COMPLETE $\mathcal{L}_{\omega_1,\omega}$-SENTENCES WITH MAXIMAL MODELS IN MULTIPLE CARDINALITIES

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Abstract. In [BKS14] examples of incomplete sentences are given with maximal models in more than one cardinality. The question was raised whether one can find similar examples of complete sentences. In this paper we give examples of complete $\mathcal{L}_{\omega_1,\omega}$-sentences with maximal models in more than one cardinality. From (homogeneous) characterizability of κ we construct sentences with maximal models in κ and in one of $\kappa^+, \kappa^\omega, 2^\kappa$ and more. Indeed, consistently we find sentences with maximal models in uncountably many distinct cardinalities.

We unite ideas from [Mal68, Bau74, Kni77, LS93, Hjo02, BFKL13, BKL14] to find complete sentences of $\mathcal{L}_{\omega_1,\omega}$ with maximal (under the substructure relation $\subset$ in the class of models of ϕ) models in multiple cardinals. This paper is part of a program outlined in [BB14] and expounded further in Section 4; various Hanf numbers described in the literature are either large cardinals or roughly in the neighborhood of $\beth_{\omega_1}$. Is this an accident?

There have been a number of papers finding complete sentences characterizing cardinals (Definition 1.1) beginning with Baumgartner, Malitz and Knight in the 70's, refined by Laskowski and Shelah in the 90's and crowned by Hjorth's characterization of all cardinals below $\aleph_{\omega_1}$ in 2002. The completeness requirement makes these constructions much more complicated than Morley's example showing the Hanf number for existence is $\beth_{\omega_1}$. These results have been refined since. But this is the first paper finding complete sentences with maximal models in two or more cardinals. All models of these sentences have cardinality less than $\beth_{\omega_1}$.

Our arguments combine and extend the techniques of building atomic models by Fraissé constructions using disjoint amalgamation, pioneered by Laskowski-Shelah and Hjorth, with the notion of homogeneous characterization and tools from Baldwin-Koerwien-Laskowski ([BFKL13]), which give a general model theoretic formulation of the key ideas of [Hjo02]. This paper uses specific techniques from [BFKL13, BKL14, Sou14, Sou13] and many proofs are adapted from these sources.

We want to stress the differences in techniques of this paper from [BKS14]. The main idea behind [BKS14] is certain combinatorial properties of bipartite graphs. Here the main construction is a refinement of old ideas, e.g. the characterization of κ^+ by a κ^+-like linear order in Section 1 and the characterization of κ^ω using results from [Sou14] in Section 2, combined with repeated use of sets of absolute indiscernibles. All the examples presented here are *complete* sentences with maximal models in more than one cardinality, which *do not* have arbitrarily large models. In [BKS14] the examples are *incomplete* sentences with maximal models in more than one cardinality, which *do* have arbitrary large models.

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We fix the following notation. For any vocabulary τ and τ-predicate R and τ-structure N we write R^N for the interpretation of R in N. An $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence means an $\mathcal{L}_{\omega_1,\omega}$-sentence in the vocabulary τ.

Structure of the paper:

In Section 1, we explain the merger techniques for combining sentences that homogeneously characterize one cardinal (possibly in terms of another). We adapt the methods of [Sou14] to get a complete sentence with maximal models in κ and κ^+.

In Section 2 we present, for each homogeneously characterizable κ, an $\mathcal{L}_{\omega_1,\omega}$-sentence with maximal models in κ and κ^ω and no larger models. The argument can be generalized to obtain maximal models in κ and $\kappa^{\aleph_\alpha}$, for all countable α.

In Section 3, we give various examples of complete sentences with maximal models in a number of cardinalities, modulo appropriate hypotheses on cardinal arithmetic. For example, Theorem 3.1 asserts that if κ is homogeneously characterizable and μ is minimal with $2^\mu \geq \kappa$ there is an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_κ such that for each λ with $\mu \leq \lambda \leq \kappa$, ϕ_κ has a maximal model in 2^λ and no models larger than 2^κ.

Section 4 both raises some additional questions and places the results in the context of further developments in the program of examining Hanf numbers.

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1. The general construction

In this section, for a cardinal κ that admits a homogeneous characterization (Definition1.1), we prove that there exists a complete sentence ϕ_κ of $\mathcal{L}_{\omega_1,\omega}$ that has maximal models in κ and κ^+ and no larger models. The proof applies the notion of a receptive model from [BFKL13] and merges a sentence homogeneously characterizing κ with a complete sentence encoding uniformly the transfer from characterizing κ to characterizing κ^+ by a κ^+-like linear order. This template is extended from successor to other cardinal functions in later sections.

We require a few preliminary definitions.

Definition 1.1. *Assume $\lambda \leq \kappa$ are infinite cardinals, ϕ is an $\mathcal{L}_{\omega_1,\omega}$-sentence in a vocabulary that contains a unary predicate U, and $\mathcal{M}$ is a countable model of ϕ. We say*

1. *a model $\mathcal{N}$ of ϕ is of type (κ, λ), if $|\mathcal{N}| = \kappa$ and $|U^{\mathcal{N}}| = \lambda$;*
2. *For a countable structure $\mathcal{M}$, $U^{\mathcal{M}}$ is a set of* absolute indiscernibles *for $\mathcal{M}$, if $U^{\mathcal{M}}$ is infinite and every permutation of $U^{\mathcal{M}}$ extends to an automorphism of $\mathcal{M}$.*
3. *ϕ characterizes κ, if ϕ has models of size κ, but no models of size κ^+. If in addition ϕ is a complete sentence, we say that ϕ* completely characterizes *κ.*[1]
4. *The complete sentence ϕ* homogeneously characterizes *(*[Bau74]*) κ, if*
   1. *(a) ϕ characterizes κ;*
   2. *(b) $U^{\mathcal{M}}$ is a set of absolute indiscernibles for the (unique) countable $\mathcal{M}$, and*
   3. *(c) there is a maximal model of ϕ of type (κ, κ).*

Next we define mergers. Before we delve into the details we describe the main idea behind the definition. We start with a "host" sentence θ and a "guest" sentence ψ. The vocabularies of θ and ψ are disjoint and there exists a distinct predicate U in the vocabulary of the host sentence θ.

In [BFKL13, Definition 1.11], the merger of a complete θ and a possibly incomplete ψ is a sentence such that: Each model of the merger consists of an expansion of a model of θ

[1] The reader should be aware that some authors, e.g. [Hjo02], when they write "ϕ characterizes κ", they mean what we define here as "ϕ *completely* characterizes κ".

so that the U-sort of the model becomes a model of ψ. If $\mathcal{M}$ is a model of θ and $\mathcal{N}$ is a model of ψ, then there is an expansion of $\mathcal{M}$ so that the U-sort of $\mathcal{M}$ becomes a model of ψ isomorphic to $\mathcal{N}$. This is called to *merger model* $(\mathcal{M}, \mathcal{N})$. If in addition, $\mathcal{M}$ is a countable model and $U^{\mathcal{M}}$ is a set of absolute indiscernibles, then there is (up to isomorphism) only one way that the merger model $(\mathcal{M}, \mathcal{N})$ can be defined.

Here we take the merger definition one step further than [BFKL13] and we require that the U-sort becomes not a model of ψ, but a definable subset of a model of ψ. In particular, we require that there is a distinct predicate Q in the vocabulary of ψ and in the merger we identify the U-sort of θ and the Q-sort of ψ. So, while in [BFKL13] the domain of the merger model $(\mathcal{M}, \mathcal{N})$ is the same as the domain of $\mathcal{M}$, in this paper the domain of the merger model will be equal to the domain of $\mathcal{M}$ union the domain of $\mathcal{N} \setminus Q^{\mathcal{N}}$.

The extended definition of a merger allows us, when θ is a complete sentence that homogeneously characterizes some cardinal κ and ψ is a sentence with arbitrarily large models, to construct a merger whose models will contain isomorphic copies of models of ψ whose size does not exceed κ.

Definition 1.2. *Fix a vocabulary τ containing a unary predicate U and let θ be a complete $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence. Fix a vocabulary τ' disjoint from τ that contains a unary predicate Q, and let ψ an arbitrary (possibly incomplete) $\mathcal{L}_{\omega_1,\omega}(\tau')$-sentence. Let τ_2 contain the symbols of $\tau \cup \tau'$, adding unary predicates R and S.*

(1) *If U defines an infinite* absolutely indiscernible set *in the countable model of θ, we call the pair (θ, U)* receptive. *We call θ* receptive *if there is a U such that (θ, U) is receptive and in that case we also call the countable model of θ a receptive model.*
(2) *The* merger *$\chi_{\theta,U,\psi,Q}$ of the pair (θ, U) and (ψ, Q) is a τ_2-sentence defined as the conjunction of the following.*
   (a) *$(S(x) \wedge R(x)) \leftrightarrow U(x)$, and that $U(x) \leftrightarrow Q(x)$.*
   (b) *the τ'-predicates hold only of tuples from R and the τ-predicates only of tuples from S,*
   (c) *the τ'-structure with domain R is a model of ψ.*
   (d) *the τ-structure with domain S is a model of θ.*
(3) *If $\mathcal{M} \models \theta$ and $\mathcal{N} \models \psi$, the* merger model *$(\mathcal{M}, \mathcal{N})$ denotes a model of $\chi_{\theta,U,\psi,Q}$ where the elements of $Q^{\mathcal{N}}$ have been identified with the elements of $U^{\mathcal{M}}$, which is the intersection of M and N.*
   $\mathcal{M}$ will be called the host *model and $\mathcal{N}$ the* guest model.
(4) *If in all models $\mathcal{N}$ of ψ, $Q^{\mathcal{N}}$ is the domain of $\mathcal{N}$, then we will drop Q and write $\chi_{\theta,U,\psi}$.*

Note that if ϕ is a complete sentence that homogeneously characterize some κ, then the countable model of ϕ is receptive. Fact 1.3 extends the argument for Theorem 1.10 in [BFKL13] to reflect our more general notion of merger.

Fact 1.3. *Let (θ, U) be receptive and ψ a sentence of $\mathcal{L}_{\omega_1,\omega}$. Then the merger $\chi_{\theta,U,\psi,Q}$ is a complete sentence if and only if ψ is complete.*

Remark 1.4. The proof of Fact 1.3 (Theorem 1.10 in [BFKL13]) is a bit quick. The completeness also depends on absolute indiscernability. Let N and N' be countable models of $\chi_{\theta,U,\psi,Q}$. Let α be a bijection between N and N' which is a τ_1-isomorphism of $S^N \restriction \tau_1$ onto $S^{N'} \restriction \tau_1$. Push-through the τ'-structure on R^N to $R^{N'}$ to give a structure N'' such that for $\mathbf{s} \in R^N$, $P \in \tau'$, $N'' \models P(\alpha(\mathbf{s}))$ if and only if $N \models P(\mathbf{s})$; so $R^{N''} \restriction \tau' \models \psi$.

Let γ be a permutation of $R^{N''} = R^{N'}$ that is an isomorphism from the τ'-structure imposed on $R^{N'}$ by α to $R^{N'} \restriction \tau'$ (and so fixes U setwise). Now by absolute indiscernability, extend $\gamma \restriction U$ to an automorphism of N'. Then $\gamma \circ \alpha$ is a τ_2-isomorphism between N and N' as required.

Using this result we show if κ is homogeneously characterizable, we can construct a complete sentence of $\mathcal{L}_{\omega_1,\omega}$ that has maximal models in κ and κ^+ and no larger models. Before we proceed with the proof we introduce the tool by which we turn homogeneously characterizable cardinals into pairs of maximal models.

Theorem 1.5. *Let the complete sentence θ homogeneously characterize κ. Then there exists a complete $\mathcal{L}_{\omega_1,\omega}$-sentence $\chi = \chi_\theta$ in a vocabulary with a new unary predicate symbol B, such that (χ, B) is receptive, χ homogeneously characterizes κ and χ has maximal models of type $(|\mathcal{M}|, |B^{\mathcal{M}}|) = (\kappa, \lambda)$, for all $\lambda \leq \kappa$.*

Proof. Fix a receptive pair (θ, U) such that θ homogeneously characterizes κ. Define a new vocabulary $\tau = \{A, B, p\}$ where A, B are unary predicates and p is a binary predicate. Let ϕ_0 be the conjunction of: (a) A, B partition the universe and (b) p is a total function from A onto B such that each $p^{-1}(x)$ is infinite. By Theorem 1.10 of [BFKL13] there is a complete sentence ϕ that implies ϕ_0 and in the countable model of ϕ, B is a set of absolute indiscernibles.

Now merge θ and ϕ by identifying U and A. The merger $\chi = \chi_{\theta,U,\phi,A}$ is a complete sentence which does not have any models of size κ^+. Let $\mathcal{M}$ be a maximal model of θ with $U^{\mathcal{M}}$ of size κ, and $\mathcal{N}$ a model of ϕ of type (κ, λ), for some $\lambda \leq \kappa$. Then the merger model $(\mathcal{M}, \mathcal{N})$ is a maximal model of χ with $|(\mathcal{M}, \mathcal{N})| = \kappa$ and $|B^{(\mathcal{M},\mathcal{N})}| = \lambda$, which proves the result. $\square$

A word of caution: In the countable model of θ, the predicate U defines a set of absolute indiscernibles in the countable model, and the same is true for the countable model of ϕ and B. So, after the first paragraph we have two models and two sets of absolute indiscernibles. In the merger $\chi_{\theta,U,\phi,A}$, the absolute indiscernibles of the host model (model of θ) fix the size of A from the guest model (model of ϕ) and bound the cardinality of the predicate B from the guest model that defines a set of absolute indiscernibles in the merger model.

The construction in the next theorem extends by the use of Theorem 1.5 the 50 year old argument that if κ is characterized then so is κ^+ to obtain maximal models in distinct cardinalities.

Theorem 1.6. *Suppose θ is a complete sentence of $\mathcal{L}_{\omega_1,\omega}$ that homogeneously characterizes κ. Then there is a complete sentence $\psi = \psi_\theta$ of $\mathcal{L}_{\omega_1,\omega}$ such that ψ characterizes κ^+ and has maximal models in κ and κ^+.*

Proof. We first replace the sentence θ that homogeneously characterizes κ by the $\tau = \tau_{\chi_\theta}$-sentence χ_θ from Theorem 1.5 that homogeneously characterizes κ, has a set of absolute indiscernibles B, and has maximal models of type (κ, λ) for each $\lambda \leq \kappa$.

Before we define the sentence ψ we give the idea behind it. We start with a linear order $(Q_2, <)$. If we ensure that $(Q_2, <)$ is κ^+-like, i.e. every initial segment has size no more than κ, then the size of Q_2 will be no more than κ^+. Towards this end we assign to each $a \in Q_2$ a model M_a of χ_θ so that B^{M_a} equals the inital segment $\{y \in Q_2 | y < a\}$. The different models M_a intersect only on their B-sorts (which are all initial segments of Q_2). Since χ_θ characterizes κ, each M_a has size no more than κ and therefore, the same is true for every initial segment of Q_2. Using (heavily) absolute indiscernability of B in the countable model, we prove that the resulting sentence is complete.

Formally now, let $\psi = \psi_\theta$ be the conjunction of the following sentences, in a vocabulary τ' that contains unary predicates Q_1, Q_2, binary predicates $<, P$, and for each k-ary $R \in \tau$ a $k+1$-ary predicate $\hat{R}$ in τ'. The axioms assert:

(1) Q_1, Q_2 partition the universe.
(2) $(Q_2, <)$ is a dense linear order without endpoints.
(3) P is the graph of a function from Q_1 to Q_2.

(4) For every predicate $R(\vec{x})$ in τ, if $\hat{R}(a,\vec{x})$ in τ' holds, then $a \in Q_2$ and all members of $\vec{x}$ belong to $D_a = P^{-1}(a) \cup \{y \in Q_2 | y < a\}$.
(5) For every a in Q_2, define a τ-structure M_a with domain D_a, such that for each $R \in \tau \setminus \{B\}$, $R^{M_a} = \hat{R}(a,\cdot)$ and $B^{M_a} = \hat{B}(a,\cdot) = \{y \in Q_2 | y < a\}$. Then M_a is a model of χ_θ.

Note that for any a and $\hat{R}$, $\hat{R}(a,\vec{c})$ holds for a vector $\vec{c}$ of distinct elements of $\{y : y < a\}$ if and only if it holds for all such tuples (by the absolute indiscernability of B in models of χ_θ).

We prove any two countable models, M, N of ψ are isomorphic. Fix an isomorphism α from $(Q_2^M, <^M)$ onto $(Q_2^N, <^N)$. As in Remark 1.4, for every $a \in Q_2$ we now extend the $\alpha \restriction \{y : y <^M a\}$ to a family of τ'-isomorphisms α_a between $M \restriction D_a^M$ and $N \restriction D_{\alpha(a)}^N$. By the categoricity of χ_θ, there exists a τ-isomorphism ρ between $M \restriction D_a^M$ and $N \restriction D_{\alpha(a)}^N$ (and ρ induces a τ'-isomorphism). But we don't know *a priori* that $\rho \restriction \{y : y <^M a\} = \alpha \restriction \{y : y <^M a\}$. Let γ be a permutation of $\{y : y <^N \alpha(a)\}$ that is an order isomorphism between the order given by ρ and the one imposed by α. Now extend γ by absolute indiscernability to an automorphism of $D_{\alpha(a)}^N$. Then $\alpha_a = \gamma \circ \rho$ is a τ'-isomorphism between D_a^M and $D_{\alpha(a)}^N$ that extends $\alpha \restriction \{y : y <^M a\}$. For $b < a$, α_a and α_b agree on their common domain, since their domains intersect only on Q_2.

Now we claim that $\bigcup_{a \in U^M} \alpha_a$ is an isomorphism from M to N. It is well-defined since we noted that any α_a and α_b agree on their common domain which is a subset of Q_2 and the union maps all of M to all of N. The relations $Q_1, Q_2, <, P$ are clearly preserved. Finally, this is a τ'-isomorphism because each atomic τ'-formula $\hat{R}(\cdot,\vec{c})$ holds on the domain of some α_a.

Moreover, note that if M is a model of ψ so that all the D_a^M are *maximal* (κ,λ)-models of χ_θ, then $(Q_2^M, <^M)$ is λ^+-like. So $|M| \leq \max(\kappa, \lambda^+)$ and there is a model in which that maximum is attained. Now when $\lambda = \kappa$ there is a maximal τ'-model M of ψ with size κ^+ and when $\lambda < \kappa$, M is a maximal model of size κ; in both cases, Q_2^M has size λ^+. $\square_{1.6}$

Note that Theorem 1.6 is a trivial corollary of Theorem 1.5 if the answer to the following question is positive. But after considerable effort trying to modify the construction of [Kni77], the question seems to be harder than Theorem 1.6.

Open Question 1.7. *Is there is a complete sentence of $\mathcal{L}_{\omega_1,\omega}$ that has a (κ^+,κ)-model in every cardinality? More strongly, is there such a first order $\aleph_0$-categorical theory?*

Particular examples of homogeneously characterizable cardinals are given by [Bau74] (where the notion is employed but not named), [Hjo02], [Sou12], [Sou13], [Sou14], [BKL14]. We use these results to construct maximal models in various pairs of cardinals.

Fact 1.8 (Theorem 4.31, [Sou13])**.** *If $\aleph_\alpha$ is a completely characterizable cardinal, then $2^{\aleph_{\alpha+\beta}}$ is homogeneously characterizable, for all $0 < \beta < \omega_1$.*

Fact 1.9. *If κ is homogeneously characterizable, then so is each*[2] *of the following:*

(1) 2^κ *;*
(2) κ^ω*;*
(3) $\kappa^{\aleph_\alpha}$*, for all countable ordinals α.*

Finally a result of slightly different character; we note a direct proof of a sentence satisfying the conclusion of Theorem 1.5 when $\kappa = \aleph_n$ for some $n < \omega$.

Fact 1.10 ([BKL14])**.** *For each $n \in \omega$, there is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that homogeneously characterizes $\aleph_n$ and has a maximal model of type $(\aleph_n, \aleph_k)$, for each $k \leq n$.*

[2] 1) [Bau74]; see also Theorem 3.4 of [Sou13]; 2) Theorem 3.6, [Sou14]; 3) Corollary 5.6, [Sou12].

Thus without appealing to Theorem 1.5, the argument of Theorem 1.6, yields, for each $n<\omega$, the existence of complete sentences ϕ_n with maximal models in $\aleph_n$ and $\aleph_{n+1}$.

2. Maximal models in κ and κ^ω

Working similarly to Section 1 we construct a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that admits maximal models in κ and κ^ω, and has no larger models. But we must define a sentence that transfers from characterizing κ to characterizing κ^ω rather than to κ^+.

Although proved earlier ([Sou14]), the following result can be viewed as an extension of the argument for Theorem 1.6. We first have to replace well-known fact that $\mathrm{Th}(Q,<)$ is first order $\aleph_0$-categorical by a proof that the tree $\lambda^{<\omega}$ along with a set of dense paths can be axiomatized in $\mathcal{L}_{\omega_1,\omega}$. Then we extend the trick illustrated in Theorem 1.6 to bound the number of successors of each node in the tree by λ and thus the number of paths by λ^ω. The detailed axiomatization of a structure with these properties, but in a different vocabulary, by a complete sentence of $\mathcal{L}_{\omega_1,\omega}$ and the proof that it completely characterizes λ^ω appears in [Sou14]. The extension to show λ^ω is *homogeneously* characterized requires the further analysis of Hjorth construction in the same paper.

Theorem 2.1. *Let ϕ be a complete $\mathcal{L}_{\omega_1,\omega}(\tau)$-sentence with a set of absolute indiscernibles U that homogeneously characterizes κ. Then there exists a vocabulary $\tau_2\supset\tau$ and a complete $\mathcal{L}_{\omega_1,\omega}(\tau_2)$-sentence ϕ^* that characterizes*[3] *κ^ω.*

Moreover, let μ be the least infinite cardinal such that $\kappa\leq\mu^\omega$. If $\mu>\aleph_0$, then ϕ^* has maximal models in κ and κ^ω, and no models larger than κ^ω. If $\mu=\aleph_0$, ϕ^* has maximal models only in $2^{\aleph_0}$.

Proof. Fix a vocabulary τ_1 with unary predicates T, P, L_n for finite n, binary predicate $\trianglelefteq$, and constant 0 (none of which are in τ). We first show the τ_1-structure with universe $M=\omega^{<\omega}\cup\{f\in\omega^\omega\colon f \text{ is eventually constant}\}$ with the following relations has a Scott sentence. The sentence ϕ_1 in $\mathcal{L}_{\omega_1,\omega}(\tau_1)$ describes the following structure on M: T (tree) and P (paths) partition the universe; T denotes $\omega^{<\omega}$ and P denotes the eventually constant sequences. $(M,\trianglelefteq)$ is a tree of height $\omega+1$ ($\trianglelefteq$ is a partial order, with initial element 0, such that the set of predecessors of any element v of M is linearly ordered and includes 0). An element has finitely many predecessors if $v\in T$, while P contains the elements of infinite height. But $v\in P$ implies every $u\trianglelefteq d$ has finite height. That is, $T^M=\bigcup_{n<\omega}L_n$, where L_n picks out the elements of 'height' n. One easily defines an 'immediate extension' predicate $E(u,v)$ on M^2 (when $v\notin P$), which holds just if $u\trianglelefteq v$ and $L_n(u)\leftrightarrow L_{n+1}(v)$. Note that for any $v\in M$, there is a unique definable restriction $v\restriction n$ (for any n not greater than the height of v).

Include in ϕ_1 the crucial axioms for τ_1-categoricity:

(1) Each v with finite height has infinitely many immediate extensions.
(2) Each v with finite height has infinitely many extensions in P.

We first prove ϕ_1 is a Scott sentence for the τ_1-structure $(M,\trianglelefteq,T,P,(L_n)_{n\in\omega},0)$. We construct a back-and-forth system between arbitrary models M and N of ϕ_1. Suppose A and B are finite subsets of M and N respectively, and $\alpha\colon A\approx B$. Take any $c\in M\setminus A$.

If $c\trianglelefteq a$ for some $a\in A$, the extension is easy. If not, there exists a unique $a_c\in A$, maximal with $a_c\trianglelefteq c$ and apply axiom 1 or 2 depending whether $c\in T$ or P.

This completes the first step in the argument. Without loss of generality we may replace ϕ by the χ_ϕ from Theorem 1.5 that has *maximal* (κ,λ) models for each $\lambda\leq\kappa$.

[3] ϕ^* does *not* homogeneously characterize κ^ω.

Now we use a slightly more complicated version of the strategy for Theorem 1.6. Form τ_2 by adding a binary symbol $D(\cdot,\cdot)$ to τ_1 and an $n+1$-ary predicate $Q(x,\cdot)$ for each n-ary τ-predicate $Q(\cdot)$.

Let ϕ^* be the conjunction of ϕ_1 with the assertions that for $u \neq v \in T$ the sets $D(u,\cdot)$ and $D(v,\cdot)$ are disjoint and they are also disjoint from T and P.

Require further that for each $u \in V$, the set $D(u,\cdot)$ (under the relations $Q(u,\cdot)$) is a model of ϕ and that the set $R(u,\cdot)$ of the immediate successors of u is also the set $U(u,\cdot)$ of absolute indiscernibles of the model $D(u,\cdot)$ of ϕ. Since ϕ homogeneously characterizes κ, if $N \models \phi^*$, $|R^N(u,\cdot)| \leq \kappa$.

To see that any countable models of M, N of ϕ^* are isomorphic, note first that we already showed their τ_1-reducts are isomorphic. The extension to a τ_2-isomorphism uses the absolute indiscernibility of $\{u : u \trianglelefteq v\}$ in $D(v,\cdot)$ as in Theorem 1.5.

If for every $u \in V$, the set $D(u,\cdot)$ is a maximal model of ϕ^* of type (κ,λ), then the resulting tree is λ-splitting and there is an associated maximal model of ϕ^* of size $\max\{\kappa,\lambda^\omega\}$.

Take μ to be the least infinite cardinal such that $\kappa \leq \mu^\omega$. Thus, ϕ^* has a maximal model of size μ^ω. Moreover, for any λ with $\mu \leq \lambda \leq \kappa$ by cardinal arithmetic $\mu^\omega \leq \lambda^\omega \leq \kappa^\omega \leq (\mu^\omega)^\omega = \mu^\omega$. Also for any $\lambda < \mu$, $\lambda^\omega < \kappa$ and ϕ^* has a maximal model of size $\max\{\kappa,\lambda^\omega\} = \kappa$. Note that the model is maximal if each $D(\cdot,\cdot)$ is a maximal (κ,λ)-model and *each* path through resulting tree on $\lambda^{<\omega}$ is realized.

For the last claim, if $\mu = \aleph_0$, then the only possible trees are on $\aleph_0^{<\omega}$ and they must be $\aleph_0$-splitting. So there is a maximal model of ϕ^* of size $\max\{\kappa,\aleph_0^\omega\} = 2^{\aleph_0}$, and every model of size less than $2^{\aleph_0}$ is not maximal. □

Replacing the construction that characterizes κ^ω from [Sou14] with the construction that characterized $\kappa^{\aleph_\alpha}$, $\alpha < \omega_1$, from [Sou12, Corollary 5.6] (cf. Fact 1.9(3)) one can prove the following theorem.

Theorem 2.2. *Assume $\alpha < \omega_1$, $2^{\aleph_\alpha} < \kappa < \kappa^{\aleph_\alpha}$ and there is a complete sentence that homogeneously characterizes κ. Then there is a complete sentence that has maximal models in κ and $\kappa^{\aleph_\alpha}$, and no models larger than $\kappa^{\aleph_\alpha}$.*

An easy application of Shoenfield's absoluteness theorem proves that for a countable vocabulary and for a sentence $\phi \in \mathcal{L}_{\omega_1,\omega}$ the existence of a countable maximal model is absolute. And the existence of a model in $\aleph_1$ is also absolute (For instance, apply Keisler's completeness theorem for $\mathcal{L}_{\omega_1,\omega}(Q)$.). But, as is evident from considering models showing the independence of the continuum hypothesis, absoluteness fails for the existence of a maximal model of size $\aleph_1$. The example of Theorem 2.1 provides more graphic illustrations of the non-absoluteness of questions around maximal models:

Corollary 2.3. *Neither the existence of a maximal model in cardinality $\kappa \geq \aleph_1$ nor the number of cardinals in which a complete sentence has a maximal model is absolute.*

Proof. For example, let $\kappa = \aleph_{\omega+1}$. Fix a model V_1 of set theory in which $\kappa = 2^{\aleph_0} = \aleph_{\omega+1}$. In this model the ϕ^* from Theorem 2.1 has maximal models only in κ.

But if in V_2, $2^{\aleph_n} < \aleph_\omega$ for all $n < \omega$ and $\aleph_\omega^{\aleph_0} = \aleph_{\omega+2}$, the μ from Theorem 2.1 is $\aleph_\omega$ and there are maximal models in both $\kappa = \aleph_{\omega+1}$ and $\kappa^\omega = \aleph_{\omega+2}$. □

3. Consistency of Maximal models in many cardinalities

In this section we construct a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that consistently admits maximal models in many cardinalities. We first give an easy argument to find maximal models in κ and 2^κ when κ is homogeneously characterized.

In [Bau74], Baumgartner used independent families of sets to prove that if κ is homogeneously characterizable, then the same is true for 2^κ. A similar result is Theorem 4.29 of [Sou13] (cf. Fact 1.8) where the assumption of homogeneously characterizability of κ is relaxed to a κ being characterized by a linear order. Given the machinery of homogeneously characterizable cardinals and mergers, our transfer theorem 3.1 has a rather elementary proof. In the process, we give a new proof that if κ is homogeneously characterized by a complete sentence, then so is 2^κ. This resulting sentence has maximal models in several cardinals.

Theorem 3.1. *Suppose that ϕ is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence that homogeneously characterizes κ with absolute indiscernibles in the predicate P and ϕ has no maximal models below κ. Then there is a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_κ that characterizes 2^κ and for every $\lambda \leq \kappa$, this sentence ϕ_κ has a maximal model of size* $\max\{\kappa, 2^\lambda\}$*, and every maximal model of ϕ_κ has one of these cardinalities.*

Thus, if μ is the least cardinal such that $2^\mu \geq \kappa$, ϕ_κ has maximal models in exactly *the cardinalities κ and 2^λ, for each λ with $\mu \leq \lambda \leq \kappa$.*

Proof. By Theorem 1.5, we can assume ϕ has maximal models of type (κ, λ) with absolute indiscernibles B, for each $\lambda \leq \kappa$.

Let T be the $\aleph_0$-categorical first order theory witnessing the independence property. That is, say U and V are disjoint infinite sets and $E \subseteq U \times V$ is extensional so that $E(\cdot, v)$ defines a family of subsets X_v of U. Requiring that every finite Boolean combination of the X_v is non-empty (and dually every finite Boolean combination of the $Y_u = \{v \colon E(u,v)\}$ is non-empty) gives an $\aleph_0$-categorical theory such that for every model M, $|V^M| \leq 2^{|U|}$. The $\aleph_0$-categoricity is a simple back-and-forth.

Merge $\sigma = \bigwedge T$ with the complete sentence ϕ from Theorem 1.5 identifying U with B. Let $\phi_\kappa = \chi_{\phi,B,\sigma,U}$. By Fact 1.3, ϕ_κ is a complete sentence.

Now M, a maximal model of ϕ with type (κ, λ), yields a maximal model of ϕ_κ with cardinality $\max\{\kappa, 2^\lambda\}$. There can be no other maximal models as if (M, N) is a maximal model of the merger ϕ_κ then M is maximal and if $|U^M| = |B^N| = \lambda$, then $|V^N|$ must be 2^λ. $\square$

Exactly what this says about the cardinality of maximal models depends on the cardinal arithmetic. The difficulty is that it is impossible to specify in ZFC the equalities/inequalities among the 2^λ's. In ZFC we cannot specify them as $\aleph$'s. Using Easton's theorem, we describe below some ways to arrange the values the powerset function assumes on the interval $[\mu, \kappa]$ to illustrate the effect of the next theorem.

Corollary 3.2. *Let $\kappa = 2^{\aleph_1}$ and ϕ_κ be from Theorem 3.1. If $\Gamma = (\alpha_i | i < \alpha_0)$ is an increasing sequence of ordinals and $\mathrm{cf}(\aleph_{\alpha_i}) > \aleph_{i+1}$, then there is a $V^\Gamma \models ZFC$ such that ϕ_κ has maximal models in exactly the cardinalities $(\aleph_{\alpha_i} | i < \alpha_0)$ along with the values of the $2^{\aleph_\gamma}$ where $\gamma < \alpha_0$ and γ is a limit ordinal.*

Proof. We apply the 'thus' clause of Theorem 3.1 with $\mu = \aleph_1$ and $\kappa = 2^{\aleph_1}$. We need the fact that $\aleph_1$ is homogeneously characterizable, but this follows from Fact 1.10, and clearly a complete sentence characterizing $\aleph_1$ can have no maximal countable model. By Theorem 3.1 the sentence ϕ_κ has maximal models in all cardinalities 2^λ, for λ with $\aleph_1 \leq \lambda \leq 2^{\aleph_1}$. Notice that ϕ_κ depends only on κ and not on the choice of Γ.

Next, we create a model V^Γ of ZFC where the set $\{2^\lambda | \aleph_1 \leq \lambda \leq 2^{\aleph_1};\ \lambda \text{ a successor}\}$ equals the set $\{\aleph_{\alpha_i} | \alpha_i \in \Gamma\}$, which proves the statement. We describe the cardinal arithmetic requirements on V^Γ carefully. Using Easton forcing, we ensure first that $2^{\aleph_1}$ equals $\aleph_{\alpha_0}$. So, $\{2^\lambda | \aleph_1 \leq \lambda \leq 2^{\aleph_1};\ \lambda \text{ a successor}\} = \{2^{\aleph_{i+1}} | i < \alpha_0\}$. Then using the assumption on $\mathrm{cf}(\aleph_{\alpha_i})$, Easton guarantees as well that in V^Γ, $2^{\aleph_{i+1}} = \aleph_{\alpha_i}$, for all $i < \alpha_0$. So Γ indexes a part of the range of the function giving the cardinality of power sets. $\square$

We know a bit more.

1. If $\alpha_0 > \omega$, the complete sentence given by Corollary 3.2 will have maximal models in other cardinalities than $(\aleph_{\alpha_i} | i < \alpha_0)$. For instance, for those i where $\lambda = \aleph_i$ is singular, Easton's theorem does not control the $\aleph$-index of 2^λ, although we know there is a maximal model in that cardinality.
2. Despite the fact that the sentence ϕ_κ given by Corollary 3.2 has maximal models in cardinalities that are bounded by $2^{2^{\aleph_1}}$, the same idea can be applied to other characterizable cardinals. However, since characterizable cardinals are bounded by $\beth_{\omega_1}$, the cardinalities where the maximal models occur are also bounded by $\beth_{\omega_1}$.
3. The complete sentences given by Theorem 3.1 and Corollary 3.2 do not have arbitrarily large models.

In Corollary 2.3 we proved that the number of cardinals in which a complete sentence has a maximal model is non-absolute. Using Theorem 3.1, we can even produce an example of a complete sentence that has maximal models in finitely many cardinalities in some ZFC-model V_1, while is has maximal models in uncountably many cardinalities in some other ZFC-model V_2.

4. Conclusion

The existence of maximal models in several cardinalities suggests the following strengthening of earlier questions concerning the number of models in a cardinal that is characterized.

Open Question 4.1. *Is there a complete $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ which has at least one maximal model in an uncountable cardinal κ, but less than 2^κ many models of cardinality κ?*

In particular, a negative answer to Open Question 4.1 implies a negative answer to the following Open Question 4.2, which was asked in [BKL14].

Open Question 4.2 ([BKL14])**.** *Is there a complete $\mathcal{L}_{\omega_1,\omega}$-sentence which characterizes an uncountable cardinal κ and has less than 2^κ many models in cardinality κ?*

All the examples in this paper have maximal models in some cardinalities and using set theory we can identify the maximality cardinals in the $\aleph$-hierarchy. Our examples cannot be used to settle whether the statement "ϕ has a maximal model" is absolute. We noticed already in the comments preceding Corollary 2.3 that existence of a countable maximal model and existence of an uncountable model are absolute notions. So, it is necessary that a proposed counterexample will consistently have a maximal model in an uncountable cardinality. By Lemma 5.8 of [Bal12], the property that an $\mathcal{L}_{\omega_1,\omega}$-sentence has arbitrarily large models is absolute. This further implies that the proposed counterexample will have arbitrarily large models in *all* models of ZFC.

Open Question 4.3. *Given an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ, is the following statement absolute for transitive models of ZFC? "ϕ has a maximal model in an uncountable cardinality".*

More precisely, do there exist two transitive models of ZFC, $V \subset W$, $\phi \in \mathcal{L}^V_{\omega_1,\omega}$, both V and W satisfy that "ϕ has arbitrarily large models", and V, W disagree on the statement "ϕ has a maximal model in an uncountable cardinality"?

Further developments: Stimulated by early versions of this paper, Baldwin and Shelah began the paper 'The Hanf number for extendability and related phenomena' [BS17]. They construct (under mild set theoretic hypotheses) a complete sentence of $\mathcal{L}_{\omega_1,\omega}$ with maximal models arbitrarily high below the first measurable. Note that every model above the first measurable has a proper $\mathcal{L}_{\omega_1,\omega}$-elementary extension. The hypotheses beyond ZFC are eliminated in [BS18]. In contrast to this result the methods discussed in this paper seem to be limited to counterexamples below $\beth_{\omega_1}$. Can one find a sentence ϕ with maximal models bounded somewhere between these bounds? If not, can one explain why there is such an immense gap? Under ZFC + 'there exists a measurable cardinal', no complete sentence of $\mathcal{L}_{\omega_1,\omega}$ has arbitrarily large maximal models. Under ZFC + 'no measurable cardinals', our only example with a maximal model of cardinality beyond $\beth_{\omega_1}$ has arbitrarily large maximal models. Is it always true that under ZFC + 'there are no measurable cardinals'Def:HomChar, if there is a maximal model of cardinality at least $\beth_{\omega_1}$, then there are arbitrarily large maximal models. Does this make the Hanf number for the existence of a maximal model (with no measurable) $\beth_{\omega_1}$ or can more counterexamples be constructed?

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