Large cardinals and strong logics

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CRM tutorial- Lecture 1

Tutorial Chapters

1. Reflection, compactness and elementary embeddings
2. Supercompacts as reflection cardinals for second order logic
3. Extendible cardinals as compactness cardinals for second order logics.
4. Vopenka's principle as the ultimate reflection/compactness principle.
5. ω_1 strongly compact and ω- logic
6. Some remarks on the set theory of generalized logics
7. Inner models constructed from generalized logics.

outline

Reflection , Compactness and elementary embeddings

Elementary embeddings

The importance being earnest about ω

Motivating axioms of strong infinity

Schemes for generating Axioms of strong infinity

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1. Reflection Principles
2. Compactness Principles
3. elementary embeddings

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2. Compactness Principles
3. elementary embeddings

Adding the axiom schema of replacement to Zermello Set Theory is adaption of a reflection principle.

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Typically "small" means *" Having cardinality less than .....*
Compactness of a certain property is essentially a localization principle. For simplicity sake we shall assume that the properties we consider are properties of mathematical structure in a countable signature and such that they are invariant under isomorphism.

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Namely a reflection principle for a given property is equivalent to a compactness principle for the negation of the property and vice versa.
In some cases it is more natural to state a principle as a reflection principle and in some cases it is more natural to state it as a compactness principle.

Combinatorial Examples

The existence of transversals: A *transversal* for a family of non empty sets is a 1-1 choice function on the family. Suppose that every smaller cardinality subfamily of the family $\mathcal{F}$ has a transversal. Does $\mathcal{F}$ has a transversal?

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Special case : $\mathcal{F}$ is a family of countable sets.

Disjointifying $\mathcal{F}$ is family of infinite sets. Disjointfying $\mathcal{F}$ means picking for every $X \in \mathcal{F}$ a finite $z_X \subseteq X$ such that the family $\{X - z_X | X \in \mathcal{F}\}$ is a family of mutually disjoint sets.

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Suppose that every smaller cardinality subfamily of $\mathcal{F}$ can be disjointified . Can $\mathcal{F}$ be disjointified ?

More combinatorics

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Algebraic examples

Freeness of Abelian groups An Abelian group $\mathcal{H}$ is free if it can be represented as the direct sum of copies of Z. $\mathcal{H} = \bigoplus_I Z$. Suppose that very smaller cardinality subgroup of $\mathcal{H}$ is free . Is $\mathcal{H}$ free?

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Free* The Abelian group $\mathcal{H}$ is said to be free* if it is a subgroup of a direct product of copies of $Z.\mathcal{H} \subseteq \bigotimes_I Z$. Suppose that every smaller cardinality subgroup of $\mathcal{H}$ is free*. Is $\mathcal{H}$ free*?

Topology- Lindelöf Property

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The statement that a certain class of spaces are all κ-Lindelöf is a reflection (or dually a compactness) property .

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Collection-wise Hausdorff $\mathcal{X}$ is a topological space. $Y \subseteq X$ is a discrete closed set. We say that Y can be *separated* if there is a family of mutually disjoint open sets $\{U_y | y \in Y\}$ such that for $y \in Y$ $y \in U_y$.
Suppose that every smaller cardinality subset of Y can be separated. Can Y be separated?

Other examples from Set Theory

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Model Theory:Chang's Conjecture

Definition

A structure $\mathcal{A}$, whose signature contains a distinguished unary predicate *R* is said to be of type (κ, λ) if $|\mathcal{A}| = \kappa$ and $|R^{\mathcal{A}}| = \lambda$. We say the Chang's conjecture $(\kappa, \lambda) \Rightarrow (\tilde{\kappa}, \tilde{\lambda})$ holds if every structure of countable signature of type (κ, λ) has a substructure of type $(\tilde{\kappa}, \tilde{\lambda})$.

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Question (Chang's question)

For which cardinals $\lambda, \tilde{\lambda}$ *we can have* $(\lambda^+, \lambda) \Rightarrow (\tilde{\lambda^+}, \tilde{\lambda})$*?*

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A related question: Suppose that the structure $\mathcal{A}$ is well ordered where the order type is a regular cardinal, Can we find a proper substructure whose order type is also a regular cardinal?

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The duality of reflection and compactness: κ is a reflection cardinal for a certain property iff it is a compactness cardinal for the negation of the property.

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Sometimes the existence of the reflection cardinal is equivalent to the existence of a certain large cardinal. Some other times the reflecting cardinals does not have to be large (Say it could be less than the first inaccessible) but the consistency of its existence may require the consistency of some large cardinal.

Weakly compact cardinals

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A weakly compact cardinal is a weak compactness cardinal for the properties we listed above.

Failure of stationary reflection implies general failure of compactness

Fact

Suppose that κ is a regular cardinal κ , S $\subseteq \kappa$ is a stationary set of κ of such that for $\alpha \in$ S $\text{cof}(\alpha) = \omega$. Suppose that S does not reflect. (S $\cap\, \alpha$ is not stationary in α for $\alpha < \kappa$.). Then for most of the properties the compactness property fails.

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Theorem

Suppose that κ is a regular cardinal κ , S $\subseteq \kappa$ is a stationary set of κ of such that for $\alpha \in$ S $\mathrm{cof}(\alpha) = \omega$. Suppose that S does not reflect. Then there is a first countable space $\mathcal{X}$ of cardinality κ such that very subspace of smaller cardinality is metric, but $\mathcal{X}$ is not metric.

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Proof.

For each limit $\alpha \in S$ pick an ω-sequence $\langle \beta_n^\alpha | n < \omega \rangle$ cofinal in α made up of successor ordinals . We define a topology τ on κ by specifying the discrete topology on $\kappa - S$. If $\alpha \in S$ then we take $U_n^\alpha = \{\beta_k^\alpha | k \geq n\}$ for $n < \omega$ be a neighborhood basis for α. □

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For every $\alpha < \kappa$ the space $\langle \alpha, \tau \rangle$ is metric.

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Let $\alpha < \kappa$. Then the collection $\alpha \cap S$ can be separated.More specifically for every $\beta < \alpha$ there is a family of mutually disjoint open sets $\langle U_\gamma | \gamma \in (\beta, \alpha) \cap S \rangle$ such that for $\gamma \in (\beta, \alpha) \cap S$ $\gamma \in U_\gamma$ and $U_\gamma \cap \beta = \emptyset$.

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Lemma

The space $\langle \kappa, \tau \rangle$ is not metric.

Corollary

1. *For all the properties above ω_1 is not compact with respect to them.*
2. *Let κ be regular. $S \subseteq \kappa$ is a stationary in κ such that $\alpha \in S \Rightarrow \mathrm{cof}(\alpha) = \omega$ and it does not reflect. Then for most of the properties above there is no strongly compact cardinal for this property which is $\leq \kappa$.*

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Since by a theorem of Jensen , in the constructible universe *L* every regular κ which is not weakly compact has a non reflecting stationary subset of points of cofinality ω we get:

Corollary (V=L)

For the most of the properties above :

1. *A regular κ is weakly compact with respect to the property iff it is weakly compact*
2. *There is no strongly compact cardinal for the property.*

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κ is weakly compact iff it is strong limit and every structure of the form $\langle \kappa, <, R_0, ...R_n....\rangle$ has a proper elementary end extension which has a minimal element in the new part. (We do not have to assume that the extension is well founded.)

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It is easily seen that κ satisfying the conditions must be a regular limit cardinal. The additional assumption of κ being strong limit guarantees that κ is inaccessible.

Proof of the $\leftarrow$ direction

Suppose that κ fails to be a reflection cardinal for the Π^1_1 property $\forall X \Phi(X)$. So let the structure $\mathcal{A} = \langle \kappa, <, R_0, R_1, \ldots, R_n \rangle$ be a counter example to κ being a reflection cardinal for this property.

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Hence for every $\alpha < \kappa$ there is $X_\alpha \subseteq \alpha$ such that

$$\langle \alpha, < \restriction \alpha R_0 \restriction \alpha, R_1 \restriction \alpha, \ldots, R_n \restriction \alpha \rangle \models \neg\Phi(X_\alpha)$$

. Enrich the structure $\mathcal{A}$ by adding a binary relation $S(\gamma, \alpha) \equiv \gamma \in X_\alpha$.

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Let $\mathcal{B}$ be the enriched the end extension given by the theorem and let c be minimal in $\mathcal{B} - \mathcal{A}$. It is easily seen that $X = \{\delta | \mathcal{B} \models S(\delta, c)$ is a subset of κ which witness that $\forall X \Phi(X)$ fails for $\mathcal{A}$.

Proof of the $\rightarrow$ direction of the theorem.

Given the structure $\mathcal{A} = \langle, \kappa, < \ldots \rangle\rangle$.We can assume that $\mathcal{A}$ is fully Skolemized. Enrich the language of $\mathcal{A}$ by adding a constant $c_g amma$ for every$\gamma < \kappa$ plus an additional constant c.

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A cofinal branch in T gives the complete diagram of a structure which is an end extension of $\mathcal{A}$. This end extension is easily seen to be well founded, so it has a minimal new ordinal.

Elementary embedding yields compactness

Theorem

Let κ be a cardinal, X a first countable topological space . Let $\lambda = |X|$. Assume that there is an elementary embedding $j : V \to M$ where $\langle M, E \rangle$ is a class model of ZFC (not necessarily well founded) such that M is correct about ω.(" M is an ω model ").

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Suppose that there is $D \in M$ such that $M \models |D| < j(\kappa)$ and for every $\alpha < \lambda$ $M \models j(\alpha) \in D$.

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Then If every subspace of X of cardinality less than κ is metric, then X is metric.

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Corollary

Assume that κ is a cardinal such that for every λ there is a class model $\langle M, E\rangle$ and j as in the statement of the theorem , then κ is a strongly compact cardinal for the property of a first countable space being metric.

Sketch of proof

Use the assumptions about M and j to get in M a subspace $Y \subseteq X$ such that $M \models |Y| < j(\kappa)$ and for all $x \in X$ $j(x) \in Y$. $j(X)$ and Y are also topological space in V. Since

$$M \models Z \subset X \wedge |Z| < j(\kappa) \to Z \text{ is metric}$$

we get $M \models$ Y is metric

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$M \models Y$ is metric , but again using the ω correctness of M we get that $V \models Y$ is metric . Hence X as a subspace of a metric space is metric.

The same argument works for many of the properties we listed above.

Theorem

Let κ be a cardinal such that for every λ there is a class model $\langle M, E\rangle$ and j as in the statement of the theorem , then κ is a strongly compact cardinal for all of the following properties:

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We shall later see that the same assumption about the cardinal κ yields results for κ-Lindlöf,stationary set reflection , Chang's model theoretic transfer etc.

Measurable cardinals and ω models .

Theorem

A cardinal κ is $\geq$ than the first measurable cardinal iff every ω-model of cardinality $\geq \kappa$ has a proper elementary extension which is still an ω -model.

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The $\rightarrow$ direction follows by taking an ultrapower of the structure by a σ complete ultrafilter.

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For the $\leftarrow$ direction, consider the structure $\mathcal{A} = \langle V_{\kappa+1}, \epsilon, \omega \rangle$. Let $\mathcal{B} = \langle B, E, \omega$ be a proper elementary extension of $\mathcal{A}$ which is an ω model. There is $b \in B - \kappa$ such that $\mathcal{B} \models b \in \kappa$. The set $U = \{X \subseteq \kappa | \mathcal{B} \models b \in X\}$ is easily seen to be a σ complete non principle ultrafilter on κ.

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Hence κ is $\geq$ the first measurable cardinal. □

Elementary embeddings from forcing extensions

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Theorem

Let κ be a cardinal such that for every $\kappa \leq \lambda$ there is a σ closed forcing notion $\mathcal{P}$ such that in $V^{\mathcal{P}}$ there is an ω class model M of ZFC , an elementary embedding $j : V \to M$ and an element $D \in M$ such that $M \models |D| < j(\kappa)$ and that for $\alpha < \lambda$ $M \models j(\alpha) \in D$. Then κ is a strongly compact cardinal for the property "The family $\mathcal{F}$ of countable sets can be disjointified.

Sketch.

Let $\mathcal{F}$ be a family of countable sets such that every subfamly of cardinality less than κ can be disjointified. Let $\lambda = |\mathcal{F}|$ .Let $\mathcal{P}$ be a σ closed forcing like in the statement of the theorem for λ.

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Let $\mathcal{F}$ be a family of countable sets such that every subfamly of cardinality less than κ can be disjointified. Let $\lambda = |\mathcal{F}|$ .Let $\mathcal{P}$ be a σ closed forcing like in the statement of the theorem for λ. Similarly to the proof above we can show that $\mathcal{F}$ can be disjointified , but in $V^{\mathcal{P}}$. The following lemma proves the theorem:

Lemma

Suppose that $\mathcal{F}$ is a family countable sets. Let $\mathcal{P}$ be a σ closed forcing notion, then if $V^{\mathcal{P}} \models \mathcal{F}$ can be disjointified then in the ground model V,$\mathcal{F}$ can be disjointified.

□