# The reasonable Effectiveness of Model Theory in Mathematics

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# Today

1. exhibit effectiveness
2. describe methodological features that partially account for the effectiveness
3. tentative thoughts: epistemological features of the model theoretic account

# Unreasonable Effectiveness

Wigner 

> *The first point is that ~~mathematical~~ model theoretic concepts turn up in entirely unexpected connections. Moreover, they often permit an unexpectedly close and accurate description of the phenomena in these connections.*
> *Wigner*

We argue that for large areas of mathematics, Hilbert-Gödel-Tarski formalization as organized by Shelah and modern model theory should be expected to be an effective tool in mathematics.

And it is.

# Detlefsen

Poincaré vs. Russell on the Role of Logic in Mathematics

## Two Notions of Synthetic Reasoning

1. Russell: logical reasoning IS mathematical reasoning – reasoning from one proposition to a different one
2. Poincaré (per Detlefsen)
   Mathematical reasoning is thus no longer to be seen as primarily a logical relationship between propositions, but rather as an epistemic relationship between judgements, the relationship between whose propositional contents is not to be gauged in logical terms but rather in terms of the relationships induced by the categories of mathematical thought.

The model theoretic perspective allows us to merge the two notions.

# Axiomatization vrs Formalization

# Bourbaki on Axiomatization:

Dieudonne  Bourbaki  Cartan 

Bourbaki wrote:

> *'We emphasize that it [formalization] is but one aspect of this [the axiomatic] method, indeed the least interesting one.'*

We reverse Bourbaki's aphorism to argue.
Full formalization is an important tool for modern mathematics.

# Formalizing a topic in mathematics

Anachronistically, *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
    - a Specify a class of well formed formulas.
    - b Specify truth of a formula from this class in a structure.
    - c Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the topic in question by sentences of the logic.

I restrict today to first order logic.

# The role of formalization

1. Specific areas of mathematics are axiomatized by first order theories.
2. Classification theory divides these theories into syntactically defined classes. Theories in the same class have analogous mathematically significant properties.
3. Formal proofs of specific formalizable results are shown to hold in the formal theories.
4. But the analogous properties may be described set theoretically.

Russell's pursuit of truth is given by the deductions in the theories.

Poincairé's mathematical reasoning underlies the overall development.

# III. The Methods of Model Theory

# The ingredients of effectiveness

1. interpretation Hilbert- Malcev-Tarski – everywhere
2. formal definability – quantifier reduction
3. theories - understanding families of related structures
4. The paradigm shift: the partition of first order theories by syntactic properties specifying mathematically significant properties of the theories;
5. structure of definable sets
   a. stable theories: rank, one based, chain conditions; geometric analysis of models
   b. o-minimal theories: cell decomposition, uniformly bounded fibrations
   c. p-adics: cell decomposition

# The Paradigm Shift

## The paradigm in 50's – 60's

*Model theory deals specifically with logical analogies among mathematical procedures and theories. It proceeds by means of an analysis of the language of theories while exploring the reciprocal relations between this language and the mathematical models that satisfy it.*

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*Model Theory is the Geography of Tame Mathematics*

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http://www.forkinganddividing.com/

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NOT QUITE

# Shelah on Dividing Lines: Shelah 

*I am grateful for this great honour. While it is great to find full understanding of that for which we have considerable knowledge, I have been attracted to trying to find some order in the darkness, more specifically,*
***finding meaningful dividing lines among general families of structures***
*This means that there are*
***meaningful things to be said on both sides of the divide**: characteristically, understanding the tame ones and giving evidence of being complicated for the chaotic ones.*

# Shelah on Dividing Lines 2

> *It is expected that this will eventually help*
>
> *in understanding even specific classes and even specific structures.*
>
> *Some others see this as the aim of model theory, not so for me.*
>
> *Still I expect and welcome such applications and interactions.*
> *It is a happy day for me that this line of thought has received such honourable recognition. Thank you*

on receiving the Steele prize for seminal contributions.

# Properties of classes of theories

## The Stability Hierarchy

Every complete first order theory falls into one of the following 4 classes.

1. $\omega$-stable
2. superstable but not $\omega$-stable
3. stable but not superstable
4. unstable

# Stability is Syntactic

## Definition

$T$ is stable if no formula has the order property in any model of $T$.

$\phi$ is unstable in $T$ just if for every $n$ the sentence
$\exists x_1, \ldots x_n \exists y_1, \ldots y_n \bigwedge_{i<j} \phi(x_i, y_i) \wedge \bigwedge_{j \geq i} \neg\phi(x_i, y_i)$ is in $T$.

This formula changes from theory to theory.

1. dense linear order: $x < y$;
2. real closed field: $(\exists z)(x + z^2 = y)$,
3. $(\mathbb{Z}, +, 0, \times)$ :$(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. infinite boolean algebras: $x \neq y$ & $(x \wedge y) = x$.

# Section III: Fuchsian Groups

# Bemis-Sinaceur: analogy in Poincare

*Mathematicians of all stripes, whether intuitionists or structuralists, have acknowledged the fundamental role played by analogy in mathematical invention. Thus for Poincare analogy is the inventor's principal "guide" (Poincare 1900, 127). He himself tells us that he was able to find the representation of a category of Fuchsian functions in terms of a series, because he was guided by an analogy with elliptic functions.*

*HOURYA BENIS-SINACEUR*
*THE NATURE OF PROGRESS IN MATHEMATICS: THE SIGNIFICANCE OF ANALOGY 1993 p 281*

# Complex Analysis

In the late 19th century three areas of mathematics coalesced.

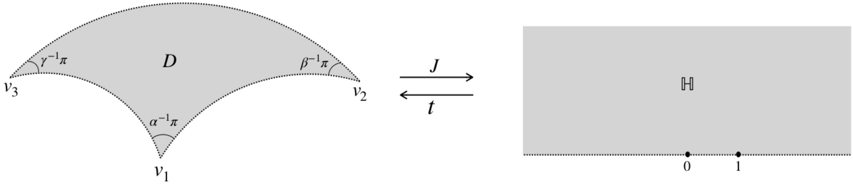
1. Riemann uniformization: $J_\Delta$ is a (bi-holomorphic) mapping from a domain $\Delta$ to the upper half plane $\mathbb{H}$.
2. $\Delta$ is the fundamental domain for a (Schwartzian triangle) group $\Gamma_\Delta$ acting on the upper half plane.
3. Integration in finite terms, transcendence results about: When are the solution sets of equations irreducible -not solvable in terms of known functions?
    1. Painleve's classification of certain 2rd order (non)-linear differential equations.
       Much Japanese work; completed by Pillay-Nagloo showing
        1. irreducible = strongly minimal
        2. the solution sets of these equations are strongly minimal and geometrically trivial
    2. $J_\Delta$ is a solution of a certain (Schwartzian) 3rd order (non)-linear differential equation.
       See below:

# Model theoretic approaches

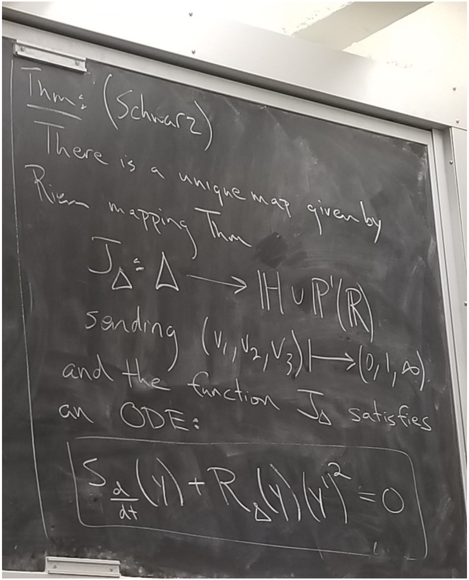
In the early 21 century there are three model theoretic ways to formalize these areas and solve classical problems.

1. Zilber's conjectured $L_{\omega_1,\omega}(Q)$ axiomatizations.
2. o-minimal complex analysis (Peterzil-Starchenko-Wilkie-Pila ....)
3. $DCF_0$ (Hrushovski, Pillay, Scanlon, Nagloo, Freitag, Casales, Blázquez-sanz)

# Riemann Mapping Theorem



# Schwartzian



# STRONGLY MINIMAL

**Definition**

A definable set $D$ is strongly minimal if every definable subset of $D$ is finite or cofinite.

e.g. acf, vector spaces, successor,
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**Definition**

1. $a$ is in the algebraic closure of $B$ ($a \in \mathrm{acl}(B)$) if for some $\phi(x, \mathbf{b})$: $\models \phi(a, \mathbf{b})$ with $\mathbf{b} \in B$ and $\phi(x, \mathbf{b})$ has only finitely many solutions.
2. algebraic closure induces a pregeometry on strongly minimal set;

Strongly minimal sets are the building blocks of structures whose first order theories are $\omega$-stable.

# Zilber trichotomy conjecture

## Conjecture

Every strongly minimal set is:

1. discrete (or trivial) ( $\mathrm{cl}(ab) = \mathrm{cl}(a) \cup cl(b)$)
   completely disintegrated $\mathrm{cl}(a) = a$.
2. modular: the lattice of closed subsets of the geometry is a modular
3. field-like

The conjecture fails in general. But it holds if

1. For $DCF_0$
2. *ACFA*
3. $T$ is 0-minimal

# Key model theoretic idea for differential equations

”An instance of Shelah’s Philosophy” (Hrushovski-Itai)

**Theorem: (Pillay)**

an *algebraically* closed differential field $K$ is *differentially* closed if every strongly minimal formula over $K$ has a solution in $K$’.

”An instance of Shelah’s Philosophy” (Hrushovski-Itai)

**strongly minimal differential equation**

For $y$ in the solution set of a third order differential equation over $F$, either

1. $y \in \mathrm{acl}(F)$ or
2. $y, y', y''$ are algebraically independent.

# Axiomatic analysis

Axiomatic analysis studies behavior of fields of functions with operators but *without* explicit attention in the formalism to continuity but rather to the algebraic properties of the functions.

The function symbols of the vocabulary act on the functions being studied; the functions are elements of the domain of the model.

Robinson introduced the notion of a universal domain for this subject; satuated models of the theory $DCF_0$.

# Specific Question: example of 'unlikely intersections'

Let $X$ and $Y$ be algebraic varieties over C and let $\phi : X^{an} \to Y^{an}$ be a complex analytic map which is not algebraic. In this case, for most algebraic subvarieties $X_0 \subseteq X$ the image $\phi(X_0)$is not algebraic.

The pairs of algebraic subvarieties $(X_0; Y_0)$ with $(X_0 \subset X)$and $(Y_0 \subset Y)$ such that $\phi(X_0) = \phi(Y_0)$ are called *bi-algebraic* for $\phi$.

Bi-algebraic subvarieties should be rare and revealing of important geometric aspects of the analytic map $\phi$.

**100 year old question**

Which circular triangle admit bialgebraic varieties (on their powers)?

# Recent answer

1. (known) If Fuchsian group $\Gamma_\Delta$ is *arithmetic* then there are non-trivial bi-algebraic varieties in $\Delta^n$ for some $n$.
(omit definition of arithmetic; There are 85 arithmetic triangle groups.)
2. (New) (Blázquez-Sanz, Casales, Freitag, Nagloo) if $X$ is the set of solutions of a Schwartzian with generic (algebraically independent) coefficients then $X$ is strongly minimal with trivial completely disintegrated geometry and so no nontrivial bialgebraic varieties over $\Delta$.
3. Many further cases to be studied.

# The wild world of mathematics

## Point

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## Counterpoint

In fact we show how to systematically make this separation in important cases.
What ”Gödel showed us that the wild infinite could not really be separated from the tame mathematical world if we insist on starting with the wild worlds of arithmetic or set theory.

The crucial contrast is between a foundationalist approach – a demand for global foundations
and a foundational approach – a search for mathematically important foundations of different topics.

# Tame theories

1. superstable
   1. ranks - that interact with ones defined by algebraists
   2. definable chain conditions on subgroups; this gives a notion of closed subgroup in model theory corresponding to the Zariski closure in algebraic geometry.
   3. NO pairing function – some chance at dimension
   4. structure of definable sets
2. stable
   1. an independence relation (non-forking)
   2. general notion of 'generic element' (realizes a non-forking extension)
   3. dimension on 'regular' types
3. o-minimal ordered structures
4. neo-stability theory: simple and NIP (no independence property)
5. local tameness – a tame piece of a model can be exploited

# Reasonable effectiveness

There have been significant applications of the themes above in:

1. groups of finite Morley rank
2. differential algebra, ode's Painlevé, Fuchsian groups
3. real algebraic geometry
4. algebraically closed valued fields
5. Banach spaces (continuous logic and metric AEC)
6. approximate groups
7. combinatorial graph theory
8. learning theory
9. Diophantine geometry
10. Stable Szmeredi

# Foundations

*So my suggestion is that we replace the claim that set theory is a (or the) foundation for mathematics with a handful of more precise observations: set theory provides* Risk Assessment *for mathematical theories, a* Generous Arena *where the branches of mathematics can be pursued in a unified setting with a* Shared Standard of Proof*, and a* Meta-mathematical Corral *so that formal techniques can be applied to all of mathematics at once.*
*P. Maddy What is a foundation for?*

1. Generous Arena: the role of ZFC as establishing a framework for traditional mathematics.
2. meta-mathematical corral: extensions of ZFC provide different and perhaps contradictory arenas (Witness V=L, Martin's axiom, and the Whitehead problem.).

# Essential Guidance

Maddy includes another criteria, *essential guidance*:
'such a foundation is to reveal the fundamental features – the essence, in practice – of the mathematics being founded, without irrelevant distractions; and it's to guide the progress of mathematics along the lines of those fundamental features and away from false alleyways.'

My argument is that model theory provides this essential guidance by means of formalization(s) and in particular the stability hierarchy.

# Empirical Philosophy

https://math.ethz.ch/news-and-events/news/d-math-news/2019/10/heinz-hopf-prize-to-a-model-theorist.html

# Mathematical Proof

Derivation of Consequences from Given Premises; Relation to Universally Valid Formulas

*Now we shall illustrate by a few examples the general methods of formal derivation in the predicate calculus. ... It is now a question of deriving the consequences from any premises whatsoever, no longer of a purely logical nature.*

*The method explained in this section of formal derivation from premises which are not universally valid logical formulas has its main application in the setting up of the primitive sentences or axioms for any particular field of knowledge and the derivation of the remaining theorems from them as consequences.*
*Hilbert-Ackermann 1938.*

We prove $T \models \psi$ by showing $\psi$ true in all models of $T$; so $T \vdash \psi$.