corollary 8.23 of [1]); we give two variants. Note that in the second case the Galois-saturated model may not be unique. Furthermore, there may be countable models that are not extendible, even when there is a unique Galois-saturated model in $\aleph_1$.

Theorem 3.7. *Let $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ be an AEC which is almost Galois ω-stable and satisfies amalgamation in $\aleph_0$.*

1. *If $\boldsymbol{K}$ satisfies joint embedding in $\aleph_0$ then there is a unique Galois-saturated model M in $\boldsymbol{K}_{\aleph_1}$.*

2. *If $N \in \boldsymbol{K}_{\aleph_0}$ has an uncountable extension in $\boldsymbol{K}$, then there is a Galois-saturated model M in $\boldsymbol{K}_{\aleph_1}$ with $N \prec_{\boldsymbol{k}} M$.*

Proof. For the first part, carefully construct an interweaving enumeration the Galois types over an increasing chain of countable models in order type ω_1 so that each Galois type over each model in the chain is realized. For uniqueness, suppose that M and M' are Galois-saturated models in $\boldsymbol{K}_{\aleph_1}$. Choose countable $M_0 \prec_{\boldsymbol{k}} M$ and $M'_0 \prec_{\boldsymbol{k}} M$. By joint embedding there is a countable M_1 that $\prec_{\boldsymbol{k}}$-extends both M_0 and M'_0. Applying Galois saturation, a countable recursive construction shows that M_1 is $\prec_{\boldsymbol{k}}$-embeddable into both M and M'. Then a recursive construction of length ω_1 using Galois saturation shows M and M' are isomorphic (over M_1).

For the second part, let $\boldsymbol{K}_N$ be the equivalence class under joint embedding of the models that are jointly embeddable with N. Apply the first argument to this class. $\square_{3.7}$

For any AEC $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$, if $M, N \in \boldsymbol{K}$ and $M \prec_K N$, then M is a substructure of N, but the definition of AEC does not require even that M be a first-order elementary submodel of N. Before proving the main result of this section, Theorem 3.18, we prove a lemma which reduces the proof to the case where $M \prec_K N$ implies $L_{\omega_1,\omega}(\tau)$-elementarity. A similar reduction appears in Theorem 3.6E of [26] and Lemma 2.5 of [20].

Definition 3.8. *Let $\boldsymbol{K}$ be an AEC in a countable similarity type τ, with Löwenheim-Skolem number $\aleph_0$, such that $\boldsymbol{K}$ has a unique Galois-saturated model M in $\aleph_1$, which is small.*

1. *For $N_0, N_1 \in \boldsymbol{K}$, define $N_0 \prec_{\boldsymbol{k}^*} N_1$ to mean that $N_0 \prec_{\boldsymbol{k}} N_1$ and $N_0 \prec_{\infty,\omega} N_1$.*
2. *Let $\boldsymbol{K}^*$ be the set of $N \in \boldsymbol{K}_{\aleph_0}$ which satisfy the Scott sentence of M.*
3. *Let $(\boldsymbol{K}', \prec_{\boldsymbol{k}'})$ be the closure of $(\boldsymbol{K}^*, \prec_{\boldsymbol{k}^*})$ under isomorphism and direct limits of arbitrary length.*

To discuss the relationship between (almost) Galois stability of $\boldsymbol{K}$ and $\boldsymbol{K}'$, we introduce some notation. We first give a standard equivalent for the definition of Galois type, but parameterized for the comparisons we need here. The class $\boldsymbol{K}_0$ below will be $\boldsymbol{K}$ or $\boldsymbol{K}'$ in our applications. This construction is implicit in [26] and in the extension of those arguments towards the construction of examples of a good frame in [25] and chapter III of [24]. The next lemma shows the properties of the induced class $\boldsymbol{K}'$. We describe a slightly more general situation from [24] in Remark 3.12

Notation 3.9. *Let $\boldsymbol{K}_0$ be an AEC with a $(\boldsymbol{K}_0, \aleph_1)$-homogenous-universal model M in $\aleph_1$.*

1. *If $M_0 \prec_{\boldsymbol{K}_0} M$, $\mathbb{S}_{\boldsymbol{K}_0}(M_0)$ is the collection of orbits of elements of M under $\mathrm{aut}_{\mathrm{M_0}}(\mathrm{M})$ (the automorphisms of M fixing M_0 pointwise).*
2. *$\alpha(\boldsymbol{K}_0) = \sup\{|\mathbb{S}_{\boldsymbol{K}_0}(M_0)| : M_0 \in \boldsymbol{K}_0, |M_0| = \aleph_0\}$.*

We need to require the joint embedding property to guarantee that $(\boldsymbol{K}, \aleph_1)$-homogeneous-universal[5] is equivalent to Galois saturated. Most of the argument for the next lemma would work if we just assume there is a unique Galois saturated model (which is small); but it might not be universal (in either $\boldsymbol{K}$ or $\boldsymbol{K}'$). (See Chapter 16 of [1] or Remark 1 of [23] for more detailed remarks.)

Lemma 3.10. *Let $\boldsymbol{K}$ be an AEC in a countable similarity type τ, with Löwenheim-Skolem number $\aleph_0$, with joint embedding and the amalgamation property in $\aleph_0$. Suppose further that unique Galois-saturated model M in $\aleph_1$ is small. Then the following hold.*

1. *$(\boldsymbol{K}', \prec_{\boldsymbol{k}'})$ is an AEC with Löwenheim-Skolem number $\aleph_0$.*
2. *M is $(\boldsymbol{K}',\aleph_1)$-homogenous-universal.*
3. *$(\boldsymbol{K}', \prec_{\boldsymbol{k}'})$ satisfies amalgamation in $\aleph_0$.*
4. *For every $M_0 \in \boldsymbol{K}'_{\aleph_0}$, $\mathbb{S}_{\boldsymbol{K}}(M_0) = \mathbb{S}_{\boldsymbol{K}'}(M_0)$.*
5. *$\alpha(\boldsymbol{K}) = \alpha(\boldsymbol{K}')$.*
6. *$\boldsymbol{K}'$ is $\aleph_0$-categorical.*

[5] M is $(\boldsymbol{K}, \aleph_1)$-homogeneous-universal it is universal for countable structures in $\boldsymbol{K}$ and $\aleph_1$-model-homogenous.

Proof. 1) The coherence and unions of chains axioms are immediate on $\boldsymbol{K}^*$. For Löwenheim-Skolem, note that M can be written as an increasing chain of $\boldsymbol{K}'$-submodels. Thus, $\boldsymbol{K}^*$ is a weak AEC in the sense of Definition 16.10 of [1] and so $(\boldsymbol{K}', \prec_{\boldsymbol{k}'})$ is an AEC applying either Exercise 16.12 of [1] or Lemma II.1.12 of [24].

2) Let $M_0 \prec_{\boldsymbol{k}'} M_1$ be countable. Then there are $\boldsymbol{K}'$-maps f and g such that $f(M_0) \prec_{\boldsymbol{k}'} M$ and $g(M_1) \prec_{\boldsymbol{k}'} M$ by the definition of $\boldsymbol{K}'$. But since M is $(\boldsymbol{K},\aleph_1)$-homogenous-universal, there is an h in $\mathrm{aut(M)}$ such that $h \circ g \restriction M_0 = f$. Since both $\prec_{\boldsymbol{k}}$ and $\prec_{\boldsymbol{k}'}$ are preserved by automorphisms, h is a $\boldsymbol{K}'$-map. So $h \circ g$ is a $\boldsymbol{K}'$ embedding of M_1 into M extending f. This shows M is $(\boldsymbol{K}', \aleph_1)$ homogeneous and it is clearly $\boldsymbol{K}'$-universal.

3) Suppose $M_0 \prec_{\boldsymbol{k}'} M_1, M_2$. Then there are $\boldsymbol{K}'$-embeddings of M_1 and M_2 over M_0 into M. So amalgamation holds.

4) The Galois types are determined by $\mathrm{aut_{M_0}M}$ which does not depend on the choice of AEC.

5) We have that $\alpha(\boldsymbol{K}) \geq \alpha(\boldsymbol{K}')$ since the supremum is taken over a smaller set. But for each $M_0 \in \boldsymbol{K}_{\aleph_0}$, there is an $M_1 \in \boldsymbol{K}'_{\aleph_0}$ with $M_0 \prec_{\boldsymbol{k}} M_1 \prec_{\boldsymbol{k}'} M$ and by the extendability of $\boldsymbol{K}$-Galois types, and part 4, $|\mathbb{S}_{\boldsymbol{K}}(M_0)| \leq |\mathbb{S}_{\boldsymbol{K}}(M_1)| = |\mathbb{S}_{\boldsymbol{K}'}(M_1)|$ so $\alpha(\boldsymbol{K}) = \alpha(\boldsymbol{K}')$.

6) Let τ' and $\tau'' = \tau' \cup \{P\}$ be the vocabularies which witness that $\boldsymbol{K}$ is $\aleph_0$-presented. Let ψ_1 be the τ' sentence whose reducts are the models in $\boldsymbol{K}$; let ψ_2 be the τ'' sentence whose reducts are pairs (N, M) with $N \prec_{\boldsymbol{k}} M$. Further suppose that ϕ is the Scott sentence of M. The following sentences witness that $(\boldsymbol{K}', \prec_{\boldsymbol{k}'})$ is $\aleph_0$-presented: $\hat{\psi}_1 = \psi_1 \wedge \phi$ and $\hat{\psi}_2 = \psi_2 \wedge \chi$ where $(M, N) \models \chi$ if $M \prec_{L^*} N$ where L^* is least countable fragment containing ϕ.

7) This is evident since N is small. $\square_{3.10}$

Conclusion 5 immediately yields.

Corollary 3.11. *Under the hypotheses of Lemma 3.10,*

- $(\boldsymbol{K}, \prec_{\boldsymbol{k}})$ *is Galois ω-stable if and only if* $(\boldsymbol{K}', \prec_{\boldsymbol{k}'})$ *is;*
- $(\boldsymbol{K}, \prec_{\boldsymbol{k}})$ *is almost Galois ω-stable if and only if* $(\boldsymbol{K}', \prec_{\boldsymbol{k}'})$ *is.*

Moreover, the hypothesis of joint embedding is in some ways only a convenience; see Corollary 3.22. If $\boldsymbol{K}$ has the amalgamation property then joint embedability is an equivalence relation and each of the equivalence classes is an AEC with joint embedding preserving the other properties defining AEC's. At least one class fails Galois ω-stability if $\boldsymbol{K}$ does. But some classes may not have any uncountable models.

Remark 3.12. In chapters I and II (e.g. II.3.4) of [24], Shelah makes a somewhat more general argument. Add to Definition 3.8 a third clause: For each countable $M \in \boldsymbol{K}$, let $\boldsymbol{K}_M = \{N \in \boldsymbol{K} : |N| = \aleph_0 \wedge M \prec_{\boldsymbol{k}'} N\}$, where $\prec_{\boldsymbol{k}'}$ is defined as before. It is again straightforward to see that each $\boldsymbol{K}_M$ is an $\aleph_0$ categorical AEC. If there are less than $2^{\aleph_1}$ models in $\aleph_1$ of $\boldsymbol{K}$ and a fortiori of each $\boldsymbol{K}_M$ then under $2^{\aleph_0} < 2^{\aleph_1}$, $\boldsymbol{K}_M$ has the amalgamation property and since all models are extension of a single one, the joint embedding property. Then Shelah argues that by way of the notion of 'materialization of types' (Chapter 1 of [24]) one can deduce almost Galois stability.

The following variants on an example of Jarden and Shelah [12] will illustrate the situation and also provide some context for Theorem II.3.4 of [24]. That theorem aims to construct a good frame from an $\aleph_0$-presentable class that has few models in $\aleph_1$, is $\aleph_0$-categorical, has amalgamation in $\aleph_0$ and is ω-Galois stable or at least ω-almost Galois stable. We show several of these conditions are necessary. In particular, these examples are not $\aleph_0$-categorical. Note that one use of Lemma 3.10 is to extract an $\aleph_0$-categorical AEC from a given AEC with few models in $\aleph_1$. Recall that there are only $\aleph_1$ countable linear orders that are one transitive (any two points are automorphic) [19].

Here is the basic example

Example 3.13. Let τ contain equality and a binary symbol $<$. Let $(\boldsymbol{K}, \prec_{\boldsymbol{k}})$ be the class of τ structures such that each $M \in \boldsymbol{K}$ is a partially ordered set such that each component is a countable 1-transitive linear order. $M \prec_{\boldsymbol{k}} N$ means $M \subseteq N$ but each element of $N - M$ is incomparable with all elements of M.

$\boldsymbol{K}$ is an $\aleph_0$-presentable AEC. It has exactly $\aleph_1$ countable models and $2^{\aleph_1}$ in $\aleph_1$. It is almost Galois ω-stable but not Galois ω-stable. $\boldsymbol{K}_{\aleph_0}$ satisfies the amalgamation property and the joint embedding property. Thus there is a unique Galois saturated model in $\aleph_1$.

Now we vary the example so there are $\aleph_1$ non-isomorphic Galois-saturated models in $\aleph_1$.

Example 3.14. Let τ consist of a binary symbol $<$ and another binary relation symbol E. Let $(\boldsymbol{K}, \prec_{\boldsymbol{k}})$ be the class of τ-structures such that each $M \in \boldsymbol{K}$ is a partially ordered set such that each component (maximal connected component) is a countable 1-transitive linear order. Further E is an equivalence relation; each class intersects each component in exactly one point. Moreover E induces an order-isomorphism between each pair of components. $M \prec_{\boldsymbol{k}} N$ means $M \subseteq N$ but each element of $N - M$ is incomparable with all elements of M.

$\boldsymbol{K}$ is an $\aleph_0$-presentable AEC as it is describable in $L(Q)$ using only assertions of the form '$\phi(x)$ is countable'. It has exactly $\aleph_1$ models in each infinite cardinality. It is almost Galois ω-stable but not Galois ω-stable. $\boldsymbol{K}_{\aleph_0}$ satisfies the amalgamation property but does *not* satisfy the joint embedding property. There are in fact $\aleph_1$, pairwise non-isomorphic Galois saturated models in $\aleph_1$; each model is $\aleph_1$ copies of a particular 1-transitive order.

There is no countable fragment L^* such that syntactic type in L^* is the same as the Galois type in $\boldsymbol{K}$.

Because the joint embedding property fails, Lemma 3.10 does not apply to this example. Applying the construction in Definition 3.8 gives rise to $\aleph_1$ distinct $\aleph_0$-presentable AEC; each is categorical in every infinite cardinality; each is ω-stable. In each derived AEC, Galois type is equivalent to syntactic type.

The refined AEC, where all components have the same order type, are indexed by ϕ_α for $\alpha < \omega_1$, which list the Scott sentences of countable transitive linear orders. Exercise 14.28 of Rosenstein [19] shows that the transitive order $\mathbb{Z}^\alpha$ has Scott rank $\omega \cdot \alpha + 1$.

Neither Lemma 3.10 nor II.3.4 (page 285) of [24] applies to either of these examples because there are too many models in $\aleph_1$ in the first case and the Galois saturated model is not *locally small* in the second. Nevertheless there are $\aleph_1$ restrictions of $\boldsymbol{K}$ to AEC $\boldsymbol{K}_\alpha$, where models in $\boldsymbol{K}_\alpha$ contain only components satisfying ϕ_α. Each of them is Galois ω-stable. In each $\boldsymbol{K}_\alpha$, Galois types are equivalent to syntactic types in an appropriate fragment L_α.

Question 3.15. *Find an example of an $\aleph_0$-presented AEC with the joint embedding and amalgamation properties that has fewer than $2^{\aleph_1}$ many models in $\aleph_1$ and is strictly almost Galois ω-stable.*

By Theorem 3.18 below, $\boldsymbol{K}$ must fail joint embedding or have at least $\aleph_1$ models in $\aleph_1$.

Given a τ-structure M and a fragment L of $L_{\omega_1,\omega}(\tau)$, we say that M is *L-atomic* if for each finite sequence $\mathbf{a}$ from M there exists an $|a|$-ary formula $\chi_{\mathbf{a}}(\mathbf{x}) \in L$ such that $M \models \chi_{\mathbf{a}}(\mathbf{a})$, and, for each $|a|$-ary formula $\lambda(\mathbf{x})$ of $L_{\omega_1,\omega}(\tau)$, if $M \models \lambda(\mathbf{a})$, then $M \models (\forall \mathbf{x})[\chi_{\mathbf{a}}(\mathbf{x}) \to \lambda(\mathbf{x})]$.

Remark 3.16. *It follows from the Scott analysis (in Section 2) that a τ-structure M is small if and only if there is a countable fragment L of $L_{\omega_1,\omega}(\tau)$ such that M is L-atomic (for instance, any fragment containing the Scott sentence of M).*

Lemma 3.17. *Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ is an AEC over a vocabulary τ, and that $M_0 \in \boldsymbol{K}_{\aleph_0}$. Suppose that $M \in \boldsymbol{K}_{\aleph_1}$ is Galois saturated, with $M_0 \prec_{\boldsymbol{k}} M$. Let $\hat{\tau}$ be the union of τ with a countably infinite collection of new constant symbols, and let M' be an expansion of M where these new symbols are used to enumerate M_0. Suppose that M' is $L_{\omega_1,\omega}(\hat{\tau})$-small. Then for some $L^*(\hat{\tau})$, M' is $L^*(\hat{\tau})$-atomic. It follows that (in $\boldsymbol{k}$) there are only countably many Galois types over M_0.*

Proof. By Remark 3.16 applied in the vocabulary $\hat{\tau}$, M' is atomic in $L^*(\hat{\tau})$, the countable fragment in which M' has a Scott sentence; this is Theorem 3.18.1. We will show that for any $a \in M$ the $L^*(\hat{\tau})$-type of a determines the Galois type (in $\boldsymbol{K}$) of a over M_0. Since M' is $L_{\omega_1,\omega}(\hat{\tau})$-small, it follows that only countably many Galois types over M_0 are realized in M. Suppose that some $a, b \in M$ realize the same $L^*(\hat{\tau})$-type in M'. Then this type is given by a formula in $L^*(\hat{\tau})$, by $L^*(\hat{\tau})$-atomicity. There exists a countable $\hat{M} \in \mathbf{K}$ such that $M_0ab \subset \hat{M} \prec_{L^*(\hat{\tau})} M$, and, as $\hat{M}$ is $L^*(\hat{\tau})$-atomic, there exists an automorphism g of $\hat{M}$, fixing M_0 pointwise with $g(a) = b$. Thus, a and b have the same Galois type over M_0. So M realizes only countably many Galois types over M_0. $\square_{3.17}$

We turn to the main result. Corollary 3.22 derives a slightly weaker conclusion than Theorem 3.18 in the absence of the joint embedding property. By Corollary 2.8, the hypotheses of Theorem 3.18 imply that all models in $\boldsymbol{K}$ are small. By Theorem 3.7, $\boldsymbol{K}_1$ contains a unique Galois-saturated model.

Theorem 3.18. *Suppose that $\boldsymbol{K}$ is an $\aleph_0$-presented AEC (over a countable vocabulary τ) which satisfies amalgamation, and JEP for countable models, such that that $\boldsymbol{K}$ is almost Galois ω-stable, and $|\boldsymbol{K}_{\aleph_1}| \leq \aleph_0$. Let M be the unique Galois-saturated model in $\boldsymbol{K}_{\aleph_1}$, and let $\boldsymbol{k}' = (\boldsymbol{K}', \prec_{\boldsymbol{k}'})$ be as in Definition 3.8. Let $\hat{\tau}$ be formed by adding ω many new constant symbols to τ. Then*

1. *for each $M_0 \in \boldsymbol{K}'_0$ such that $M_0 \prec_{\boldsymbol{k}'} M$, if M' is a $\hat{\tau}$-structure expanding M in which the interpretations of the new constant symbols in $\hat{\tau}$ enumerate M_0, then M' is small. This implies*
2. *$\boldsymbol{K}$ is Galois ω-stable.*

Proof. There are three cases, as follows.

1. For some countable fragment $L^*(\hat{\tau})$ of $L_{\omega_1,\omega}(\hat{\tau})$ and some n, there are uncountably many $L^*(\hat{\tau})$-n-types realized in M' .
2. For every countable fragment $L_0(\hat{\tau})$ of $L_{\omega_1,\omega}(\hat{\tau})$ and every n, only countably many $L_0(\hat{\tau})$-n-types are realized in M'. Then one of the following holds.
   (a) The model M' is not $L_{\omega_1,\omega}(\hat{\tau})$-small.
   (b) The model M' is $L_{\omega_1,\omega}(\hat{\tau})$-small, so for some countable fragment $L^*(\hat{\tau})$, M' has a Scott sentence in $L^*(\hat{\tau})$.

We will show that case 1) contradicts the assumption of almost Galois ω-stability of $\boldsymbol{k}'$ (which by Corollary 3.11 is equivalent to that of $\boldsymbol{k}$), and that case 2a) contradicts the assumption that $|\boldsymbol{K}_{\aleph_1}| \leq \aleph_0$. We are reduced to case 2b) and Lemma 3.17 gives that $\boldsymbol{k}$ is Galois ω-stable.

For case 1, we use the following fact[6].

Fact 3.19. *If for some n, there is a fragment L^1 of $L_{\omega_1,\omega}(\hat{\tau})$ such that there are a perfect set of L^1- n-types over a countable model N, then there is a fragment L^* of $L_{\omega_1,\omega}(\hat{\tau})$ containing L_1 such that there are a perfect set of L^*-1-types over N.*

[6]We just note that there is no need here for the Boney result, discussed in footnote 3 [6], although it could have been applied directly.

Proof. From the hypothesis there must be an $n-1$-type p such that there are a perfect set $\{q_\eta(x,\mathbf{y}) : \eta \in 2^\omega\}$ of n-types extending p. So $q'_\eta(x) = \{(\exists \mathbf{y})[\phi(x,\mathbf{y}) \wedge \bigwedge p(\mathbf{y})] : \phi(x,\mathbf{y}) \in q_\eta(x,\mathbf{y})\}$ for $\eta \in 2^\omega$ is the required collection of L^*-1-types over N, where L^* adds the conjunction of p to L_1. $\square_{3.19}$

By Fact 3.19, in Case 1 there exists a perfect set of syntactic 1-types in $L^*(\hat{\tau})$ that are realized in countable $\hat{\tau}$-structures whose τ-reducts are in $\boldsymbol{K}'_{\aleph_0}$ and for which the interpretation of the c_i's enumerates M_0 in the same manner that M' does. Since $\prec_{\boldsymbol{k}'}$ implies $L_{\omega_1,\omega}(\tau)$-elementarity, this implies the existence of a perfect set of Galois 1-types over M_0, contradicting the almost Galois ω-stability of $\boldsymbol{k}'$.

The bulk of the proof derives a contradiction from Case 2a. Let ϕ be the Scott sentence for M. Let $\overline{M} = \langle M_\alpha : \alpha < \omega_1\rangle$ be such that (as above) M_0 is the model introduced in the statement of the theorem, $M = \bigcup_{\alpha<\omega_1} M_\alpha$ and the following hold for each $\alpha < \omega_1$:

- M_α is a countable element of $\boldsymbol{K}$;
- $M_\alpha \prec_{\boldsymbol{k}} M$;
- $M_\alpha \models \phi$;
- M_α is a proper subset of $M_{\alpha+1}$;
- if α is a limit ordinal, then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.

The models M_α are all isomorphic, as they satisfy the same Scott sentence. As M is Galois saturated, there is a set $\overline{F} = \{F_\alpha : \alpha < \omega_1\}$ such that each F_α is an automorphism of M mapping M_0 setwise to M_α. For each pair $\alpha, \beta < \omega_1$, let $F_{\alpha,\beta}$ denote $F_\beta \circ F_\alpha^{-1}$.

Let τ^+ be the expansion of our vocabulary τ' to the τ' of Theorem 2.5 (i.e., add the symbols E_n, f_n $(n \in \omega)$, and a binary relation ordering the domain of M in order type ω_1; alternately, using Theorem 2.20 we could skip this step). Fix a regular cardinal θ large enough so that M', τ^+, $\overline{M}$ and $\overline{F}$ are elements of $H(\theta)$ (to apply the methods of [5], we need θ to be larger than $2^{2^{\aleph_1}}$).

Let $\langle X_\alpha : \alpha < \omega_1\rangle$ be a $\subseteq$-increasing continuous chain of countable elementary submodels of $H(\theta)$ such that $M', \tau^+, \overline{M}$ and $\overline{F}$ are elements of X_0, and such that for each $\alpha < \omega_1$ there is a countable ordinal $\beta \in X_{\alpha+1} - X_\alpha$. For each $\alpha < \omega_1$, let P_α be the transitive collapse of X_α, and let $\rho_\alpha\colon X_\alpha \to P_\alpha$ be the corresponding collapsing map. Then $\rho_\alpha(\omega_1) = \omega_1^{P_\alpha}$ is the ordinal $X_\alpha \cap \omega_1$.

The following is a paraphrase of Theorem 2.1 of [10] (Hutchinson built on work of Keisler and Morley [14]; Enayat provides a useful source on this work in [7]). It can be proved via iterated ultrapowers as in [5]. Section 4 of [10] describes the fragment of ZFC needed for Fact 3.20; this fragment is easily seen to follow from the theory ZFC° of [5].

Fact 3.20. *Let $\mathcal{B}$ be a countable model of ZFC and c a regular cardinal in $\mathcal{B}$. Then there is a countable elementary extension $\mathcal{C}$ of $\mathcal{B}$ such that each a such that $\mathcal{B} \models a \in c$ is fixed (i.e. has no new elements in $\mathcal{C}$) but c is enlarged and there is a least new element of $\mathcal{C}$.*

Construct a family $\{P'_\alpha : \alpha < \omega_1\}$ of uncountable models of set theory so that, for each $\alpha < \omega_1$, there is an elementary extension of P_α to P'_α (with corresponding elementary embedding $\chi_\alpha\colon P_\alpha \to P'_\alpha$) such that

1. the critical point of χ_α is $\omega_1^{P_\alpha}$, so $\omega_1^{P_\alpha}$ is an initial segment of $\omega_1^{P'_\alpha}$;
2. $\omega_1^{P'_\alpha}$ is ill-founded;
3. in V, there is a continuous increasing ω_1-sequence $\langle t^\alpha_\gamma : \gamma < \omega_1\rangle$ consisting of elements of $\omega_1^{P'_\alpha}$.

Item 3 above implies in particular that each $\omega_1^{P'_\alpha}$ is uncountable. Each P'_α can be realized as the union of a increasing elementary chain of models $\langle P^\alpha_\gamma : \gamma < \omega_1\rangle$, where $P^\alpha_0 = P_\alpha$,

$$P'_\alpha = \bigcup_{\gamma<\omega_1} P^\alpha_\gamma$$

for limit α, and each $P^\alpha_{\gamma+1}$ can be obtained by applying Fact 3.20 to P^α_γ. Then each t^α_γ (the c of Fact 3.20) can be taken to be $\omega_1^{P^\alpha_\gamma}$.

Recall that M is the union of the continuous $\subseteq$-increasing chain $\langle M_\alpha : \alpha < \omega_1\rangle$. It follows then for each $\alpha < \omega_1$, that $M_{\omega_1^{P_\alpha}} = \rho_\alpha(M) \subset P_\alpha$, and that $M_{\omega_1^{P_\alpha}}$ has cardinality $\aleph_1$ in P_α. For each $\alpha < \omega_1$, let $N_\alpha = \chi_\alpha(M_{\omega_1^{P_\alpha}})$ and let $N'_\alpha = \chi_\alpha(\rho_\alpha(M'))$. Then each N'_α is an expansion of N_α via the given enumeration of M_0 by the constants c_i, and it has cardinality $\aleph_1$ in P'_α.

In the argument for Theorem 2.5 replace the appeal to Lopez-Escobar (Theorem 5.3.8 of [1]) with the observation that the induced ordering on N'_α is not well-founded by construction. The rest of the argument for Theorem 2.5 (or Theorem 2.20) shows that, in V, each N'_α is small for $L_{\omega_1,\omega}(\tau')$. Nevertheless, by the elementarity of $\chi_\alpha \circ \rho_\alpha$, each P'_α thinks that N'_α is not $L_{\omega_1,\omega}(\tau')$-small.

Since $\overline{M}$ is a sequence indexed by ω_1 in V (or in X_α), $\chi_\alpha(\rho_\alpha(\overline{M}))$ is a sequence indexed by $\omega_1^{P'_\alpha}$ in P'_α. So, in P'_α, for each element t of its ω_1, there is a t-th element of the sequence, which we denote by M^α_t. Furthermore, in P'_α, $\chi_\alpha(\rho_\alpha(\overline{F}))$ is a set $\{F^\alpha_t : t \in \omega_1^{P'_\alpha}\}$ consisting of automorphisms of N_α, such that each $F^\alpha_t \in P'_\alpha$ is an automorphism of N_α sending M_0 setwise to M^α_t. Each F^α_t is then an automorphism of N_α in V also.

Since each N'_α is small, each N_α is as well. Since we are assuming that there are *only countably many models* in $\mathbf{K}$ of cardinality $\aleph_1$, there exists an uncountable set $S \subseteq \omega_1$ such that N_{α_0} and N_{α_1} are isomorphic (in V) for all α_0, α_1 in S. Fix for a moment a pair of elements α_0, α_1 of S and an isomorphism $\pi\colon N_{\alpha_0} \to N_{\alpha_1}$. Applying item 3 above and the continuity (in the sense of P'_{α_j}, for $j = 0, 1$) of the sequences $\langle M^{\alpha_0}_t : t \in \omega_1^{P'_{\alpha_0}}\rangle$ and $\langle M^{\alpha_1}_t : t \in \omega_1^{P'_{\alpha_1}}\rangle$, there must be $s_0 \in \omega_1^{P'_{\alpha_0}}$ and $s_1 \in \omega_1^{P'_{\alpha_1}}$ such that π maps $M^{\alpha_0}_{t_0}$ setwise to $M^{\alpha_1}_{t_1}$. To see this, start with $\gamma_0 = 0$ and, for each $n \in \omega$, let γ_{n+1} be large enough so that

$$\pi[M^{\alpha_0}_{t^{\alpha_0}_{\gamma_n}}] \subseteq M^{\alpha_1}_{t^{\alpha_1}_{\gamma_{n+1}}}$$

and

$$\pi^{-1}[M^{\alpha_1}_{t^{\alpha_1}_{\gamma_n}}] \subseteq M^{\alpha_0}_{t^{\alpha_0}_{\gamma_{n+1}}}.$$

Then let $s_0 = t^{\alpha_0}_{\sup_{n\in\omega}\gamma_n}$ and let $s_1 = t^{\alpha_1}_{\sup_{n\in\omega}\gamma_n}$. By the continuity in item 3, the s_j's are in the respective P'_{α_j}, for $j \in \{0,1\}$. So, for each j, by the continuity in P'_{α_j} of $M^{\alpha_j}_t$,

$$M^{\alpha_j}_{s_j} = \bigcup_{n<\omega} M^{\alpha_j}_{t^{\alpha_j}_{\gamma_n}}.$$

Then $(F^{\alpha_1}_{s_1})^{-1} \circ \pi \circ F^{\alpha_0}_{s_0}$ is an isomorphism of N_{α_0} and N_{α_1} fixing M_0 setwise, though not necessarily pointwise.

Finally, we show that for each $\alpha_0 < \omega_1$ such an isomorphism is impossible for sufficiently large $\alpha_1 < \omega_1$.

Each model P'_α thinks that N'_α is small for every countable fragment of $L_{\omega_1,\omega}(\tau')$ but not $L_{\omega_1,\omega}(\tau')$-small. Thus, from the point of view of P'_α, there is no ordinal t such that $\phi_{\overline{a},t}(\overline{x}) \equiv \phi_{\overline{a},t+1}(\overline{x})$ (in the terms of the Scott construction) for all finite tuples $\overline{a}$ of N'_α. For each well-founded ordinal γ of P'_α (this includes the members of $\omega_1^{P'_\alpha} = \omega_1 \cap X_\alpha$, by item 1 above), and each finite tuple $\overline{a}$ of N'_α, P'_α sees the same formula $\phi_{\overline{a},\gamma}(\overline{x})$ that the true universe V does, which means that the Scott sentence for N'_α has rank at least $\omega_1 \cap X_\alpha$ (and slightly more than this, in fact, in the approach from [5])[7].

Now choose $\alpha_0, \alpha_1 \in S$ such that $\omega_1 \cap X_{\alpha_1}$ is greater than the Scott rank (in V) of N'_{α_0}. Since permuting the constants c_i in terms of their enumeration of M_0 has no effect on the rank of the Scott sentence for N'_{α_1}, there cannot be then an isomorphism of N_{α_0} and N_{α_1} fixing M_0 setwise, since this would imply that N'_{α_0} and N'_{α_1} have the same Scott rank (Indeed, their Scott sentences would differ only by a permutation of the c_i's). Thus we have a contradiction in case 2a.

We have ruled out cases 1) and 2a) and are left with case 2b). Again, Lemma 3.17 gives the second conclusion of the theorem. $\square_{3.18}$

Remark 3.21. Note that argument ruling out case 2a) uses the set theoretic argument to find $\aleph_1$ τ'-small models in $\aleph_1$ with distinct τ'-Scott rank. By the automorphism argument, this contradicts the assumption that there are only $\aleph_0$ τ-models in $\aleph_1$.

We return to the slightly more complicated situation where joint embedding is not assumed.

Corollary 3.22. *Suppose $\boldsymbol{K}$ is an AEC satisfying the hypotheses of Theorem 3.18 except the joint embedding property. Then $\boldsymbol{K}$ is the union of a countable family of sub-AEC $\boldsymbol{K}_i$, which each satisfy Theorem 3.18.*

Proof. Since there are only countably many models in $\aleph_1$, the equivalence relation of common extension has at most countably many classes. Each satisfies the hypothesis and therefore the conclusion of Theorem 3.18 $\square_{3.22}$

4 Absoluteness of Categoricity

It is shown in [5] (see Theorems 2.1 and 6.2) that, given an $\aleph_0$-presented AEC $\boldsymbol{K}$, the statement that $\boldsymbol{K}$ has an uncountable model is Σ^1_1 in a real coding $\boldsymbol{K}$, and the statement that $\boldsymbol{K}$ is almost Galois ω-stable is Π^1_1 in a real coding $\boldsymbol{K}$. These statements are therefore absolute. Amalgamation of countable models for such a $\boldsymbol{K}$ is easily seen to be Π^1_2 in a code for $\boldsymbol{K}$. In this section we apply Theorem 3.18 to prove the following theorem.

Theorem 4.1. *Suppose that $\boldsymbol{K}$ is an $\aleph_0$-presented almost Galois ω-stable AEC with Löwenheim-Skolem number $\aleph_0$, satisfying amalgamation for countable models, and having a model of cardinality $\aleph_1$. The assertion that $\boldsymbol{K}$ is $\aleph_1$-categorical is then absolute, as it is equivalent to a statement of the form $\phi_1 \wedge \phi_2$, where ϕ_1 and ϕ_2 are Π^1_2 and Σ^1_2, respectively, in a code for $\boldsymbol{K}$.*

The first step of our proof of Theorem 4.1 removes from $\boldsymbol{K}$ all countable models lacking uncountable $\prec_{\boldsymbol{k}}$-extensions. Before beginning the proof we make an observation about this new class.

Remark 4.2. Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ is an analytically presented AEC, and that $M \in \boldsymbol{K}$ is countable. Then there exists an uncountable $N \in \boldsymbol{K}$ with $M \prec_{\boldsymbol{k}} N$ if and only if there exists a countable ω-model P of ZFC$^\circ$ (containing a real parameter for $\boldsymbol{K}$) such that $M \in P$ and P thinks there exists an uncountable $N \in \boldsymbol{K}$ with $M \prec N$. This follows from the same argument as for Theorem 2.1 of [5] (one direction consists of taking the transitive collapse of a countable elementary submodel; the other consists of building an iterated generic elementary embedding of length ω_1). Note that the latter clause is Σ^1_1 in the given parameter for $\boldsymbol{K}$.

[7] Alternatively, Lemma 2.21 implies that the Scott rank of N'_α is exactly the well-founded part of $\omega_1^{P'_\alpha}$

We rely on the following fact from [5].

Fact 4.3. *Suppose that $\boldsymbol{k} = (\boldsymbol{K}, \prec_{\boldsymbol{k}})$ is an $\aleph_0$-presented AEC. The following assertion is equivalent to a Σ^1_2 statement in a parameter for $\boldsymbol{k}$: There exist $M \in \boldsymbol{K}_{\aleph_0}$ and $N \in \boldsymbol{K}_{\aleph_1}$ such that*

- *$M \prec_{\mathbf{k}} N$;*
- *the set of Galois types over M realized in N is countable;*
- *some Galois type over M is not realized in N.*

Proof of Theorem 4.1. Let $\boldsymbol{K}'$ be the result of removing from $\boldsymbol{K}$ all countable models lacking uncountable $\prec_{\boldsymbol{k}}$-extensions in $\boldsymbol{K}$. Then $\boldsymbol{k}' = (\boldsymbol{K}', \prec_{\boldsymbol{k}})$ is an AEC. The first part of Lemma 3.4 guarantees closure under increasing $\prec_{\boldsymbol{k}}$ chains, and the remaining clauses are clear. By Remark 4.2 $\boldsymbol{k}'$ is still $\aleph_0$-presented, with the same real parameter as $\boldsymbol{k}$. The first part of Lemma 3.4 implies that $\boldsymbol{k}'$ satisfies amalgamation for countable models. Thus, two points which realize the same Galois type for $\boldsymbol{k}$ realize the same Galois type for $\boldsymbol{k}'$, and $\boldsymbol{k}'$ is almost Galois ω-stable. It suffices then to prove the theorem for $\boldsymbol{K}'$.

Let ϕ_1 be the conjunction of the following statements.

1. Joint embedding holds for $\boldsymbol{K}'_{\aleph_0}$.
2. There do not exist $N \prec_{\boldsymbol{k}} M$ with N countable and M uncountable ($N, M \in \boldsymbol{K}$ and hence in $\boldsymbol{K}'$), such that only countably many Galois types over N are realized in M, and some Galois type over N is not realized in M.

Clause (1) is easily seen to be Π^1_2 in a code for $\boldsymbol{k}$, and Clause (2) is as well, by Fact 4.3. If $\boldsymbol{k}$ is Galois ω-stable, then clause (2) is equivalent to the assertion that every element of $\boldsymbol{K}_{\aleph_1}$ is Galois saturated. We will show that $\boldsymbol{k}$ (and $\boldsymbol{k}'$) are Galois ω-stable in both directions of the proof below.

Let ϕ_2 be the conjunction of the following statements.

3. All uncountable models in $\boldsymbol{K}$ (equivalently, $\boldsymbol{K}'$) satisfy the same Scott sentence in $L_{\omega_1,\omega}$.
4. There exist a countable $N \in \boldsymbol{K}'$ and a countable fragment L_1 of the expanded language where constants are added for each member of N such that for every $M \in \boldsymbol{K}'$ with $N \prec_{\boldsymbol{k}} M$ and $N \prec_{\infty,\omega} M$, M is L_1-atomic.

Clause (3) is equivalent to a Σ^1_2 statement in parameter for $\boldsymbol{k}$, by Lemma 2.22. Clause (4) is easily seen to be Σ^1_2 in a parameter for $\boldsymbol{k}$.

Suppose now that $\boldsymbol{K}$ (and thus $\boldsymbol{K}'$) is $\aleph_1$-categorical. Then clause (1) clearly holds. Corollary 2.8 implies that all uncountable models of any $\aleph_1$-categorical $\aleph_0$-presented AEC satisfy the same Scott sentence in $L_{\omega_1,\omega}$, giving clause (3). Theorem 3.7 implies that there is a Galois saturated model of cardinality $\aleph_1$ in $\boldsymbol{K}$ (which is unique since we have JEP and amalgamation). By the first part of Lemma 3.18, we have clause (4) for all countable $N \in \boldsymbol{K}'$. The second part of Lemma 3.18 implies Galois ω-stability for $\boldsymbol{K}'$. Now the $\aleph_1$-categoricity of $\boldsymbol{K}$ (hence $\boldsymbol{K}'$) and the Galois ω-stability of $\boldsymbol{K}'$ imply clause (2).

For the other direction, since $\boldsymbol{K}$ and $\boldsymbol{K}'$ have the same uncountable models, it suffices to show that $\boldsymbol{K}'$ is $\aleph_1$-categorical. From clause (1) and the first part of Theorem 3.7, we get that $\boldsymbol{K}'$ has a unique small uncountable Galois saturated model. Applying Lemma 3.17 to $\boldsymbol{K}'$, clause (4) implies that $\boldsymbol{K}'$ is Galois ω-stable. Then clause (2) implies that the Galois saturated model is the only model in $\boldsymbol{K}'$ (*a fortiori* $\boldsymbol{K}$) of cardinality $\aleph_1$. $\quad\square_{4.1}$

References

[1] John T. Baldwin. *Categoricity*. Number 51 in University Lecture Notes. American Mathematical Society, Providence, USA, 2009. www.math.uic.edu/˜ jbaldwin.

[2] John T. Baldwin. Amalgamation, absoluteness, and categoricity. In Arai, Feng, Kim, Wu, and Yang, editors, *Proceedings of the 11th Asian Logic Conference, 2009*, pages 22–50. World Sci. Publ., Hackensack, NJ, 2012.

[3] J.T. Baldwin, Sy Friedman, M. Koerwien, and C. Laskowski. Three red herrings around vaught's conjecture. preprint, 2013.

[4] J.T. Baldwin, T. Hyttinen, and M. Kesälä. Beyond first order logic: From number of structures to structure of numbers part II. *Bulletin of Iranian Math. Soc.*, 2011. Available online at `http://www.iranjournals.ir/ims/bulletin/` in press.

[5] J.T. Baldwin and Paul Larson. Iterated elementary embeddings and the model theory of infinitary logic. preprint `http://homepages.math.uic.edu/~jbaldwin/pub/galois_typesjan25.pdf`.

[6] W. Boney. Computing the number of types of infinite length. preprint.

[7] A. Enayat. On certain elementary extensions of models of set theory. *Transactions of the American Mathematical Society*, 283:705–715, 1984.

[8] Su Gao. *Invariant Descriptive Set Theory*. CRC Press, 2009.

[9] V. Harnik and M. Makkai. A tree argument in infinitary model theory. *Proceeedings of the American Mathematical Society*, 67:309–314, 1977.

[10] John E. Hutchinson. Elementary extensions of countable models of set theory. *J. Symbolic Logic*, 41:139–145, 1976.

[11] T. Hyttinen and M. Kesälä. Independence in finitary abstract elementary classes. *Ann. Pure Appl. Logic*, 143(1-3):103–138, 2006.

[12] A. Jarden and S. Shelah. Good frames with a weak stability. preprint: Shelah archive Sh875, 201x.

[13] H.J Keisler. *Model theory for Infinitary Logic*. North-Holland, 1971.

[14] H.J Keisler and M. Morley. Elementary extensions of models of set theory. *Israel Journal of Mathematics*, 5:331–348, 1968.

[15] D. W. Kueker. Abstract elementary classes and infinitary logics. *Annals of Pure and Applied Logic*, 156:274–286, 2008.

[16] P.B. Larson. Scott processes. preprint.

[17] D. Marker. Model theory and infinitary logic. Notes from 2013; `http://homepages.math.uic.edu/~marker/math512-F13/`.

[18] D. Marker. Scott ranks of counterexamples to Vaught's conjecture. Notes from 2011; `http://homepages.math.uic.edu/~marker/harrington-vaught.pdf`.

[19] J. Rosenstein. *Linear Orderings*. Academic Press, New York, 1982.

[20] S. Shelah. Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$. *Israel Journal of Mathematics*, 20:127–148, 1975. paper 48.

[21] S. Shelah. *Classification Theory and the Number of Nonisomorphic Models*. North-Holland, 1978.

[22] S. Shelah. Classification theory for nonelementary classes. I. the number of uncountable models of $\psi \in L_{\omega_1\omega}$ part A. *Israel Journal of Mathematics*, 46:3:212–240, 1983. paper 87a.

[23] S. Shelah. Categoricity of abstract elementary class in two successive cardinals. *Israel Journal of Mathematics*, 126:29–128, 2001. paper 576.

[24] S. Shelah. *Classification Theory for Abstract Elementary Classes*. Studies in Logic. College Publications `www.collegepublications.co.uk`, 2009. Binds together papers 88r, 600, 705, 734 with introduction E53.

[25] S. Shelah. Categoricity of abstract elementary classes: going up inductive step. preprint 600, 200x.

[26] Saharon Shelah. Classification of nonelementary classes II, abstract elementary classes. In J.T. Baldwin, editor, *Classification theory (Chicago, IL, 1985)*, pages 419–497. Springer, Berlin, 1987. paper 88: Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; volume 1292 of *Lecture Notes in Mathematics*.