Philosophical implications of the paradigm shift in model theory

John T. Baldwin
University of Illinois at Chicago
Sèminaire de Philosophie des Mathḿatiques et de la Logique
March 28, 2018

March 30, 2018

Provocation

The announcement for a conference on Philosophy and Model Theory in 2010 began:

> *Model theory seems to have reached its zenith in the sixties and the seventies, when it was seen by many as virtually identical to mathematical logic. The works of Gödel and Cohen on the continuum hypothesis, though falling only indirectly within the domain of model theory, did bring to it some reflected glory. The works of Montague or Putnam bear witness to the profound impact of model theory, both on analytical philosophy and on the foundations of scientific linguistics.*

Response

My astonished reply to the organizers[1] began:

> *It seems that I have a very different notion of the history of model theory. As the paper at (Review of Badesa) points out, I would say that modern model theory begins around 1970 and the most profound mathematical results including applications in many other areas of mathematics have occurred since then, using various aspects of Shelah's paradigm shift. I must agree that, while in my view, there are significant philosophical implications of the new paradigm, they have not been conveyed to philosophers.*

[1] Letter to Halimi, September 20, 2009.

Forthcoming book

Model Theory and the Philosophy of Mathematical Practice:

Formalization without Foundationalism

Association for the Philosophy of Mathematical Practice

Goals include

Foster the philosophy of mathematical practice, that is, a broad outward-looking approach to the philosophy of mathematics which engages with mathematics in practice (including issues in history of mathematics, the applications of mathematics, cognitive science, etc.).

`http://www.philmathpractice.org/about/`

Two Theses

1. Contemporary model theory makes *formalization* of specific mathematical areas a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).

Two Theses

1. Contemporary model theory makes *formalization* of specific mathematical areas a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).
2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

What is the role of Logic?

Logic is the analysis of methods of reasoning

versus

Logic is a tool for doing mathematics.

What is the role of Logic?

Logic is the analysis of methods of reasoning

versus

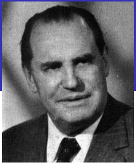
Logic is a tool for doing mathematics.

More precisely,
Mathematical logic is tool to solve not only its own problems but to organize and do traditional mathematics.

Section I.

Axiomatization vrs Formalization

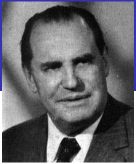
Bourbaki on Axiomatization:



Bourbaki wrote:

> *'We emphasize that it [formalization] is but one aspect of this [the axiomatic] method, indeed the least interesting one.'*

Bourbaki on Axiomatization:



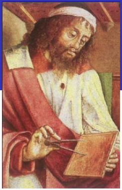
Bourbaki wrote:

> *'We emphasize that it [formalization] is but one aspect of this [the axiomatic] method, indeed the least interesting one.'*

We reverse Bourbaki's aphorism to argue.

Full formalization is an important tool for modern mathematics.

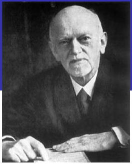
Euclid-Hilbert formalization 1900:



The Euclid-Hilbert (the Hilbert of the Grundlagen der Geometrie) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.

But the arguments and statements take place in natural language.
For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski formalization 1917-1956:



In the Hilbert (the founder of proof theory)-Gödel-Tarski framework, logic is a mathematical subject.

There are explicit rules for defining a formal language and proof.
Semantics is defined set-theoretically.

Formalization

Anachronistically, *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
   a. Specify a class of well formed formulas.
   b. Specify truth of a formula from this class in a structure.
   c. Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

Item 2c) is the least important from our standpoint.

Structures and Definability

A vocabulary τ is collection of constant, relation, and function symbols.

A τ-structure is a set in which each τ-symbol is interpreted.

A subset A of a τ-structure M is definable in M if there is $\mathbf{n} \in M$ and a τ-formula $\phi(x, \mathbf{y})$ such that

$$A = \{m \in M : M \models \phi(m, \mathbf{n})\}.$$

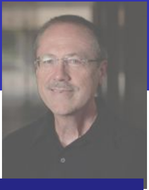
Note that if property is defined without parameters in M, then it is uniformly defined in all models of $\mathrm{Th}(M)$.

Theories

Contemporary model theory focuses on theories not logics.

One can learn about mathematically natural structures by studying related theories (model completion) or related structures (saturated, 2-cardinal etc.)

Detlefsen asked:



Question A

Which view is the more plausible—that theories are the better the more nearly they are categorical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?

Question B

Is there a single answer to the preceding question? Or is it rather the case that categoricity is a virtue in some theories but not in others?

What is virtue?

Pragmatic Criterion

A property of a theory T is virtuous if it has significant mathematical consequences for T or its models.

Under this criteria

1. categoricity of an informative axiomatization of a 2nd order theory is virtuous.
2. Categoricity of $\mathrm{Th}^2(M)$ is not virtuous.
3. Virtuous properties of first order theories include: model completeness, completeness, categoricity in power, ω-stability, π_2-axiomatizability, o-minimality etc.

Complete Theories

Complete theories are the main object of study.
Kazhdan:

> *On the other hand, the Model theory is concentrated on [the] gap between an abstract definition and a concrete construction. Let T be a complete theory. On the first glance one should not distinguish between different models of T, since all the results which are true in one model of T are true in any other model.*
> *One of the main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty.*
> *Different models of complete theories are of different flavors and support different intuitions.*

Historical Issues

First order logic and fixing vocabulary

Until the late 40's first order logic was normally viewed as 'restricted predicate calculus - infinitely many relation predicates of each arity. Church (1956) still thinks of first order logic as a subsystem of a higher order functional calculus.

Tarski, Robinson, and Henkin (based on the 1935 definition of a class of algebras by Garrett Birkhoff) are moving towards the modern concept fully stated in the 50's: Specify a list of primitive notions – vocabulary or similarity type.

Section II. The Methods of Model Theory

How does formalization impact mathematics?

1. interpretation Hilbert- Malcev-Tarski – everywhere
2. formal definability – quantifier reduction
3. theories - understanding families of related structures
4. The paradigm shift: the partition of first order theories by syntactic properties specifying mathematically significant properties of the theories;
5. structure of definable sets
   a. stable theories: rank, one based, chain conditions; geometric analysis of models
   b. o-minimal theories: cell decomposition, uniformly bounded fibrations
   c. p-adics: cell decomposition

Interpretation I:

In Borovik-Nesin: Groups of Finite Morley Rank:

> *The notion of interpretation in model theory corresponds to a number of familiar phenomena in algebra which are often considered distinct: coordinatization, structure theory, and constructions like direct product and homomorphic image.*

1. a Desarguesian projective plane is coordinatized by a division ring
2. Artinian semisimple rings are finite direct products of matrix rings over division rings;
3. classifying abstract groups as a standard family of matrix groups

Interpretation II

All of these examples have a common feature: certain structures of one kind are somehow encoded in terms of structures of another kind. All of these examples have a further feature which plays no role in algebra but which is crucial for us: in each case the encoded structures can be recovered from the encoding structures definably.

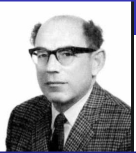
'plays no role': written in 1994 -no longer true

The Significance of Classes of Theories : Definability

Quantifier Elimination and Model Completeness

Every definable formula is equivalent to quantifier-free (resp. existential) formula.

The Significance of Classes of Theories : Definability



Quantifier Elimination and Model Completeness

Every definable formula is equivalent to quantifier-free (resp. existential) formula.

Tarski proved quantifier elimination of the reals in 1931.

Robinson provides a unified treatment of Hilbert's Nullstellensatz and the Artin-Schreier theorem.
He introduced model completeness and proved model completeness/q.e. of algebraically closed fields.

The Paradigm Shift

The paradigm around 1950

the study of logics; the principal results were completeness, compactness, interpolation and joint consistency theorems.

Various semantic properties of theories were given syntactic characterizations but there was no notion of partitioning all theories by a family of properties.

The Paradigm Shift

The paradigm around 1950

the study of logics; the principal results were completeness, compactness, interpolation and joint consistency theorems.

Various semantic properties of theories were given syntactic characterizations but there was no notion of partitioning all theories by a family of properties.

After the paradigm shift

There is a systematic search for a small set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties.

After the shift one can compare different areas of mathematics by checking where theories formalizing them lie in the classification.

The significance of classes of Theories

The breakthroughs of model theory as a tool for organizing mathematics come in several steps.

1. The significance of (complete) first order theories.
2. The significance of classes of (complete) first order theories: Quantifier reduction

The significance of classes of Theories

The breakthroughs of model theory as a tool for organizing mathematics come in several steps.

1. The significance of (complete) first order theories.
2. The significance of classes of (complete) first order theories: Quantifier reduction

East Coast: Quantifier reduction in a natural vocabulary is crucial for applications.

West Coast: Quantifier elimination by fiat exposes the fundamental model theoretic structure.

The significance of classes of Theories

The breakthroughs of model theory as a tool for organizing mathematics come in several steps.

1. The significance of (complete) first order theories.
2. The significance of classes of (complete) first order theories: Quantifier reduction

   East Coast Quantifier reduction in a natural vocabulary is crucial for applications.

   West Coast Quantifier elimination by fiat exposes the fundamental model theoretic structure.

3. The significance of classes of (complete) first order theories: syntactic dividing lines

Section III: Dividing Lines

I am grateful for this great honour. While it is great to find full understanding of that for which we have considerable knowledge, I have been attracted to trying to find some order in the darkness, more specifically,
finding meaningful dividing lines among general families of structures.
This means that there are
meaningful things to be said on both sides of the divide:
characteristically, understanding the tame ones and giving evidence of being complicated for the chaotic ones.

Shelah on Dividing Lines

It is expected that this will eventually help

in understanding even specific classes and even specific structures.

Some others see this as the aim of model theory, not so for me.

Still I expect and welcome such applications and interactions. It is a happy day for me that this line of thought has received such honourable recognition. Thank you

on receiving the Steele prize for seminal contributions.

Shelah classification strategy

A property P is a dividing line if
both P and $\neg P$ are virtuous.

Stable and superstable are dividing lines

ω-stable and $\aleph_1$-categorical are virtuous but not dividing lines.

Properties of classes of theories

The Stability Hierarchy

Every complete first order theory falls into one of the following classes.

1. ω-stable
2. superstable but not ω-stable
3. stable but not superstable
4. unstable: Several approaches:
    1. refine the classification: nip, simple, neostability theory
    2. o-minimality
    3. enforce 'enough' stability

This classification is set theoretically absolute

Stability is Syntactic

Definition

T is stable if no formula has the order property in any model of T.

ϕ is unstable in T just if for every n the sentence $\exists x_1, \ldots x_n \exists y_1, \ldots y_n \bigwedge_{i<j} \phi(x_i, y_i) \wedge \bigwedge_{j \geq i} \neg\phi(x_i, y_i)$ is in T.

Stability is Syntactic

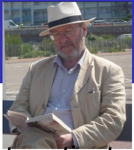
Definition

T is stable if no formula has the order property in any model of *T*.

ϕ is unstable in T just if for every n the sentence $\exists x_1, \ldots x_n \exists y_1, \ldots y_n \bigwedge_{i<j} \phi(x_i, y_i) \wedge \bigwedge_{j\geq i} \neg\phi(x_i, y_i)$ is in T.

This formula changes from theory to theory.

1. dense linear order: $x < y$;
2. real closed field: $(\exists z)(x + z^2 = y)$,
3. $(\mathbb{Z}, +, 0, \times)$:$(\exists z_1, z_2, z_3, z_4)(x + (z_1^2 + z_2^2 + z_3^2 + z_4^2) = y)$.
4. infinite boolean algebras: $x \neq y$ & $(x \wedge y) = x$.



Wilkie to Bourbaki:

It [o-minimality] is best motivated as being a candidate for Grothendieck's idea of tame topology as expounded in his Esquisse d'un Programme. It seems to me that such a candidate should satisfy (at least) the following criteria.

A. A flexible framework to carry out geometrical and topological constructions on real functions and on subsets of real euclidean spaces.
B. It should have built in restrictions to block pathological phenomena. There should be a meaningful notion of dimension for all sets under consideration and any that can be constructed from these by use of the operations allowed under (A).

o-minimality continued

C One must be able to prove finiteness theorems that are uniform over fibred collections.

Rather than enumerate analytic conditions on sets and functions sufficient to guarantee the criteria (A), (B) and (C) however, we shall give one succinct axiom, the o-minimality axiom, which implies them.

Above paraphased/quoted from a Wilkie Bourbaki seminar.

o-minimality continued

C One must be able to prove finiteness theorems that are uniform over fibred collections.

Rather than enumerate analytic conditions on sets and functions sufficient to guarantee the criteria (A), (B) and (C) however, we shall give one succinct axiom, the o-minimality axiom, which implies them.

Above paraphased/quoted from a Wilkie Bourbaki seminar.

Note Bene

o-minimality is not an axiom.

It is a syntactic property defining a class of theories – just as the stability conditions above.

Every definable set is a Boolean combination of intervals

Section 4: The Role of Set Theory

Löwenheim Skolem for 2 cardinals Vaught

Vaught: Can we vary the cardinality of a definable subset as we can vary the cardinality of the model?

Löwenheim Skolem for 2 cardinals Vaught

Vaught: Can we vary the cardinality of a definable subset as we can vary the cardinality of the model?

Two Cardinal Models

1. A two cardinal model is a structure M with a definable subset D with $\aleph_0 \leq |D| < |M|$.
2. We say a first order theory T in a vocabulary with a unary predicate P *admits* (κ, λ) if there is a model M of T with $|M| = \kappa$ and $|P^M| = \lambda$.

   We write $(\kappa, \lambda) \rightarrow (\kappa', \lambda')$

   if *every theory* that admits (κ, λ) also admits (κ', λ').

Set Theory Becomes Central in the 60's

Vaught asked a 'big question', 'For what quadruples of cardinals does $(\kappa, \lambda) \rightarrow (\kappa', \lambda')$ hold?'

Set Theory Becomes Central in the 60's

Vaught asked a 'big question', 'For what quadruples of cardinals does $(\kappa, \lambda) \to (\kappa', \lambda')$ hold?'

Hypotheses included:

1. replacement: Erdos-Rado theorem below $\beth_{\omega_1}$.
2. GCH
3. $V = L$
4. Jensen's notion of a morass
5. Erdös cardinals,
6. Foreman [1982] showing the equivalence between such a two-cardinal theorem and 2-huge cardinals AND ON

1-5 Classical work in 60's and early 70's; continuing importance in set theory.

Revised Theorem: solved in ZFC

Suppose

1. [Shelah, Lachlan $\approx$ 1972] T is stable
2. or [Bays 1998] T is *o*-minimal

then $\forall(\kappa > \lambda, \kappa' \geq \lambda')$
if T admits (κ, λ)

then T also admits (κ', λ').

Ask the right question

$P(\kappa, \lambda, T)$ means, 'there is a (κ, λ)-model of T.'

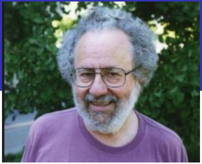
Reversing the question

before the shift:
For which cardinals (κ, λ) does $P(\kappa, \lambda, T)$ hold for every theory T?

after the shift:
For which theories T does $P(\kappa, \lambda, T)$ hold for all cardinals (κ, λ)?

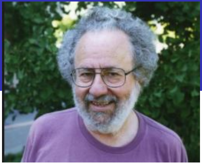
Section V. Wild and Tame

The wild world of mathematics



Martin Davis: ‘Gödel showed us that the wild infinite could not really be separated from the tame mathematical world where most mathematicians may prefer to pitch their tents.”

The wild world of mathematics



Martin Davis: 'Gödel showed us that the wild infinite could not really be separated from the tame mathematical world where most mathematicians may prefer to pitch their tents."

We systematically make this separation in important cases. What "Gödel showed us is that the wild infinite could not really be separated from the tame mathematical world if we insist on starting with the wild worlds of arithmetic or set theory.

The crucial contrast is between:
a foundationalist approach – a demand for global foundations
and a foundational approach – a search for mathematically important foundations of different topics.

Two further Theses

3. The choice of vocabulary and logic appropriate to the particular topic are central to the success of a formalization. The technical developments of first order logic have been more important in other areas of modern mathematics than such developments for other logics.

4. The study of geometry is not only the source of the idea of axiomatization and many of the fundamental concepts of model theory, but geometry itself (through the medium of geometric stability theory) plays a fundamental role in analyzing the models of tame theories and solving problems in other areas of mathematics.

Two kinds of geometry

1. first order formalizations of real and complex algebraic geometry
2. combinatorial geometry

Dimension: the essence of geometry

Dimension is a natural generalization of the notion of two and three dimensional space.

With coordinatization, the dimension tells us how many coordinates are needed to specify a point.

unidimensionality and categoricity in power

This dimension (for a countable language) and uncountable strongly minimal (more generally $\aleph_1$-categorical) structure is the same as the cardinality of the model.

A shift in view

Morley's categoricity theorem Morley 

A countable first order theory is categorical in $\aleph_1$ if and only if it is categorical in every uncountable cardinal.

B-Lachlan characterization

A countable first order theory is categorical in $\aleph_1$ if and only if it is i) ω-stable and ii) has no two-cardinal model

That is, Each model is determined by the dimension of a strongly minimal set.

The role of geometry

If T is a stable theory then there is a notion 'non-forking independence' which has major properties of an independence notion in the sense of van den Waerden.

It imposes a dimension on the realizations of regular types.

For many models of appropriate stable theories it assigns a dimension to the model.

This is the key to being able to describe structures.

The role of geometry

If T is a stable theory then there is a notion 'non-forking independence' which has major properties of an independence notion in the sense of van den Waerden.

It imposes a dimension on the realizations of regular types.

For many models of appropriate stable theories it assigns a dimension to the model.

This is the key to being able to describe structures.

Bourbaki's 3 great mother structures

order, groups, topology

ADD geometry

Geometric Stability Theory

Classification

The geometries of strongly minimal sets (regular types) fall into 4 classes:

1. discrete (trivial) $(\mathrm{cl}(ab) = \mathrm{cl}(a) \cup cl(b))$
2. modular or vector space like: (the lattice of closed subsets of the geometry is a modular lattice).
3. field-like (somehow bi-interpretable with a field).
4. none of the above: non-desarguesian but not vector space like.

Theorem (B-Paolini, 2018)

There is an infinite strongly minimal plane; all lines of the same finite length.

Geometry and Algebra are inevitable

Zilber / Hrushovski

Abstract model theoretic conditions imply algebraic consequences.

e.g. A group is definable in any $\aleph_1$-categorical theory that is not almost strongly minimal.

More technical hypothesis imply

1. the group is an abelian or a matrix group over an ACF of rank at most 3 or
2. there is a definable field.

The hypothesis do not mention anything algebraic.

Hart, Hrushovski, Laskowski

Any model of a complete theory, whose uncountable spectrum is

$$I(\aleph_\alpha, T) = \min(2^{\aleph_\alpha}, \beth_{d-1}(|\alpha+\omega| + \beth_2))$$

for some finite $d > 1$, interprets an infinite group.

Tame theories

1. superstable
   1. ranks - that interact with ones defined by algebraists
   2. definable chain conditions on subgroups;
   3. NO pairing function – some chance at dimension
   4. structure of definable sets
2. stable
   1. an independence relation (non-forking)
   2. general notion of 'generic element'
   3. dimension on 'regular' types
3. unstable theories: approaches
   1. o-minimal ordered structures
   2. neo-stability theory: simple and NIP
   3. local tameness – a tame piece of a model can be exploited

Pillay explains the diophantine geometry connection as follows.

> *The use of model-theoretic and stability-theoretic methods should not be so surprising, as the full Lang conjecture itself is equivalent to a purely model-theoretic statement. The structure $(\mathbb{Q}, +, \cdot)$ is wild (undecidable, definable sets have no structure, etc.), as is the structure $(\mathcal{C}, +, \cdot)$ with a predicate for the rationals. What comes out of the diophantine type conjectures however is that* certain *enrichments of the structure $(\mathcal{C}, +, \cdot)$ ... are not wild, in particular are stable.*

Consider $(\mathcal{C}, +, \cdot, \Gamma)$ where Γ is the finitely generated group from Mordell-Weil.

Model Theory and Analysis

First order analysis

1. Axiomatic analysis:
   Models are fields of functions:
   Solves problems dating back to Painlevè 1900
   Applications to Hardy Fields, and asymptotic analysis

Model Theory and Analysis

First order analysis

1. Axiomatic analysis:
   Models are fields of functions:
   Solves problems dating back to Painlevè 1900
   Applications to Hardy Fields, and asymptotic analysis
2. Definable analysis
   Functions are defined implicitly:
   real exponentiation, number theory

Why does this matter to philosophers?

A Technical result with philosophical significance

Either there is a uniform way to assign invariants to models of T or T has the maximal number of models in every uncountable power.

A Technical result with philosophical significance

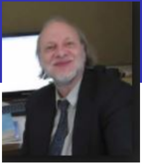
Either there is a uniform way to assign invariants to models of T or T has the maximal number of models in every uncountable power.

The Main Gap: No Structure or structure

Let T be a countable complete first order theory.

1. Either $I(T, \aleph_\alpha) = 2^{\aleph_\alpha}$ or
2. T is superstable without the *omitting types order property* or the *dimensional order property* and is shallow whence
   1. each model of cardinality λ is decomposed into countable models indexed by a tree of countable height and width λ.
   2. and thus, for any ordinal $\alpha > 0$, $I(T, \aleph_\alpha) < \beth_\delta(|\alpha|)$ (for a countable ordinal δ depending on T);

Reliability or Clarity



...a long-term look at achievements in mathematics shows that genuine mathematical achievement consists primarily in making clear by using new concepts ... We look for uses of mathematical logic in bringing out these roles of concepts in mathematics. Representations and methods from the reliability programs are not always appropriate. We need to be able to emphasize special features of a given mathematical area and its relationship to others, rather than how it fits into an absolutely general pattern. (Manders 1987)