Strongly Minimal Steiner Systems Helsinki

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Zilber's hope

> *The initial hope of this author in [Zil84] that any uncountably categorical structure comes from a classical context (the trichotomy conjecture), was based on the belief that logically perfect structures could not be overlooked in the natural progression of mathematics.*
> *[PS98]. ([Zil05])*

We show some exotics are a bit more classical than expected.
What are classical structures?

Overview

1. Strongly Minimal Theories
2. Classifying strongly minimal sets and their geometries
3. Coordinatization by varieties of algebras
4. Interactions with Combinatorics

Thanks to Joel Berman, Gianluca Paolini, and Omer Mermelstein.

Latin Squares

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

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Stein 4-quasigroup

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Stein 4-quasigroup

A Latin square is an $n \times n$ square matrix whose entries consist of n symbols such that each symbol appears exactly once in each row and each column.

By definition, this is the multiplication table of a quasigroup.

Definitions

A Steiner system with parameters t, k, n written $S(t, k, n)$ is an n-element set S together with a set of k-element subsets of S (called blocks) with the property that each t-element subset of S is contained in exactly one block.

We always take $t = 2$.

Steiner systems are 'coordinatized' by Latin squares.

Some History

Steiner triple systems were defined for the first time by W.S.B. Woolhouse in 1844 in the Lady's and Gentlemen's Diary and he posed the question.

For which v's does an $S(2, k, v)$ exist?

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Keevash 2014: for any t and sufficiently large v, if k is not obviously blocked, there are (t, k, v)-Steiner systems.

Strongly Minimal Theories

STRONGLY MINIMAL

Definition

T is strongly minimal if every definable set is finite or cofinite.

e.g. acf, vector spaces, successor

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Definition

a is in the algebraic closure of B ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$:
$\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.

$\aleph_1$-categorical theories

Morley  **Lachlan**  **Zilber** 

Theorem

A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

These two conditions assign a unique dimension which determines each model of T.

Strongly minimal sets are the building blocks of structures whose first order theories are categorical in uncountable power.

$\aleph_1$-categorical theories

Definition

A model M of a complete theory T is prime over a subset X if every morphism from X into a model N of T extends to a morphism of M into N.

Theorem (Baldwin-Lachlan)

If T is categorical in some uncountable power, there is a definable strongly minimal set D such that every model M of T is prime over $D(M)$.

Thus, the dimension of $D(M)$ determines the isomorphism type of M.

Combinatorial Geometry: Matroids

The abstract theory of dimension: vector spaces/fields etc.

Definition

A closure system is a set G together with a dependence relation

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : \; X' \subseteq_{fin} X\}$
A2. $X \subseteq cl(X)$
A3. $cl(cl(X)) = cl(X)$

(G, cl) is pregeometry if in addition:
A4. If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.

If $\mathrm{cl}(x) = x$ the structure is called a geometry.

Classifying strongly minimal sets and their geometries

The trichotomy

Zilber Conjecture

The acl-geometry of every model of a strongly minimal first order theory is

1. disintegrated (lattice of subspaces distributive)
2. vector space-like (lattice of subspaces modular)
3. ‘bi-interpretable’ with an algebraically closed field (non-locally modular)

Hrushovski’s example showed there are non-locally modular which are far from being fields; the examples don’t even admit a group structure.

Aside: Strongly minimal sets and analysis

Axiomatic analysis studies behavior of fields of functions with operators but *without* explicit attention in the formalism to continuity but rather to the algebraic properties of the functions. The function symbols of the vocabulary act on the functions being studied; the functions are elements of the domain of the model.

Differential Algebra

The axioms for *differentially closed fields* are a first order sentences in the vocabulary $(+, \times, 0, 1, \partial)$ (where ∂f is interpreted as the derivative The first order formulation is particularly appropriate because many of the fields involved are non-Archimedean.

Differentially closed fields II

Hrushovski and Itai lay out as an application of ‘Shelah’s philosophy’ the following model theoretic fact (based on Buechler’s Dichotomy) is fundamental to the study of differential fields:

‘an *algebraically* closed differential field K is *differentially* closed if every strongly minimal formula over K has a solution in K’.

Even more, by the general theory of superstability, their study reduces to the study of strongly minimal sets and definable simple FMR groups that are associated with strongly minimal sets.

Consider the theory of differentially closed fields with constant field the complex numbers $\mathcal{C}$.

Painlevè equations

In 1900 Painlevè began the study of nonlinear second order ordinary differential equations (ODE) satisfying the Painlevè property (no movable singularities). In general such an equation has the form

$$y'' = f(y, y')$$

with f a rational function (i.e. in $\mathbb{C}(t_1, t_2)$).

He classified such equations into 50 canonical forms and showed that 44 of these were solvable in terms of 'previously known' functions. Here is a canonical form for the third of the remaining classes; the Greek letters are the constant coefficients; t is the independent variable satisfying $t' = 1$ and the goal is to solve for y.

$$P_{III}(\alpha, \beta, \gamma, \delta): \qquad \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

Problem 1

Show that a generic equation (i.e. the constant coefficients are algebraically independent) of each of the six forms is irreducible.

For this, one must take on the logicians task: 'What does *not reducible* mean'?

By reducible Painlevè meant, solvable from 'known functions'.

The Japanese school clarified 'solvable' to mean, roughly speaking: generated from solutions to order one ordinary differential equations (ODE) and algebraic functions through a fixed family of constructions (integration, exponentiation, etc.).

In the formal setting, this is equivalent to showing that:

If an order two differential equation is strongly minimal; then there can be no *classical solutions*.

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This problem was solved (without the formalization) in each of the six cases by the Japanese school (led by Umemura) in the late 1980's.

Problem II

Conjecture

If there are *n* algebraically independent solutions of a generic strongly minimal Painlevè equation then that set along with its first derivatives is also algebraically independent.

The model theoretic step is to invoke the Zilber trichotomy which holds for differentially closed fields.
To reduce to a disintegrated strongly minimal set. Pillay and Nagloo show the other alternatives are impossible in this situation and indeed that the strongly minimal set is $\aleph_0$-categorical. Using the geometric triviality (from the Zilber trichotomy) heavily and tools from the Japanese analysts

Theorem: Nagloo-Pillay

The conjecture is true.

Classify non-locally modular geometries of SM sets

Definition: Flat geometries

A geometry given by a dimension function d is flat if the dimension of any set E covered by d-closed sets $E_1, \ldots E_n$ is bounded by applying the inclusion exclusion principle to the E_i.

Fact

If the geometry of a strongly minimal set M is flat.

1. Forking on M is not 2-ample.
2. M does not interpret an infinite group.
3. Thus, the geometry is not locally modular and so not disintegrated.

Classifying Hrushovski Construction

The acl-geometry associated with Hrushovski constructions

Work of Evans, Ferreira, Hasson, Mermelstein suggests that up to arity or more precisely, purity, (and modulo some natural conditions)

any two geometries associated with Hrushovski constructions are locally isomorphic.

Locally isomorphic means that after localizing one or both at a finite set, the geometries are isomorphic.
[EF11, EF12, HM18]

We are concerned not with the acl-geometry but with the Object language geometry.

‘Object Language’ geometries

Strong minimality asserts the ‘rank’ of the universe is one and imposes a combinatorial geometry whose dimension varies with the model. We study here structures which are ‘geometries’ in the object language. E.g.

Projective Planes: [Bal94]

There is an almost strongly minimal (rank 2) projective plane.
An example with the least possible structure in the Lenz-Barlotti class was constructed [Bal95].
In particular, the ternary function of the coordinatizing field cannot be decomposed into an ‘addition’ and a ‘multiplication’.

How is algebraic structure lost?

Algebraic view

1. field
2. integral domain (lose inverses)
3. matrix ring (lose commutativity)
4. alternative ring (weaken associativity)

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geometric view

1. field (Pappian plane)
2. division ring(Desarguesian plane: lose commutativity)
3. nearfield (lose left distributive)
4. quasifield(multiplication is a quasigroup with identity)
5. alternative algebra (Moufang plane: lose full associativity)
6. ternary ring (lose associativity and distributivity and even compatible binary functions, but still have inverse)

Linear Spaces

Definition (2-sorted)

A linear space [BB93] is a collection of points and lines such that
2 points determine a line;
consequently two lines intersect in at most one point.

Definition (2-sorted)

The vocabulary τ contains a single ternary predicate R, interpreted as collinearity.
$\boldsymbol{K}_0^*$ denotes the collection of finite 3-hypergraphs that are linear systems. $\boldsymbol{K}^*$ includes infinite linear spaces.

1. R is a predicate of sets (hypergraph)
2. Two points determine a line

There are natural generalizations:

1. k-points determine a line.
2. allow a finite number of line lengths

2-sorted vrs 1-sorted

In a two-sorted formulation, i.e. points and lines, clearly no strongly minimal theory has both infinitely many points and infinitely many lines.

Even in 1-sort, there cannot be two lines with infinitely many points.

Note that this does not preclude bi-interpretability between 1-sorted and 2-sorted descriptions. Because, interpretations do not need to preserve Morley rank.
In this case the universe of the two-sorted structure is interpreted as a set of pairs in the 1-sorted structure.

Theorem [BP18]

$\boldsymbol{K}_0^*$ and the class of two-sorted linear spaces are biinterpretable.

Strongly minimal linear spaces I

Fact

Suppose (M, R) is a strongly minimal linear space where all lines have at least 3 points. There can be no infinite lines.

Suppose ℓ is an infinite line. Choose A not on ℓ. For each B_i, B_j on ℓ the lines AB_i and AB_j intersect only in A. But each has a point not on ℓ and not equal to A. Thus ℓ has an infinite definable complement, contradicting strong minimality.

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Corollary

There can be no strongly minimal affine or projective plane, since in such planes the number of lines must equal the number of planes (mod $aleph_0$).

Strongly minimal linear spaces II

An easy compactness argument establishes

The fundamental corollary of strong minimality

If M is strongly minimal, then for every formula $\varphi(x, \overline{y})$, there is an integer $k = k_\phi$ such that for any $\overline{a} \in M$, $(\exists^{>k_\phi} x)\varphi(x, \overline{a})$ implies there are infinitely many solutions of $\varphi(x, \overline{a})$ and thus finitely many solutions of $\neg\varphi(x, \overline{a})$.

Corollary

If (M, R) is a strongly minimal linear system, for some k, all lines have length at most k. So it is a K-Steiner system.
$K = \{3, 4 \ldots k\}$.

Specific Strongly minimal Steiner Systems

Definition

A *Steiner* $(v, 2, k)$*-system* is a linear system with v points such that each line has k points.

Theorem (Baldwin-Paolini)[BP18]

For each $k \geq 3$, there are an uncountable family T_μ of strongly minimal $(\infty, k, 2)$ Steiner-systems.
The theory is 1-ample (not locally modular) and CM-trivial (not 2-ample).

IN ENGLISH

There is no infinite group definable in any T_μ. More strongly, Associativity is forbidden.

Hrushovski construction for linear spaces

$\mathbf{K}_0^*$ denotes the collection of finite linear systems in the vocabulary $\tau = \{R\}$.
A line in M is a maximal R-clique
$L(A)$, the lines based in A, is the collections of lines in (M, R) that contain 2 points from A.

Definition: Paolini's δ

[Pao] For $A \in \mathbf{K}_0^*$, let:

$$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$

$\boldsymbol{K}_0$ is the $A \in \boldsymbol{K}_0^*$ such that $B \subseteq A$ implies $\delta(B) \geq 0$.

Mermelstein [Mer13] has independently investigated Hrushovki functions based on the cardinality of maximal cliques.

Amalgamation and Generic model

Definition

Let $A \cap B = C$ with $A, B, C \in \boldsymbol{K}_0$. We define $D := A \oplus_C B$ as follows:

1. the domain of D is $A \cup B$;
2. a pair of $a \in A - C$ and $b \in B - C$ are on a line ℓ' in D if and only if there is a line $\ell \subseteq D$ based in C such that $a \in \ell$ (in A) and $b \in \ell$ (in B). Thus $\ell' = \ell$ (in D).

Definition

The countable model $M \in \hat{\boldsymbol{K}}_0$ is $(\boldsymbol{K}_0, \leq)$-*generic* if

1. If $A \leq M, A \leq B \in \boldsymbol{K}_0$, then there exists $B' \leq M$ such that $B \cong_A B'$,
2. M is a union of finite closed subsets ($A_i \leq M$).

Theorem: Paolini [Pao]

There is a generic model for $\boldsymbol{K}_0$; it is ω-stable with Morley rank ω.

Primitive Extensions and Good Pairs

Definition

Let $A, B, C \in \mathbf{K}_0$.

1. $A \leq B$ if $A \subseteq B$ and there is no B_0, $A \subsetneq B_0 \subsetneq B$ with $\delta(B_0/A) < 0$.
2. B is a 0-*primitive extension* of A if $A \leq B$ and there is no $A \subsetneq B_0 \subsetneq B$ such that $A \leq B_0 \leq B$ and $\delta(B/A) = 0$.
3. We say that the 0-primitive pair B/A is *good* if for every $A' \subsetneq A$ we have that $\delta(B'/A) > 0$.
4. For any good pair (A, B), $\chi_M(A, B)$ is the number of copies of B over A appearing in M.

α is the isomorphism type of $(\{a, b\}, \{c\})$.

Overview of construction

1. $\boldsymbol{K}_0^*$: all finite linear τ-spaces.
2. $\boldsymbol{K}_0 \subseteq \boldsymbol{K}_0^*$: $\delta(A)$ hereditarily ≥ 0.
3. $\boldsymbol{K}_\mu \subseteq \boldsymbol{K}_0$: μ bounds number of 'good pairs'.
4. $\boldsymbol{K}_{\mu,d} = \mathrm{mod}(T_\mu)$ strongly minimal.

Basic case

$\boldsymbol{\alpha}$ is the isomorphism of the good pair $(\{a, b\}, \{c\}$ with $R(a, b, c)$.

Context

Let $\mathbf{U} = \mathcal{U}$ be the collection of functions μ assigning to every isomorphism type β of a good pair C/B in $\boldsymbol{K}_0$:

1. (i) a natural number $\mu(\beta) = \mu(B, C) \geq \delta(B)$, if $|C - B| \geq 2$;
2. (ii) a number $\mu(\beta) \geq 1$, if $\beta = \boldsymbol{\alpha}$

The length of a line in T_μ is $\mu(\boldsymbol{\alpha}) + 2$.

T_μ is the theory of a strongly minimal Steiner $(\mu(\boldsymbol{\alpha}) + 2)$-system

If $\mu(\boldsymbol{\alpha}) = 1$, T_μ is the theory of a Steiner triple system bi-interpretable with a Steiner quasigroup.

$\boldsymbol{K}_\mu$

Definition

1. For $\mu \in \mathcal{U}$, $\boldsymbol{K}_\mu$ is the collection of $M \in \boldsymbol{K}_0$ such that $\chi_M(A, B) \leq \mu(A, B)$ for every good pair (A, B).
2. X is d-closed in M if $d(a/X) = 0$ implies $a \in X$ (Equivalently, for all finite $Y \subset M - X$, $d(Y/X) > 0$.).
3. Let $\boldsymbol{K}_d^\mu$ consist of those $M \in \boldsymbol{K}_\mu$ such that $M \leq N$ and $N \in \hat{\boldsymbol{K}}_\mu$ implies M is d-closed in N.
   Moreover, if $M \in \boldsymbol{K}_d^\mu$, and $B \leq M$, for any good pair (A, B), $\chi_M(A, B) = \mu(A, B)$.

Main existence theorem

Theorem (Baldwin-Paolini)[BP18]

For any $\mu \in \mathcal{U}$, there is a generic strongly minimal structure $\mathcal{G}_\mu$ with theory T_μ.
If $\mu(\alpha) = k$, all lines in any model of T_μ have cardinality $k + 2$.
Thus each model of T_μ is a Steiner k-system and $\mu(\alpha)$ is a fundamental invariant.

Proof follows Holland's [Hol99] variant of Hrushovski's original argument.
New ingredients: choice of amalgamation, analysis of primitives, treatment of good pairs as invariants (e.g. α).

Pure Steiner systems: $(M, R) \models T_\mu$

Definition

$A \subseteq M \models T$ has *essentially unary definable closure* if $\mathrm{dcl}(A) = \bigcup_{a \in A} \mathrm{dcl}(a)$.

Theorem

If $A \leq M \models T_\mu$, $\mu \in \mathcal{U}$ and $\mu(\alpha) > 1$, then A has, at worst, essentially unary definable closure.
In particular, it does not interpret a quasigroup.

Proof sketch:
Let $A \leq M$ and $c \in \mathrm{dcl}(A) \subset \mathrm{acl}(A)$. Without loss of generality, $c \notin A$.
So $d(c/A) = 0$ and for some good pair (B, C) with $B \subseteq A$, $c \in C$.
If $|B| \geq 2$, by Definition 26, c is not definable over A.
If $|B| = 1$, there may be (depending on μ) a definable unary function.

Coordinatization by varieties of algebras

Coordinatizing Steiner Systems

Definition

A collection of algebras V ”weakly coordinatizes” a class $\mathcal{S}$ of $(2, k)$-Steiner systems if

1. Each algebra in V definably expands to a member of $\mathcal{S}$
2. The universe of each member of $\mathcal{S}$ is the underlying system of some (perhaps many) algebras in V.

Coordinatizing Steiner triple systems

Example

A Steiner quasigroup (squag) is a groupoid (one binary function) which satisfies the equations:

$x \circ x = x, \qquad x \circ y = y \circ x, \qquad x \circ (x \circ y) = y.$

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Steiner triple systems and Steiner quasigroups are biinterpretable.
Proof: For distinct a, b, c:

$$R(a, b, c) \text{ if and only if } a * b = c$$

Theorem

Every strongly minimal Steiner (2,3)-system given by T_μ with $\mu \in \mathcal{U}$ is coordinatized by the theory of a Steiner quasigroup definable in the system.

2 VARIABLE IDENTITIES

Definition

A variety is binary if all its equations are 2 variable identities: [Eva82]

Definition

Let q be a prime power.
Given a (near)field $(F, +, \cdot, -, 0, 1)$ of cardinality q and an element $a \in F$, define a multiplication $*$ of F by $x * y = y + (x - y)a$. An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a block algebra over $(F, *)$

Coordinatizing Steiner Systems

Key fact: weak coordinatization [Ste64, Eva76]

If V is a variety of binary, idempotent algebras and each block of a Steiner system $\mathcal{S}$ admits an algebra from V then so does $\mathcal{S}$.

Consequently

If V is a variety of binary, idempotent algebras such that each 2-generated algebra has cardinality k, each $A \in V$ determines a Steiner k-system.
(The 2-generated subalgebras.)
And each Steiner k-system admits such a coordinatization.

But we showed the coordinatization cannot be defined in the pure Steiner system.

Forcing a prime power

Theorem

If a $(2, q)$ Steiner systems is weakly coordinatized k must be a prime power.

Proof: As, if an algebra A is freely generated by every 2-element subset, it is immediate that its automorphism group is strictly 2-transitive. And as [Ś61] points out an argument of Burnside [Bur97], [Rob82, Theorem 7.3.1] shows this implies that $|A|$ is a prime power.

Are there any strongly minimal quasigroups (block algebras)?

Interpretability

Theorem

For every prime power q there is a strongly minimal Steiner q-system whose theory is interpretable in a strongly minimal block algebra.

Theorem

Let $q = p^n$ and let V be a specified variety of $(2, q)$-block algebras over F_q. Let τ' contain ternary relations R and F. For each $\mu \in \mathcal{U}$, there is a strongly minimal τ'-theory $T_{\mu',V}$ such that the reducts to R are strongly minimal q-Steiner systems and the reducts to F are strongly minimal block algebras in the variety V with each line being a copy of $F_2(V)$, the free V-algebra on two generators.

Interpretability: Details if appropriate

Fix a vocabulary $\hat{\tau}$ with ternary predicates F, R.

Theorem

Fix a variety V of block algebra with $F_2(V) = q$ If $\mu \in \mathcal{U}$, and the lines in T_μ have length $q = p^n$. There is a strongly minimal theory $T_{\mu',V}$ such that if $(A, F, R) \models T_{\mu',V}$ then $A \restriction R$ is a Steiner q-system and $A \restriction F$ is in V.

Proof: Do the construction for structures (A, F, R) in a vocabulary $\hat{\tau}$ with $\delta(A, F, R) = \delta(A, R)$.
Modify $\boldsymbol{K}_0$ to $\hat{\boldsymbol{K}}_0$ by including only structures such that every line has length 2 or q.
Expand each line by interpreting the relation F as the graph of $F_2(V)$.

For each $\hat{\tau}$ isomorphism type of $\mu(A, F, R)$ with reduct (A, R) that represents a good pair, let $\mu'(A, F, R) = \mu(A, R)$.

With this modification, we return to the usual proof.

What do we know about coordinatizing algebra

Fact

Steiner quasigroups are congruence permutable, regular, and uniform. The variety of Steiner quasigroups is not residually small. Finite members are directly decomposable.

We can show that models of the T_μ are not locally finite.

Question

1. Are these $\aleph_1$-categorical block algebas subdirectly irreducible or even simple? Surely they are not free?!
2. How does the variety associated with T_μ depend on μ? Note that we can certainly get different theories $\hat{T}_\mu$ for the same μ because we had to specify the variety of the block algebra.

III. Interactions with Combinatorics

Infinite linear spaces

There is no theory of infinite linear spaces comparable to the enormous amount known about finite linear spaces. This is due to two contrasting factors. First, techniques which are crucial in the finite case (notably counting) are not available. Second, infinite linear spaces are too easy to construct; instead of having to force our configurations to close up, we just continue adding points and lines infinitely often! The result is a proliferation of examples without any set of tools to deal with them.
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1. Find families (namely the models of a complete first order theory) of infinite linear spaces that are similar both combinatorially and model theoretically.
2. Borrow one from one field to the other.

Hrushovski sm-classes

A *Hrushovski sm-class* is determined by a triple $(\boldsymbol{L}^*, \epsilon, \mathbf{U})$.

$\boldsymbol{L}^*$ is a collection of finite structures in a vocabulary σ, not necessarily closed under substructure.

ϵ is a function from members of $\boldsymbol{L}^*$ to natural numbers satisfying the conditions that gives a 'predimension'.

$\boldsymbol{L}_0$ is the subset of L_* consisting of those with hereditarily non-negative ϵ-rank, perhaps restricted by other conditions using ϵ.

$\boldsymbol{L}_\mu$ is the subset of $\boldsymbol{L}_0$ such that:
Each isomorphism type β of (m.s.a., good pair, 0-primitive) models and each $B \in A \in \boldsymbol{L}_\mu$, there are less than $\mu(\beta)$ copies of C over B in A.

If $\boldsymbol{L}_\mu$ satisfies amalgamation there is a strongly minimal theory T_μ and a generic structure $\mathcal{G}_\mu$.

The cycle graph of a Steiner triple system

This is a standard topic in finite combinatorics, extended to infinite system by e.g. Cameron and Webb. [CW12]

Definition

1. Fix any two points a, b of a Steiner triple system $\mathcal{S} = (P, L)$. The cycle graph $G(a, b)$ has vertex set $P - \{a, b, c\}$ where (a, b, c) is a block. There is an edge coloured a (resp., b) joining x to y if and only if axy is a block (resp., bxy is a block).

The cycle graph of a Steiner triple system

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2. a proper initial segment of an (a, b)-cycle is called an (a, b)-chain.
3. It is uniform if the graphs $G(a, b)$ are pair-wise isomorphic.

Corollary

If $(M, R) \models T_\mu$ *is* 2*-transitive* (M, R) *is uniform.*

Getting 2-transitivity of models of T_μ:

Lemma

If for any $B \in \boldsymbol{K}_0$ with $|B| \geq n$, $\delta(B) \geq n$ then, for any μ the number of n-types in $\mathcal{G}_\mu$ is bounded by the number of quantifier free n-types.

Corollary: Hrushovski example 5.2

If every $B \in \boldsymbol{K}_0^-$ (don't assume two point determine a line) satisfies $|B| \geq 3$, $\delta(B) \geq 3$, then every model of T_μ is a 2-transitive Steiner triple system.

2-transitivity of models of T_μ: Consequences

Lemma

For any $\mu \in \mathcal{U}$, if $(M, R) \models T_\mu$, $A \subset \mathcal{G}_\mu$ and $|A| = 2$ implies $A \leq \mathcal{G}_\mu$ then the automorphism group of (M, R) acts 2-transitively on (M, R).

Proof.

Since all pairs (a, b) are isomorphic and each sits strongly in the generic $\mathcal{G}$, the result is immediate for $\mathcal{G}$. But this property extends to all models since if one model of a complete theory has a single 2-type, all models do. And each model of a strongly minimal theory is finitely homogeneous. □

From combinatorics to model theory

Lemma

There are infinitely many mutually non-imbeddible primitives in $\boldsymbol{K}_0$ over a two element set. In fact, there are infinitely many mutually non-embeddible primitives in $\boldsymbol{K}_0$ over the empty set and similarly over a 1-element set.

Proof.

Over any a, b for each k build an (a, b)-cycle C_k , $c_1, c_2, \ldots c_{4k}$ of length $4k$ with $c_1 b c_{4k}$ and $c_1 a c_2$. C_k has $4k$ points and $(\{a, b\}, C_k) \in \boldsymbol{K}_0$ has $4k$ 3-element lines. So $\delta(\{a, b\}, C_k)) = 2 = \delta(\{a, b\})$. Primitivity easily follows since if the cycle is broken, the δ-rank goes up.
Minor variants for $\emptyset$ and singletons. □

The length of cycles

Definition

We denote the isomorphism type of $(\{a, b\}, C_n)$ by $\gamma_{\boldsymbol{n}}$.

Since for any n, $\mu(\gamma_{\boldsymbol{n}})$ is finite, we have

Lemma

For any $\mu \in \mathcal{U}$ and any $M \models T_\mu$, for every n, and every (a, b) there are only finitely many (a, b)-cycles of length n. Since $G(a, b)$ is infinite, there must be arbitrarily long finite (a, b)-chains. Since $\mathcal{G}_\mu$ is saturated there is also an infinite cycle.

More careful analysis shows only the prime model can omit infinite chains.

Generalize to Strongly minimal q-Steiner Systems

Words: Cycle graph renamed path graph: For d not on the line a, b consider paths beginning with d and alternating $a * d_{odd}$ and $b * d_{even}$. Repeat the analysis.

Propositions

Let $(M, F, R) \models T_{\mu', V}$.

1. Except in the prime model M_0 of T_μ there is a $G^{M_j}(a, b)$ with at least one infinite cycle.
2. If $\mu \in \mathcal{U}$, the dimension of the prime model of either T_μ or $T_{\mu', V}$ is at most 2. If the Fano plane is in $\boldsymbol{K}_\mu$ then it is 0.

Avoiding Finite Cycles I

Definition

Let $\mathcal{B}$ denote the set of $\mu \in \mathcal{U}$ except that for every n, $\mu(\gamma_{\boldsymbol{n}}) = 0$.
We denote by $\boldsymbol{K}_{\mathcal{B},\mu}$ the class of finite structures such that for all B:

$$(*)\ |B| > 1 \text{ implies } \delta(B) > 1 \text{ and } \mu \in \mathcal{B}.$$

When $\boldsymbol{K}_{\mathcal{B},\mu}$, we call the associated theory $T_{\mathcal{B},\mu}$.

$(*)$ implies that every two element subset of the generic is strong and so every model is 2-transitive.

Avoiding Finite Cycles II

Lemma

If $\mu \in \mathcal{B}$, $\boldsymbol{K}_{\mathcal{B},\mu}$ has the amalgamation property.

If $\mu \in \mathcal{B}$ then for any model, (M, R), of $T_{\mathcal{B},\mu}$ and any (a, b),
all (a, b)-cycles are infinite
and (M, R) is uniform.

Setting finitely many of the $\mu(\gamma_{\boldsymbol{i}}) = m_i$ for finitely many i allows finitely many cycles.

Questions

1. Verify that the Steiner spaces coordinatized as near vector spaces or near Boolean algebras [GW80] cannot be obtained by Hrushovski constructions. (There is a definable associative binary operation.)
2. Are there other approaches to coordinatization (e.g. by r-ary functions for larger r), that provide greater algebraization?
3. Our constructions show there are continuum many first order theories of strongly minimal block algebras. Do they represent continuum many distinct varieties? I.e are the classes $HSP(\mathcal{G}_\mu)$ distinct for (sufficiently) distinct μ?
4. Is it possible to characterize those μ such that T_μ can be interpreted in a quasigroup?
5. Can anything be gained from these properties of the quasigroup? What is the algebraic structure of a strongly minimal block algebra?

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