Hanf Numbers and Presentation Theorems in AECs

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1 Introduction

This paper addresses a number of fundamental problems in logic and the philosophy of mathematics by considering some more technical problems in model theory and set theory. The interplay between syntax and semantics is usually considered the hallmark of model theory. At first sight, Shelah's notion of abstract elementary class shatters that icon. As in the beginnings of the modern theory of structures ([Cor92]) Shelah studies certain classes of models and relations among them, providing an axiomatization in the Bourbaki ([Bou50]) as opposed to the Gödel or Tarski sense: mathematical requirements, not sentences in a formal language. This formalism-free approach ([Ken13]) was designed to circumvent confusion arising from the syntactical schemes of infinitary logic; if a logic is closed under infinite conjunctions, what is the sense of studying types? However, Shelah's presentation theorem and more strongly Boney's use [Bon] of aec's as theories of $L_{\kappa,\omega}$ (for κ strongly compact) reintroduce syntactical arguments. The issues addressed in this paper trace to the failure of infinitary logics to satisfy the *upward* Löwenheim-Skolem theorem or more specifically the compactness theorem. The compactness theorem allows such basic algebraic notions as amalgamation and joint embedding to be easily encoded in first order logic. Thus, all complete first order theories have amalgamation and joint embedding in all cardinalities. In contrast

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these and other familiar concepts from algebra and model theory turn out to be heavily cardinal-dependent for infinitary logic and specifically for abstract elementary classes. This is especially striking as one of the most important contributions of modern model theory is the freeing of first order model theory from its entanglement with axiomatic set theory ([Bal15a], chapter 7 of [Bal15b]).

Two main issues are addressed here. We consider not the interaction of syntax and semantics in the usual formal language/structure dichotomy but methodologically. What are reasons for adopting syntactic and/or semantic approaches to a particular topic? We compare methods from the very beginnings of model theory with semantic methods powered by large cardinal hypotheses. Secondly, what then are the connections of large cardinal axioms with the cardinal dependence of algebraic properties in model theory. Here we describe the opening of the gates for potentially large interactions between set theorists (and incidentally graph theorists) and model theorists. More precisely, can the combinatorial properties of small large cardinals be coded as structural properties of abstract elementary classes so as to produce Hanf numbers intermediate in cardinality between 'well below the first inaccessible' and 'strongly compact'?

Most theorems in mathematics are either true in a specific small cardinality (at most the continuum) or in all cardinals. For example all, *finite* division rings are commutative, thus all finite Desarguesian planes are Pappian. But *all* Pappian planes are Desarguean and not conversely. Of course this stricture does not apply to set theory, but the distinctions arising in set theory are combinatorial. First order model theory, to some extent, and Abstract Elementary Classes (AEC) are beginning to provide a deeper exploration of Cantor's paradise: algebraic properties that are cardinality dependent. In this article, we explore whether certain key properties (amalgamation, joint embedding, and their relatives) follow this line. These algebraic properties are structural in the sense of [Cor04].

Much of this issue arises from an interesting decision of Shelah. Generalizing Fraïssé [Fra54] who considered only finite and countable stuctures, Jónsson laid the foundations for AEC by his study of universal and homogeneous relation systems [Jón56, Jón60]. Both of these authors assumed the amalgamation property (AP) and the joint embedding property (JEP), which in their context is cardinal independent. Variants such as disjoint or free amalgamation (DAP) are a well-studied notion in model theory and universal algebra. But Shelah omitted the requirement of amalgamation in defining AEC. Two reasons are evident for this: it is cardinal dependent in this context; Shelah's theorem (under weak diamond) that categoricity in κ and few models in κ^+ implies amalgamation in κ suggests that amalgamation might be a dividing line.

Grossberg [Gro02, Conjecture 9.3] first raised the question of the existence of Hanf numbers for joint embedding and amalgamation in Abstract Elementary Classes (AEC). We define four kinds of amalgamation properties (with various cardinal parameters) in Subsection 1.1 and a fifth at the end of Section 3.1. The first three notions are staples of the model theory and universal algebra since the fifties and treated for first order logic in a fairly uniform manner by the methods of Abraham Robinson. It is a rather striking feature of Shelah's presentation theorem that issues of disjointness require careful study for AEC, while disjoint amalgamation is trivial for complete first order theories.

Our main result is the following:

Theorem 1.0.1. *Let κ be strongly compact and $\boldsymbol{K}$ be an AEC with Löwenheim-Skolem number less than κ. If $\boldsymbol{K}$ satisfies*[1] *AP/JEP/DAP/DJEP/NDJEP for models of size $[\mu, < \kappa)$, then $\boldsymbol{K}$ satisfies AP/JEP/DAP/DJEP/NDJEP for all models of size $\geq \mu$.*

We conclude with a survey of results showing the large gap for many properties between the largest cardinal where an 'exotic' structure exists and the smallest where eventual behavior is determined. Then we provide specific question to investigate this distinction.

Our starting place for this investigation was second author's work [Bon] that emphasized the role of large cardinals in the study of AEC. A key aspect of the definition of AEC is as a mathematical definition with no formal syntax - class of structures satisfying certain closure properties. However, Shelah's Presentation Theorem says that AECs are expressible in infinitary languages, $L_{\kappa,\omega}$, which allowed a proof via sufficiently complete ultraproducts that, assuming enough strongly compact cardinals, all AEC's were eventually tame in the sense of [GV06].

Thus we approached the problem of finding a Hanf number for amalgamation, etc. from two directions: using ultraproducts to give purely semantic arguments and using Shelah's Presentation Theorem to give purely syntactic arguments. However, there was a gap: although syntactic arguments gave characterizations similar to those found in first order, they required looking at the *disjoint* versions of properties, while the semantic arguments did not see this difference.

The requirement of disjointness in the syntactic arguments stems from a lack of canonicity in Shelah's Presentation Theorem: a single model has many expansions which means that the transfer of structural properties between an AEC $\boldsymbol{K}$ and it's expansion can break down. To fix this problem, we developed a new presentation theorem, called the *relational presentation theorem* because the expansion consists of relations rather than the Skolem-like functions from Shelah's Presentation Theorem.

Theorem 1.0.2 (The relational presentation theorem, Theorem 3.2.3)**.** *To each AEC $\boldsymbol{K}$ with $LS(K) = \kappa$ in vocabulary τ, there is an expansion of τ by predicates of arity κ and a theory T^* in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$ such that $\boldsymbol{K}$ is exactly the class of τ reducts of models of T^*.*

Note that this presentation theorem works in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$ and has symbols of arity κ, a far cry from the $\mathbb{L}_{(2^\kappa)^+,\omega}$ and finitary language of Shelah's Presentation Theorem. The benefit of this is that the expansion is canonical or functorial (see

[1]This alphabet soup is decoded in Definition 1.1.1.

Definition 3.0.1). This functoriality makes the transfer of properties between $\boldsymbol{K}$ and $(\mathrm{Mod}\, T^*, \subset_{\tau^*})$ trivial (see Proposition 3.0.2). This allows us to formulate natural syntactic conditions for our structural properties.

Comparing the relational presentation theorem to Shelah's, another well-known advantage of Shelah's is that it allows for the computation of Hanf numbers for existence (see Section 4) because these exist in $\mathbb{L}_{\kappa,\omega}$. However, there is an advantage of the relational presentation theorem: Shelah's Presentation Theorem works with a sentence in the logic $\mathbb{L}_{(2^{LS(\boldsymbol{K})})^+,\omega}$ and there is little hope of bringing that cardinal down[2]. On the other hand, the logic and size of theory in the relational presentation theorem can be brought down by putting structure assumptions on the class $\boldsymbol{K}$, primarily on the number of nonisomorphic extensions of size $LS(\boldsymbol{K})$, $|\{(M, N)/\cong : M \prec_{\boldsymbol{K}} N \text{ from } \boldsymbol{K}_{LS(\boldsymbol{K})}\}|$.

We would like to thank Spencer Unger and Sebastien Vasey for helpful discussions regarding these results.

1.1 Preliminaries

We discuss the relevant background of AECs, especially for the case of disjoint amalgamation.

Definition 1.1.1. *We consider several variations on the joint embedding property, written JEP or JEP*$[\mu, \kappa)$.

1. *Given a class of cardinals* $\mathcal{F}$ *and an AEC* $\boldsymbol{K}$*,* $\boldsymbol{K}_{\mathcal{F}}$ *denotes the collection of* $M \in \boldsymbol{K}$ *such that* $|M| \in \mathcal{F}$*. When* $\mathcal{F}$ *is a singleton, we write* $\boldsymbol{K}_\kappa$ *instead of* $\boldsymbol{K}_{\{\kappa\}}$*. Similarly, when* $\mathcal{F}$ *is an interval, we write* $< \kappa$ *in place of* $[\mathrm{LS}(\boldsymbol{K}), \kappa)$*;* $\leq \kappa$ *in place of* $[\mathrm{LS}(\boldsymbol{K}), \kappa]$*;* $> \kappa$ *in place of* $\{\lambda \mid \lambda > \kappa\}$*; and* $\geq \kappa$ *in place of* $\{\lambda \mid \lambda \geq \kappa\}$.
2. *An AEC* $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *has the joint embedding property, JEP, (on the interval* $[\mu, \kappa)$*) if any two models (from* $\boldsymbol{K}_{[\mu,\kappa)}$*) can be* $\boldsymbol{K}$*-embedded into a larger model.*
3. *If the embeddings witnessing the joint embedding property can be chosen to have disjoint ranges, then we call this the* disjoint embedding property *and write* $DJEP$.
4. *An AEC* $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ *has the amalgamation property, AP, (on the interval* $[\mu, \kappa)$*) if, given any triple of models* $M_0 \prec M_1, M_2$ *(from* $\boldsymbol{K}_{[\mu,\kappa)}$*),* M_1 *and* M_2 *can be* $\boldsymbol{K}$*-embedded into a larger model by embeddings that agree on* M_0.
5. *If the embeddings witnessing the amalgamation property can be chosen to have disjoint ranges except for* M_0*, then we call this the* disjoint amalgamation property *and write* DAP.

[2]Indeed an AEC $\boldsymbol{K}$ where the sentence is in a smaller logic would likely have to have satisfy the very strong property that there are $< 2^{LS(\boldsymbol{K})}$ many $\tau(\boldsymbol{K})$ structures that **are not** in $\boldsymbol{K}$

Definition 1.1.2. *1. A* finite diagram *or* $EC(T, \Gamma)$-class *is the class of models of a first order theory* T *which omit all types from a specified collection* Γ *of complete types in finitely many variables over the empty set.*

2. Let Γ *be a collection of first order types in finitely many variables over the empty set for a first order theory* T *in a vocabulary* τ_1*. A* $PC(T, \Gamma, \tau)$ class *is the class of reducts to* $\tau \subset \tau_1$ *of models of a first order* τ_1*-theory* T *which omit all members of the specified collection* Γ *of partial types.*

2 Semantic arguments

It turns out that the Hanf number computation for the amalgamation properties is immediate from Boney's "Łoś' Theorem for AECs" [Bon, Theorem 4.3]. We will sketch the argument for completeness. For convenience here, we take the following of the many equivalent definitions of strongly compact; it is the most useful for ultraproduct constructions.

Definition 2.0.1 ([Jec06].20)**.** *The cardinal* κ *is strongly compact iff for every* S *and every* κ*-complete filter on* S *can be extended to a* κ*-complete ultrafilter. Equivalently, for every* $\lambda \geq \kappa$*, there is a fine*[3]*,* κ*-complete ultrafilter on* $P_\kappa \lambda = \{\sigma \subset \lambda : |\sigma| < \kappa\}$*.*

For this paper, "essentially below κ" means "$LS(K) < \kappa$."

Fact 2.0.2 (Łoś' Theorem for AECs)**.** *Suppose* K *is an AEC essentially below* κ *and* U *is a* κ*-complete ultrafilter on* I*. Then* K *and the class of* K*-embeddings are closed under* κ*-complete ultraproducts and the ultrapower embedding is a* K*-embedding.*

The argument for Theorem 2.0.3 has two main steps. First, use Shelah's presentation theorem to interpret the AEC into $L_{\kappa,\omega}$ and then use the fact that $L_{\kappa,\omega}$ classes are closed under ultraproduct by *κ-complete* ultraproducts.

Theorem 2.0.3. *Let* κ *be strongly compact and* $\boldsymbol{K}$ *be an AEC with Löwenheim-Skolem number less than* κ*.*

- *If* $\boldsymbol{K}$ *satisfies* $AP(< \kappa)$ *then* $\boldsymbol{K}$ *satisfies* AP*.*
- *If* $\boldsymbol{K}$ *satisfies* $JEP(< \kappa)$ *then* $\boldsymbol{K}$ *satisfies* JEP*.*
- *If* $\boldsymbol{K}$ *satisfies* $DAP(< \kappa)$ *then* $\boldsymbol{K}$ *satisfies* DAP*.*

Proof: We first sketch the proof for the first item, AP, and then note the modifications for the other two.

[3] U is fine iff $G(\alpha) := \{z \in P_\kappa(\lambda) | \alpha \in z\}$ is an element of U for each $\alpha < \lambda$.

Suppose that $\boldsymbol{K}$ satisfies $AP(<\ \kappa)$ and consider a triple of models (M, M_1, M_2) with $M \prec_{\boldsymbol{K}} M_1, M_2$ and $|M| \leq |M_1| \leq |M_2| = \lambda \geq \kappa$. Now we will use our strongly compact cardinal. An *approximation* of (M, M_1, M_2) is a triple $\boldsymbol{N} = (N^{\boldsymbol{N}}, N_1^{\boldsymbol{N}}, N_2^{\boldsymbol{N}}) \in (K_{<\kappa})^3$ such that $N^{\boldsymbol{N}} \prec M, N_\ell^{\boldsymbol{N}} \prec M_\ell, N^{\boldsymbol{N}} \prec N_\ell^{\boldsymbol{N}}$ for $\ell = 1, 2$. We will take an ultraproduct indexed by the set X below of approximations to the triple (M, M_1, M_2). Set

$$X := \{\boldsymbol{N} \in (\boldsymbol{K}_{<\kappa})^3 : \boldsymbol{N} \text{ is an approximation of } (M, M_1, M_2)\}$$

For each $\boldsymbol{N} \in X$, $AP(<\kappa)$ implies there is an amalgam of this triple. Fix $f_\ell^{\boldsymbol{N}} : N_\ell^{\boldsymbol{N}} \to N_*^{\boldsymbol{N}}$ to witness this fact. For each $(A, B, C) \in [M]^{<\kappa} \times [M_1]^{<\kappa} \times [M_2]^{<\kappa}$, define

$$G(A, B, C) := \{\boldsymbol{N} \in X : A \subset N^{\boldsymbol{N}}, B \subset N_1^{\boldsymbol{N}}, C \subset N_2^{\boldsymbol{N}}\}$$

These sets generate a κ-complete filter on X, so it can be extended to a κ-complete ultrafilter U on X; note that this ultrafilter will satisfy the appropriate generalization of fineness, namely that $G(A, B, C)$ is always a U-large set.

We will now take the ultraproduct of the approximations and their amalgam. In the end, we will end up with the following commuting diagram, which provides the amalgam of the original triple.

$$\begin{tikzcd}
 & M_1 \arrow[rr, "h_1"] & & \Pi N_1^{\boldsymbol{N}}/U \arrow[dr, "\Pi f_1^{\boldsymbol{N}}"] & \\
M \arrow[ur] \arrow[rr, "h"'] \arrow[dr] & & \Pi N^{\boldsymbol{N}}/U \arrow[ur] \arrow[dr] & & \Pi N_*^{\boldsymbol{N}}/U \\
 & M_2 \arrow[rr, "h_2"'] & & \Pi N_2^{\boldsymbol{N}}/U \arrow[ur, "\Pi f_2^{\boldsymbol{N}}"'] &
\end{tikzcd}$$

First, we use Łoś' Theorem for AECs to get the following maps:

$$h : M \to \Pi N^{\boldsymbol{N}}/U$$
$$h_\ell : M_\ell \to \Pi N_\ell^{\boldsymbol{N}}/U \qquad \text{for } \ell = 1, 2$$

h is defined by taking $m \in M$ to the equivalence class of constant function $\boldsymbol{N} \mapsto x$; this constant function is not always defined, but the fineness-like condition guarantees that it is defined on a U-large set (and h_1, h_2 are defined similarly). The uniform definition of these maps imply that $h_1 \restriction M = h \restriction M = h_2 \restriction M$.

Second, we can average the $f_\ell^{\boldsymbol{N}}$ maps to get ultraproduct maps

$$\Pi f_\ell^{\boldsymbol{N}} : \Pi N_\ell^{\boldsymbol{N}}/U \to \Pi N_*^{\boldsymbol{N}}/U$$

These maps agree on $\Pi N^{\boldsymbol{N}}/U$ since each of the individual functions do. As each M_ℓ embeds in $\Pi N_\ell^{\boldsymbol{N}}/U$ the composition of the f and h maps gives the amalgam.

There is no difficulty if one of M_0 or M_1 has cardinality $< \kappa$; many of the approximating triples will have the same first or second coordinates but this causes no harm. Similary, we get the JEP transfer if $M_0 = \emptyset$. And we can transfer disjoint amalgamation since in that case each $N_1^{\boldsymbol{N}} \cap N_2^{\boldsymbol{N}} = N^{\boldsymbol{N}}$ and this is preserved by the ultraproduct. $\dagger_{2.0.3}$

3 Syntactic Approaches

The two methods discussed in this section both depend on expanding the models of $\boldsymbol{K}$ to models in a larger vocabulary. We begin with a concept introduced in Vasey [Vasa, Definition 3.1].

Definition 3.0.1. *A* functorial expansion *of an AEC $\boldsymbol{K}$ in a vocabulary τ is an AEC $\hat{\boldsymbol{K}}$ in a vocabulary $\hat{\tau}$ extending τ such that*

1. *each $M \in \boldsymbol{K}$ has a* unique *expansion to a $\hat{M} \in \hat{\boldsymbol{K}}$,*
2. *if $f : M \cong M'$ then $f : \hat{M} \cong \hat{M}'$, and*
3. *if M is a strong substructure of M' for $\boldsymbol{K}$, then $\hat{M}$ is strong substructure of $\hat{M}'$ for $\hat{\boldsymbol{K}}$.*

This concept unifies a number of previous expansions: Morley's adding a predicate for each first order definable set, Chang adding a predicate for each $L_{\omega_1,\omega}$ definable set, T^{eq}, [CHL85] adding predicates $R_n(\mathbf{x}, y)$ for closure (in an ambient geometry) of $\mathbf{x}$, and the expansion by naming the orbits in Fraîssè model[4].

An important point in both [Vasa] and our relational presentation is that the process does not just reduce the complexity of already definable sets (as Morley, Chang) but adds new definable sets. But the crucial distinction here is that the expansion in Shelah's presentation theorem is not 'functorial' in the sense here: each model has several expansions, rather than a single expansion. That is why there is an extended proof for amalgamation transfer in Section 3.1, while the transfer in Section 3.2 follows from the following result which is easily proved by chasing arrows.

Proposition 3.0.2. *Let $\boldsymbol{K}$ to $\hat{\boldsymbol{K}}$ be a functorial expansion. $(K, \prec)$ has λ-amalgamation [joint embedding, etc.] iff $\hat{\boldsymbol{K}}$ has λ-amalgamation [joint embedding, etc.].*

[4]This has been done for years but there is a slight wrinkle in e.g. [BKL15] where the orbits are not first order definable.

3.1 Shelah's Presentation Theorem

In this section, we provide syntactic characterizations of the various amalgamation properties in a finitary language. Our first approach to these results stemmed from the realization that the amalgamation property has the same syntactic characterization for $L_{\kappa,\kappa}$ as for first order logic if κ is strongly compact, i.e., the compactness theorem hold for $L_{\kappa,\kappa}$. Combined with Boney's recognition that one could code each AEC with Löwenheim-Skolem number less than κ in $L_{\kappa,\kappa}$ this seemed a path to showing amalgamation. Unfortunately, this path leads through the trichotomy in Fact 3.1.1. The results depend directly (or with minor variations) on Shelah's Presentation Theorem and illustrate its advantages (finitary language) and disadvantage (lack of canonicity).

Fact 3.1.1 (Shelah's presentation theorem)**.** *If $\boldsymbol{K}$ is an AEC (in a vocabulary τ with $|\tau| \leq \mathrm{LS}(\boldsymbol{K})$) with Löwenheim-Skolem number $\mathrm{LS}(\boldsymbol{K})$, there is a vocabulary $\tau_1 \supseteq \tau$ with cardinality $|\mathrm{LS}(\boldsymbol{K})|$, a first order τ_1-theory T_1 and a set Γ of at most $2^{\mathrm{LS}(\boldsymbol{K})}$ partial types such that*

1. $\boldsymbol{K} = \{M'|\tau : M' \models T_1 \text{ and } M' \text{ omits } \Gamma\}$;
2. *if M' is a τ_1-substructure of N' where M', N' satisfy T_1 and omit Γ then $M'|\tau \prec_{\boldsymbol{K}} N'|\tau$; and*
3. *if $M \prec N \in \boldsymbol{K}$ and $M' \in EC(T_1, \Gamma)$ such that $M'|\tau = M$, then there is $N' \in EC(T_1, \Gamma)$ such that $M' \subset N'$ and $N'|\tau = N$.*

The exact assertion for part 3 is new in this paper; we don't include the slight modification in the standard proofs (e.g. [Bal09, Theorem 4.15]). Note that we have a weakening of Definition 3.0.1 caused by the possibility of multiple 'good' expansion of a model M.

Here are the syntactic conditions equivalent to DAP and DJEP.

Definition 3.1.2.

- Ψ *has* $< \lambda$-DAP satisfiability *iff for any expansion by constants $\boldsymbol{c}$ and all sets of atomic and negated atomic formulas (in $\tau(\Psi) \cup \{\boldsymbol{c}\}$) $\delta_1(\mathbf{x}, \boldsymbol{c})$ and $\delta_2(\mathbf{y}, \boldsymbol{c})$ of size $< \lambda$, if $\Psi \wedge \exists \mathbf{x} \left(\bigwedge \delta_1(\mathbf{x}, \boldsymbol{c}) \wedge \bigwedge x_i \neq c_j\right)$ and $\Psi \wedge \exists \mathbf{y} \left(\bigwedge \delta_2(\mathbf{y}, \boldsymbol{c}) \wedge \bigwedge y_i \neq c_j\right)$ are separately satisfiable, then so is*

$$\Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_1(\mathbf{x}, \boldsymbol{c}) \wedge \bigwedge \delta_2(\mathbf{y}, \boldsymbol{c}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

- Ψ *has* $< \lambda$-DJEP satisfiability *iff for all sets of atomic and negated atomic formulas (in $\tau(\Psi)$) $\delta_1(\mathbf{x})$ and $\delta_2(\mathbf{y})$ of size $< \lambda$, if $\Psi \wedge \exists \mathbf{x} \bigwedge \delta_1(\mathbf{x})$ and $\Psi \wedge \exists \mathbf{y} \bigwedge \delta_2(\mathbf{y})$ are separately satisfiable, then so is*

$$\Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_1(\mathbf{x}) \wedge \bigwedge \delta_2(\mathbf{y}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

We now outline the argument for $DJEP$; the others are similar. Note that $(2) \to (1)$ for the analogous result with DAP replacing DJEP has been shown by Hyttinen and Kesälä [HK06, 2.16].

Lemma 3.1.3. *Suppose that $\boldsymbol{K}$ is an AEC, $\lambda > LS(\boldsymbol{K})$, and T_1 and Γ are from Shelah's Presentation Theorem. Let Φ be the $L_{LS(\boldsymbol{K})^+,\omega}$ theory that asserts the satisfaction of T_1 and omission of each type in Γ. Then the following are equivalent:*

1. *$\boldsymbol{K}_{<\lambda}$ has $DJEP$.*
2. *$(EC(T_1,\Gamma), \subset)_{<\lambda}$ has DJEP.*
3. *Φ has $< \lambda$-$DJEP$-satisfiability.*

Proof:

$(1) \leftrightarrow (2)$: First suppose that $\boldsymbol{K}_{<\lambda}$ has DJEP. Let $M_0^*, M_1^* \in EC(T_1,\Gamma)_{<\lambda}$ and set $M_\ell := M_\ell^* \mid \tau$. By disjoint embedding for $\ell = 0,1$, there is $N \in \boldsymbol{K}$ such that each $M_\ell \prec N$. Our goal is to expand N to be a member of $EC(T_1,\Gamma)$ in a way that respects the already existing expansions.

Recall from the proof of Fact 3.1.1 that expansions of $M \in \boldsymbol{K}$ to models $M^* \in EC(T_1,\Gamma)$ exactly come from writing M as a directed union of $LS(\boldsymbol{K})$-sized models indexed by $P_\omega|M|$, and then enumerating the models in the union. Thus, the expansion of M_ℓ to M_ℓ^* come from $\{M_{\ell,\mathbf{a}} \in \boldsymbol{K}_{LS(\boldsymbol{K})} \mid \mathbf{a} \in M_\ell\}$, where $|M_{\ell,\mathbf{a}}| = \{\left(F^i_{|\mathbf{a}|}\right)^{M_\ell^*}(\mathbf{a}) \mid i < LS(\boldsymbol{K})\}$ and the functions F_n^i are from the expansion. Because M_1 and M_2 are disjoint strong submodels of N, we can write N as a directed union of $\{N_\mathbf{a} \in \boldsymbol{K}_{LS(\boldsymbol{K})} \mid \mathbf{a} \in N\}$ such that $\mathbf{a} \in M_\ell$ implies that $M_{\ell,\mathbf{a}} = N_\mathbf{a}$. Now, any enumeration of the universes of these models of order type $LS(\boldsymbol{K})$ will give rise to an expansion of N to $N^* \in EC(T_1,\Gamma)$ by setting $\left(F^i_{|\mathbf{a}|}\right)^{N^*}(\mathbf{a})$ to be the ith element of $|N_\mathbf{a}|$.

Thus, choose an enumeration of them that agrees with the original enumerations from M_ℓ^*; that is, if $\mathbf{a} \in M_\ell$, then the ith element of $|N_\mathbf{a}| = |M_{\ell,\mathbf{a}}|$ is $\left(F^i_{|\mathbf{a}|}\right)^{M_\ell^*}(\mathbf{a})$ (note that, as used before, the disjointness guarantees that there is at most one ℓ satisfying this). In other words, our expansion N^* will have

$$\mathbf{a} \in M_\ell \to \left(F^i_{|\mathbf{a}|}\right)^{M_\ell^*}(\mathbf{a}) = \left(F^i_{|\mathbf{a}|}\right)^{N^*}(\mathbf{a}) \text{ for all } i < LS(\boldsymbol{K})$$

This precisely means that $M_\ell^* \subset N^*$, as desired. Furthermore, we have constructed the expansion so $N^* \in EC(T_1,\Gamma)$. Thus, $(EC(T_1,\Gamma), \subset)_{<\lambda}$ has DJEP.

Second, suppose that $EC(T_1,\Gamma)$ has λ-DJEP. Let $M_0, M_1 \in \boldsymbol{K}$; WLOG, $M_0 \cap M_1 = \emptyset$. Using Shelah's Presentation Theorem, we can expand to $M_0^*, M_1^* \in$

$EC(T_1, \Gamma)$. Then we can use disjoint embedding to find $N^* \in EC(T_1, \Gamma)$ such that $M_1^*, M_2^* \subset N^*$. By Shelah's Presentation Theorem 3.1.1.(1), $N := N^* \mid \tau$ is the desired model.

(2) $\leftrightarrow$ (3): First, suppose that Φ has $< \lambda$-DJEP satisfiability. Let $M_0^*, M_1^* \in EC(T_1, \Gamma)$ be of size $< \lambda$. Let $\delta_0(\mathbf{x})$ be the quantifier-free diagram of $M_0^*$and $\delta_1(\mathbf{y})$be the quantifier-free diagram of M_1^*. Then $M_0^* \vDash \Phi \wedge \exists \mathbf{x} \bigwedge \delta_0(\mathbf{x})$; similarly, $\Phi \wedge \exists \mathbf{y} \bigwedge \delta_1(\mathbf{y})$ is satisfiable. By the satisfiability property, there is N^* such that

$$N^* \vDash \Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_0(\mathbf{x}) \wedge \bigwedge \delta_1(\mathbf{y}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

Then $N^* \in EC(T_1, \Gamma)$ and contains disjoint copies of M_0^* and M_1^*, represented by the witnesses of $\mathbf{x}$ and $\mathbf{y}$, respectively.

Second, suppose that $(EC(T_1, \Gamma), \subset)_{<\lambda}$ has DJEP. Let $\Phi \wedge \exists \mathbf{x} \bigwedge \delta_1(\mathbf{x})$ and $\Phi \wedge \exists \mathbf{y} \bigwedge \delta_2(\mathbf{y})$ be as in the hypothesis of $< \lambda$-DJEP satisfiability. Let M_0^* witness the satisfiability of the first and M_1^* witness the satisfiability of the second; note both of these are in $EC(T_1, \Gamma)$. By DJEP, there is $N \in EC(T_1, \Gamma)$ that contains both as substructures. This witnesses

$$\Psi \wedge \exists \mathbf{x}, \mathbf{y} \left(\bigwedge \delta_1(\mathbf{x}) \wedge \bigwedge \delta_2(\mathbf{y}) \wedge \bigwedge_{i,j} x_i \neq y_j \right)$$

Note that the formulas in δ_1 and δ_2 transfer up because they are atomic or negated atomic.

†

The following is a simple use of the syntactic characterization of strongly compact cardinals.

Lemma 3.1.4. *Assume κ is strongly compact and let $\Psi \in L_{\kappa,\omega}(\tau_1)$ and $\lambda > \kappa$. If Ψ has $< \kappa$-DJEP-satisfiability, then Ψ has $< \lambda$-DJEP-satisfiability.*

Proof: $< \lambda$-DJEP satisfiability hinges on the consistency of a particular $L_{\kappa,\omega}$ theory. If Ψ has $< \kappa$-DJEP-satisfiability, then every $< \kappa$ sized subtheory is consistent, which implies the entire theory is by the syntactic version of strong compactness we introduced at the beginning of this section.

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Obviously the converse (for $\Psi \in L_{\infty,\omega}$) holds without any large cardinals.

Proof of Theorem 1.0.1 for DAP and $DJEP$: We first complete the proof for DJEP. By Lemma 3.1.3, $< \kappa$-DJEP implies that Φ has $< \kappa$-DJEP satisfiability. By

Lemma 3.1.4, Φ has $< \lambda$-DJEP satisfiability for every $\lambda \geq \kappa$. Thus, by Lemma 3.1.3 again, $\boldsymbol{K}$ has DJEP. The proof for DAP is exactly analogous. †

3.2 The relational presentation theorem

We modify Shelah's Presentation Theorem by eliminating the two instances where an arbitrary choice must be made: the choice of models in the cover and the choice of an enumeration of each covering model. Thus the new expansion is functorial (Definition 3.0.1). However, there is a price to pay for this canonicity. In order to remove the choices, we must add predicates of arity $LS(K)$ and the relevant theory must allow $LS(K)$-ary quantification, potentially putting it in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$, where $\kappa = LS(K)$; contrast this with a theory of size $\leq 2^\kappa$ in $\mathbb{L}_{\kappa^+,\omega}$ for Shelah's version. As a possible silver lining, these arities can actually be brought down to $\mathbb{L}_{(I(\boldsymbol{K},\kappa)+\kappa)^+,\kappa^+}$. Thus, properties of the AEC, such as the number of models in the Löwenheim-Skolem cardinal are reflected in the presentation, while this has no effect on the Shelah version.

We fix some notation. Let $\boldsymbol{K}$ be an AEC in a vocabulary τ and let $\kappa = LS(\boldsymbol{K})$. We assume that $\boldsymbol{K}$ contains no models of size $< \kappa$. The same arguments could be done with $\kappa > LS(\boldsymbol{K})$, but this case reduces to applying our result to $\boldsymbol{K}_{\geq\kappa}$.

We fix a collection of compatible enumerations for models $M \in K_\kappa$. *Compatible enumerations* means that each M has an enumeration of its universe, denoted $\mathbf{m}^M = \langle m_i^M : i < \kappa \rangle$, and, if $M \cong M'$, there is some fixed isomorphism $f_{M,M'} : M \cong M'$ such that $f_{M,M'}(m_i^M) = m_i^{M'}$ and if $M \cong M' \cong M''$, then $f_{M,M''} = f_{M',M''} \circ f_{M,M'}$.

For each isomorphism type $[M]_\cong$ and $[M \prec N]_\cong$ with $M, N \in K_\kappa$, we add to τ

$$R_{[M]}(\mathbf{x}) \text{ and } R_{[M \prec N]}(\mathbf{x}; \mathbf{y})$$

as κ-ary and $\kappa 2$-ary predicates to form τ^*.

A skeptical reader might protest that we have made many arbitrary choices so soon after singing the praises of our choiceless method. The difference is that all choices are made prior to defining the presentation theory, T^*.

Once T^* is defined, no other choices are made.

The goal of the theory T^* is to recognize every strong submodel of size κ and every strong submodel relation between them via our predicates. This is done by expressing in the axioms below concerning sequences $\mathbf{x}$ of length at most κ the following properties connecting the canonical enumerations with structures in $\boldsymbol{K}$.

$$R_{[M]}(\mathbf{x}) \text{ holds iff } x_i \mapsto m_i^M \text{ is an isomorphism}$$

$R_{[M\prec N]}(\mathbf{x},\mathbf{y})$ holds iff $x_i \mapsto m_i^M$ and $y_i \mapsto m_i^N$ are isomorphisms and $x_i = y_j$ iff $m_i^M = m_j^N$

Note that, by the coherence of the isomorphisms, the choice of representative from $[M]_{\cong}$ doesn't matter. Also, we might have $M \cong M'$; $N \cong N'$; $M \prec N$ and $M' \prec N'$; but not $(M, N) \cong (M', N')$. In this case $R_{[M\prec N]}$ and $R_{[M'\prec N']}$ are different predicates.

We now write the axioms for T^*. *A priori* they are in the logic $\mathbb{L}_{(2^\kappa)^+,\kappa^+}(\tau^*)$ but the theorem states a slightly finer result. To aid in understanding, we include a description prior to the formal statement of each property.

Definition 3.2.1. *The theory T^* in $\mathbb{L}_{(I(\boldsymbol{K},\kappa)+\kappa)^+,\kappa^+}(\tau^*)$ is the collection of the following schema:*

1. If $R_{[M]}(\mathbf{x})$ holds, then $x_i \mapsto m_i^M$ should be an isomorphism.
   If $\phi(z_1, \ldots, z_n)$ is an atomic or negated atomic τ-formula that holds of $m_{i_1}^M, \ldots, m_{i_n}^M$, then include
   $$\forall \mathbf{x} \left(R_{[M]}(\mathbf{x}) \to \phi(x_{i_1}, \ldots, x_{i_n})\right)$$
2. If $R_{[M\prec N]}(\mathbf{x},\mathbf{y})$ holds, then $x_i \mapsto m_i^M$ and $y_i \mapsto m_i^N$ should be isomorphisms and the correct overlap should occur.
   If $M \prec N$ and $i \mapsto j_i$ is the function such that $m_i^M = m_{j_i}^N$, then include
   $$\forall \mathbf{x},\mathbf{y} \left(R_{[M\prec N]}(\mathbf{x},\mathbf{y}) \to \left(R_{[M]}(\mathbf{x}) \wedge R_{[N]}(\mathbf{y}) \wedge \bigwedge_{i<\kappa} x_i = y_{j_i}\right)\right)$$
3. Every κ-tuple is covered by a model.
   Include the following where $\lg(\mathbf{x}) = \lg(\mathbf{y}) = \kappa$
   $$\forall \mathbf{x} \exists \mathbf{y} \left(\bigvee_{[M]_{\cong} \in K_\kappa/\cong} R_{[M]}(\mathbf{y}) \wedge \bigwedge_{i<\kappa} \bigvee_{j<\kappa} x_i = y_{j_i}\right)$$
4. If $R_{[N]}(\mathbf{x})$ holds and $M \prec N$, then $R_{[M\prec N]}(\mathbf{x}^\circ, \mathbf{x})$ should hold for the appropriate subtuple $\mathbf{x}^\circ$ of $\mathbf{x}$.
   If $M \prec N$ and $\pi : \kappa \to \kappa$ is the unique map so $m_i^M = m_{\pi(i)}^N$, then denote $\mathbf{x}^\pi$ to be the subtuple of $\mathbf{x}$ such that $x_i^\pi = x_{\pi(i)}$ and include
   $$\forall \mathbf{x} \left(R_{[N]}(\mathbf{x}) \to R_{[M\prec N]}(\mathbf{x}^\pi, \mathbf{x})\right)$$
5. Coherence: If $M \subset N$ are both strong substructures of the whole model, then $M \prec N$.
   If $M \prec N$ and $m_i^M = m_{j_i}^N$, then include
   $$\forall \mathbf{x},\mathbf{y} \left(R_{[M]}(\mathbf{x}) \wedge R_{[N]}(\mathbf{y}) \wedge \bigwedge_{i<\kappa} x_i = y_{j_i} \to R_{[M\prec N]}(\mathbf{x},\mathbf{y})\right)$$

Remark 3.2.2. *We have intentionally omitted the converse to Definition 3.2.1.(1), namely*

$$\forall \mathbf{x} \left(\bigwedge_{\phi(z_{i_1},\ldots,z_{i_n}) \in tp_{qf}(M/\emptyset)} \phi(x_{i_1}, \ldots, x_{i_n}) \to R_{[M]}(\mathbf{x}) \right)$$

because it is not true. The "toy example" of a nonfinitary AEC–the $L(Q)$-theory of an equivalence relation where each equivalence class is countable–gives a counterexample.

For any $M^* \models T^*$, denote $M^* \restriction \tau$ by M.

Theorem 3.2.3 (Relational Presentation Theorem)**.** *1. If $M^* \models T^*$ then $M^* \restriction \tau \in K$. Further, for all $M_0 \in K_\kappa$, we have $M^* \models R_{[M_0]}(\mathbf{m})$ implies that $\mathbf{m}$ enumerates a strong substructure of M.*

2. *Every $M \in K$ has a unique expansion M^* that models T^*.*

3. *If $M \prec N$, then $M^* \subset N^*$.*

4. *If $M^* \subset N^*$ both model T^*, then $M \prec N$.*

5. *If $M \prec N$ and $M^* \models T$ such that $M^* \mid \tau = M$, then there is $N^* \models T$ such that $M^* \subset N^*$ and $N^* \mid \tau = N$.*

Moreover, this is a functorial expansion in the sense of Vasey [Vasa, Definition 3.1] and $(\mathrm{Mod}\, T^*, \subset)$ is an AEC except that it allows κ-ary relations.

Note that although the vocabulary τ^* is κ-ary, the structure of objects and embeddings from $(\mathrm{Mod}\, T^*, \subset)$ still satisfies all of the category theoretic conditions on AECs, as developed by Lieberman and Rosicky [LR]. This is because $(\mathrm{Mod}\, T^*, \subset)$ is equivalent to an AEC, namely $\boldsymbol{K}$, via the forgetful functor.

Proof: (1): We will build a $\prec$-directed system $\{M_{\mathbf{a}} \subset M : \mathbf{a} \in {}^{<\omega}M\}$ that are members of K_κ. We don't (and can't) require in advance that $M_{\mathbf{a}} \prec M$, but this will follow from our argument.

For singletons $a \in M$, taking $\mathbf{x}$ to be $\langle a : i < \kappa\rangle$ in (3.2.1.3), implies that there is $M'_a \in K_\kappa$ and $\mathbf{m}^a \in {}^\kappa M$ with $a \in \mathbf{m}^a$ such that $M \models R_{[M'_a]}(\mathbf{m}^a)$. By (1), this means that $m^a_i \mapsto m^{M'_a}_i$ is an isomorphism. Set $M_a := \mathbf{m}^a$.[5]

Suppose $\mathbf{a}$ is a finite sequence in M and $M_{\mathbf{a}'}$ is defined for every $\mathbf{a}' \subsetneq \mathbf{a}$. Using the union of the universes as the $\mathbf{x}$ in (3.2.1.3), there is some $N \in K_\kappa$ and $\mathbf{m}^{\mathbf{a}} \in {}^\kappa M$ such that

[5]We mean that we set M_a to be τ-structure with universe the range of $\mathbf{m}^a$ and functions and relations inherited from M'_a via the map above.

- $|M_{\mathbf{a}'}| \subset \mathbf{m}^{\mathbf{a}}$ for each $\mathbf{a}' \subsetneq \mathbf{a}$.
- $M \vDash R_{[N]}(\mathbf{m}^{\mathbf{a}})$.

By (3.2.1.4), this means that $M \vDash R_{\overline{M}_{\mathbf{a}'} \prec \overline{N}}(\mathbf{m}^{\mathbf{a}'}, \mathbf{m}^{\mathbf{a}})$, after some permutation of the parameters. By (2) and (1), this means that $M_{\mathbf{a}'} \prec N$; set $M_{\mathbf{a}} := \mathbf{m}^{\mathbf{a}}$.

Now that we have finished the construction, we are done. AECs are closed under directed unions, so $\cup_{\mathbf{a} \in M} M_{\mathbf{a}} \in K$. But this model has the same universe as M and is a substructure of M; thus $M = \cup_{\mathbf{a} \in M} M_{\mathbf{a}} \in K$.

For the further claim, suppose $M^* \vDash R_{[M_0]}(\mathbf{m})$. We can redo the same proof as above with the following change: whenever $\mathbf{a} \in M$ is a finite sequence such that $\mathbf{a} \subset \mathbf{m}$, then set $\mathbf{m}^{\mathbf{a}} = \mathbf{m}$ directly, rather than appealing to (3.2.1.3) abstractly. Note that $\mathbf{m}$ witnesses the existential in that axiom, so the rest of the construction can proceed without change. At the end, we have

$$\mathbf{m} = M_{\mathbf{a}} \prec \bigcup_{\mathbf{a}' \in {}^{<\omega} M} M_{\mathbf{a}'} = M$$

(2): First, it's clear that $M \in K$ has an expansion; for each $M_0 \prec M$ of size κ, make $R_{[M_0]}(\langle m_i^{M_0} : i < \kappa \rangle)$ hold and, for each $M_0 \prec N_0 \prec M$ of size κ, make $R_{[M_0 \prec N_0]}(\langle m_i^{M_0} : i < \kappa \rangle, \langle m_i^{N_0} : i < \kappa \rangle)$ hold. Now we want to show this expansion is the unique one.
Suppose $M^+ \vDash T^*$ is an expansion of M. We want to show this is in fact the expansion described in the above paragraph. Let $M_0 \prec M$. By (3.2.1.3) and (1) of this theorem, there is $N_0 \prec M$ and $\mathbf{n} \in {}^{\kappa}M$ such that

- $M^+ \vDash R_{[N_0]}(\mathbf{n})$
- $|M_0| \subset \mathbf{n}$

By coherence, $M_0 \prec \mathbf{n}$. Since $n_i \mapsto m_i^{N_0}$ is an isomorphism, there is $M_0^* \cong M_0$ such that $M_0^* \prec N_0$. Note that $T^* \models \forall \mathbf{x} R_{[M_0^*]}(\mathbf{x}) \leftrightarrow R_{[M_0]}(\mathbf{x})$. By (3.2.1.4),

$$M^+ \vDash R_{[M_0^* \prec N_0]}(\langle m_i^{M_0} : i < \kappa \rangle, \mathbf{n})$$

By (3.2.1.2), $M^+ \vDash R_{[M_0^*]}(\langle m_i^{M_0} : i < \kappa \rangle)$, which gives us the conclusion by the further part of (1) of this theorem.

Similarly, if $M_0 \prec N_0 \prec M$, it follows that

$$M^+ \vDash R_{[M_0 \prec N_0]}(\langle m_i^{M_0} : i < \kappa \rangle, \langle m_i^{N_0} : i < \kappa \rangle)$$

Thus, this arbitrary expansion is actually the intended one.

(3): Apply the uniqueness of the expansion and the transitivity of $\prec$.

(4): As in the proof of (1), we can build $\prec$-directed systems $\{M_{\mathbf{a}} : \mathbf{a} \in {}^{<\omega}M\}$ and $\{N_{\mathbf{b}} : \mathbf{b} \in {}^{<\omega}N\}$ of submodels of M and N, so that $M_{\mathbf{a}} = N_{\mathbf{a}}$ when $\mathbf{a} \in {}^{<\omega}M$. From the union axioms of AECs, we see that $M \prec N$.

(5): This follows from (3), (4) of this theorem and the uniqueness of the expansion.

Recall that the map $M^* \in \operatorname{Mod} T^*$ to $M^* \mid \tau \in \boldsymbol{K}$ is a an abstract Morleyization if it is a bijection such that every isomorphism $f : M \cong N$ in $\boldsymbol{K}$ lifts to $f : M^* \cong N^*$ and $M \prec N$ implies $M^* \subset N^*$. We have shown that this is true of our expansion. †

Remark 3.2.4. The use of infinitary quantification might remind the reader of the work on the interaction between AECs and $\mathbb{L}_{\infty,\kappa^+}$ by Shelah [She09, Chapter IV] and Kueker [Kue08] (see also Boney and Vasey [BV] for more in this area). The main difference is that, in working with $\mathbb{L}_{\infty,\kappa^+}$, those authors make use of the semantic properties of equivalence (back and forth systems and games). In contrast, particularly in the following transfer result we look at the syntax of $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$.

The functoriality of this presentation theorem allows us to give a syntactic proof of the amalgamation, etc. transfer results without assuming disjointness (although the results about disjointness follow similarly). We focus on amalgamation and give the details only in this case, but indicate how things are changed for other properties.

Proposition 3.0.2 applied to this context yields the following result.

Proposition 3.2.5. *$(K, \prec)$ has λ-amalgamation [joint embedding, etc.] iff $(\operatorname{Mod} T^*, \subset)$ has λ-amalgamation [joint embedding, etc.].*

Now we show the transfer of amalgamation between different cardinalities using the technology of this section.

Notation 3.2.6. *Fix an AEC $\boldsymbol{K}$ and the language τ^* from Theorem 3.2.3.*

1. *Given τ^*-structures $M_0^* \subset M_1^*, M_2^*$, we define the* amalgamation diagram *$AD(M_1^*, M_2^*/M_0^*)$ to be*

$$\{\phi(\boldsymbol{c}_{\mathbf{m}_0}, \boldsymbol{c}_{\mathbf{m}_1})) : \phi \text{ is quantifier-free from } \tau^* \text{ and for } \ell = 0 \text{ or } 1, \\ M_\ell^* \vDash \phi(\boldsymbol{c}_{\mathbf{m}_0}, \boldsymbol{c}_{\mathbf{m}_1}), \text{ with } \mathbf{m}_0 \in M_0^* \text{ and } \mathbf{m}_1 \in M_\ell^*\}$$

   in the vocabulary $\tau^* \cup \{c_m : m \in M_1^* \cup M_2^*\}$ where each constant is distinct except for the common submodel M_0 and $\mathbf{c_m}$ denotes the finite sequence of constants $c_{m_1}, \ldots, c_{m_n}$.

The disjoint amalgamation diagram $DAD(M^*_1, M^*_2/M^*_0)$ *is*

$$AD(M^*_1, M^*_2/M^*_0) \cup \{c_{m_1} \neq c_{m_2} : m_\ell \in M^*_\ell - M^*_0\}$$

2. *Given τ^*-structures M^*_0, M^*_1, we define the* joint embedding diagram $JD(M^*_0, M^*_1)$ *to be*

   $$\{\phi(\mathbf{c_m})) : \phi \text{ is quantifier-free from } \tau^* \text{ and for } \ell = 0 \text{ or } 1, M^*_\ell \models \phi(\mathbf{c_m}) \text{ with } \mathbf{m} \in M^*_\ell\}$$

   in the vocabulary $\tau^* \cup \{c_m : m \in M^*_1 \cup M^*_2\}$ where each constant is distinct.
   The disjoint amalgamation diagram $DJD(M^*_0, M^*_1)$ *is*

   $$AD(M^*_1, M^*_2/M^*_0) \cup \{c_{m_1} \neq c_{m_2} : m_\ell \in M^*_\ell - M^*_0\}$$

The use of this notation is obvious.

Claim 3.2.7. *Any amalgam of M_1 and M_2 over M_0 is a reduct of a model of*

$$T^* \cup AD(M^*_1, M^*_2/M^*_0)$$

Proof: An amalgam of $M_0 \prec M_1, M_2$ is canonically expandable to an amalgam of $M^*_0 \subset M^*_1, M^*_2$, which is precisely a model of $T^* \cup AD(M^*_1, M^*_2/M^*_0)$. Conversely, a model of that theory will reduct to a member of $\boldsymbol{K}$ with embeddings of M_1 and M_2 that fix M_0. †

There are similar claims for other properties. Thus, we have connected amalgamation in $\boldsymbol{K}$ to amalgamation in $(\operatorname{Mod} T^*, \subset)$ to a syntactic condition, similar to Lemma 3.1.3. Now we can use the compactness of logics in various large cardinals to transfer amalgamation between cardinals. To do this, recall the notion of an amalgamation base.

Definition 3.2.8. *For a class of cardinals $\mathcal{F}$, we say $M \in K_{\mathcal{F}}$ is a $\mathcal{F}$-*amalgamation base *($\mathcal{F}$-a.b.) if any pair of models from $K_{\mathcal{F}}$ extending M can be amalgamated over M. We use the same rewriting conventions as in Definition 1.1.1.(1), e. g., writing $\leq \lambda$-a.b. for $[LS(K), \lambda]$-amalgamation base.*

We need to specify two more large cardinal properties.

Definition 3.2.9. 1. *A cardinal κ is weakly compact if it is strongly inaccessible and every set of κ sentence in $L_{\kappa,\kappa}$ that is $< \kappa$-satisfiable is satisfiable is satisfiable*[6].

2. *A cardinal κ is measurable if there exists a κ-additive, non-trivial, $\{0, 1\}$-valued measure on the power set of κ.*

[6] At one time strong inaccessiblity was not required, but this is the current definition

3. *κ is (δ, λ)-strongly compact for $\delta \leq \kappa \leq \lambda$ if there is a δ-complete, fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$.*

   κ is λ-strongly compact if it is (κ, λ)-strongly compact.

This gives us the following results syntactically.

Proposition 3.2.10. *Suppose $LS(K) < \kappa$.*

- *Let κ be weakly compact and $M \in K_\kappa$. If M can be written as an increasing union $\cup_{i<\kappa} M_i$ with each $M_i \in K_{<\kappa}$ being a $< \kappa$-a.b., then M is a κ-a.b.*
- *Let κ be measurable and $M \in K$. If M can be written as an increasing union $\cup_{i<\kappa} M_i$ with each M_i being a λ_i-a.b., then M is a $(\sup_{i<\kappa} \lambda_i)$-a.b.*
- *Let κ be λ-strongly compact and $M \in K$. If M can be written as a directed union $\cup_{x \in P_\kappa \lambda} M_x$ with each M_x being a $< \kappa$-a.b., then M is a $\leq \lambda$-a.b.*

Proof: The proof of the different parts are essentially the same: take a valid amalgamation problem over M and formulate it syntactically via Claim 3.2.7 in $\mathbb{L}_{\kappa,\kappa}(\tau^*)$. Then use the appropriate syntactic compactness for the large cardinal to conclude the satisfiability of the appropriate theory.

First, suppose κ is weakly compact and $M = \cup_{i<\kappa} M_i \in K_\kappa$ where $M_i \in K_{<\kappa}$ is a $< \kappa$-a.b. Let $M \prec M^1, M^2$ is an amalgamation problem from K_κ. Find resolutions $\langle M_i^\ell \in K_{<\kappa} : i < \kappa \rangle$ with $M_i \prec M_i^\ell$ for $\ell = 1, 2$. Then

$$T^* \cup AD(M^{1*}, M^{2*}/M^*) = \bigcup_{i<\kappa} \left(T^* \cup AD(M_i^{1*}, M_i^{2*}/M_i^*) \right)$$

and is of size κ. Each member of the union is satisfiable (by Claim 3.2.7 because M_i is a $< \kappa$-a.b.) and of size $< \kappa$, so $T^* \cup AD(M^{1*}, M^{2*}/M^*)$ is satisfiable. Since $M^1, M^2 \in K_\kappa$ were arbitrary, M is a κ-a.b.

Second, suppose that κ is measurable and $M = \cup_{i<\kappa} M_i$ where M_i is a λ_i-a.b. Set $\lambda = \sup_{i<\kappa} \lambda_i$ and let $M \prec M^1, M^2$ is an amalgamation problem from K_λ. Find resolutions $\langle M_i^\ell \in K : i < \kappa \rangle$ with $M_i \prec M_i^\ell$ for $\ell = 1, 2$ and $\|M_i^\ell\| = \lambda_i$. Then

$$T^* \cup AD(M^{1*}, M^{2*}/M^*) = \bigcup_{i<\kappa} \left(T^* \cup AD(M_i^{1*}, M_i^{2*}/M_i^*) \right)$$

Each member of the union is satisfiable because M_i is a λ_i-a.b. By the syntactic characterization of measurable cardinals (see [CK73, Exercise 4.2.6]), the union is satisfiable. Thus, M is λ-a.b.

Third, suppose that κ is λ-strongly compact and $M = \cup_{x \in P_\kappa \lambda} M_x$ with each M_x being a $< \kappa$-a.b. Let $M \prec M^1, M^2$ be an amalgamation problem from K_λ. Find directed systems $\langle M_x^\ell \in K_{<\kappa} \mid x P_\kappa \lambda \rangle$ with $M_x \prec M_x^\ell$ for $\ell = 1, 2$. Then

$$T^* \cup AD(M^{1*}, M^{2*}/M^*) = \bigcup_{x \in P_\kappa \lambda} \left(T^* \cup AD(M_x^{1*}, M_x^{2*}/M_x^*)\right)$$

Every subset of the left side of size $< \kappa$ is contained in a member of the right side because $P_\kappa \lambda$ is $< \kappa$-directed, and each member of the union is consistent because each M_x is an amalgamation base. Because κ is λ-strongly compact, this means that the entire theory is consistent. Thus, M is a λ-a.b. †

From this, we get the following corollaries computing upper bounds on the Hanf number for the $\leq \lambda$-AP.

Corollary 3.2.11. *Suppose $LS(K) < \kappa$.*

- *If κ is weakly compact and K has $< \kappa$-AP, then K has $\leq \kappa$-AP.*
- *If κ is measurable,* cf $\lambda = \kappa$, *and K has $< \lambda$-AP, then K has $\leq \lambda AP$.*
- *If κ is λ-strongly compact and K has $< \kappa$-AP, then K has $\leq \lambda$-AP.*

Moreover, when κ is strongly compact, we can imitate the proof of [MS90, Corollary 1.6] to show that being an amalgamation base follows from being a $< \kappa$-existentially closed model of T^*. This notion turns out to be the same as the notion of $< \kappa$-universally closed from [Bon], and so this is an alternate proof of [Bon, Lemma 7.2].

4 The Big Gap

This section concerns examples of 'exotic' behavior in small cardinalities as opposed to behavior that happens unboundedly often or even eventually. We discuss known work on the spectra of existence, amalgamation of various sorts, tameness, and categoricity.

Intuitively, Hanf's principle is that if a certain property can hold for only set-many objects then it is eventually false. He refines this twice. First, if $\mathcal{K}$ a *set* of collections of structures $\boldsymbol{K}$ and $\phi_P(X, y)$ is a formula of set theory such $\phi(\boldsymbol{K}, \lambda)$ means some member of $\boldsymbol{K}$ with cardinality λ satisfies P then there is a cardinal κ_P such that for any $\boldsymbol{K} \in \mathcal{K}$, if $\phi(\boldsymbol{K}, \kappa')$ holds for some $\kappa' \geq \kappa_P$, then $\phi(\boldsymbol{K}, \lambda)$ holds for arbitrarily large λ. Secondly, he observed that if the property P is closed down for sufficiently large members of each $\boldsymbol{K}$, then 'arbitrarily large' can be replaced by 'on a tail' (i.e. eventually).

Existence: Morley (plus the Shelah presentation theorem) gives a decisive concrete example of this principle to AEC's. Any AEC in a countable vocabulary with countable Löwenheim-Skolem number with models up to $\beth_{\omega_1}$ has arbitrarily large models. And Morley [Mor65] gave easy examples showing this bound was tight for arbitrary sentences of $L_{\omega_1,\omega}$. But it was almost 40 years later that Hjorth [Hjo02, Hjo07] showed this bound is also tight for *complete*-sentences of $L_{\omega_1,\omega}$. And a fine point in his result is interesting.

We say a ϕ *characterizes* κ, if there is a model of ϕ with cardinality κ but no larger. Further, ϕ *homogeneously* [Bau74] characterizes κ if ϕ is a complete sentence of $L_{\omega_1,\omega}$ that characterizes κ, contains a unary predicate U such that if M is the countable model of ϕ, every permutation of $U(M)$ extends to an automorphism of M (i.e. $U(M)$ is a set of absolute indiscernibles.) and there is a model N of ϕ with $|U(N)| = \kappa$.

In [Hjo02], Hjorth found, by an inductive procedure, for each $\alpha < \omega_1$, a countable (finite for finite α) set S_α of complete $L_{\omega_1,\omega}$-sentences such that some $\phi_\alpha \in S_\alpha$ characterizes $\aleph_\alpha$[7]. This procedure was nondeterministic in the sense that he showed one of (countably many if α is infinite) sentences worked at each $\aleph_\alpha$; it is conjectured [Sou13] that it may be impossible to decide in ZFC which sentence works. In [BKL15], we show a modification of the Laskowski-Shelah example (see [LS93, BFKL16]) gives a family of $L_{\omega_1,\omega}$-sentences ϕ_r, such that ϕ_r homogeneously characterizes $\aleph_r$ for $r < \omega$. Thus for the first time [BKL15] establishes in ZFC, the existence of specific sentences ϕ_r characterizing $\aleph_r$.

Amalgamation: In this paper, we have established a similar upper bound for a number of amalgamation-like properties. Moreover, although it is not known beforehand that the classes are eventually downward closed, that fact falls out of the proof. In all these cases, the known lower bounds (i. e., examples where AP holds initially and eventually fails) are far smaller. We state the results for countable Löwenheim-Skolem numbers, although the [BKS09, KLH14] results generalize to larger cardinalities.

The best lower bounds for the disjoint amalgamation property is $\beth_{\omega_1}$ as shown in [KLH14] and [BKS09]. In [BKS09], Baldwin, Kolesnikov, and Shelah gave examples of $L_{\omega_1,\omega}$-definable classes that had disjoint embedding up to $\aleph_\alpha$ for every countable α (but did not have arbitrarily large models). Kolesnikov and Lambie-Hanson [KLH14] show that for the collection of all coloring classes (again $L_{\omega_1,\omega}$-definable when α is countable) in a vocabulary of a fixed size κ, the Hanf number for amalgamation (equivalently in this example disjoint amalgamation) is precisely $\beth_{\kappa^+}$ (and many of the classes have arbitrarily large models). In [BKL15], Baldwin, Koerwein, and Laskowski construct, for each $r < \omega$, a *complete* $L_{\omega_1,\omega}$-sentence ϕ^r that has disjoint 2-amalgamation up to and including $\aleph_{r-2}$; disjoint amalgamation and even amalgama-

[7] Malitz [Mal68] (under GCH) and Baumgartner [Bau74] had earlier characterized the $\beth_\alpha$ for countable α.

tion fail in $\aleph_{r-1}$ but hold (trivially) in $\aleph_r$; there is no model in $\aleph_{r+1}$.

The joint embedding property and the existence of maximal models are closely connected[8]. The main theorem of [BKS16] asserts: If $\langle \lambda_i : i \leq \alpha < \aleph_1 \rangle$ is a strictly increasing sequence of characterizable cardinals whose models satisfy JEP($< \lambda_0$), there is an $L_{\omega_1,\omega}$-sentence ψ such that

1. The models of ψ satisfy JEP($< \lambda_0$), while JEP fails for all larger cardinals and AP fails in all infinite cardinals.
2. There exist $2^{\lambda_i^+}$ non-isomorphic maximal models of ψ in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality; and
3. ψ has arbitrarily large models.

Thus, a lower bound on the Hanf number for either maximal models of the joint embedding property is again $\beth_{\omega_1}$. Again, the result is considerably more complicated for complete sentences. But [BS15b] show that there is a sentence ϕ in a vocabulary with a predicate X such that if $M \models \phi$, $|M| \leq |X(M)|^+$ and for every κ there is a model with $|M| = \kappa^+$ and $|X(M)| = \kappa$. Further they note that if there is a sentence ϕ that homogenously characterizes κ, then there is a sentence ϕ' with a new predicate B such that ϕ' also characterizes κ, B defines a set of absolute indiscernibles in the countable model, and there are models M_λ for $\lambda \leq \kappa$ such that $(|M|, |B(M_\lambda)|) = (\kappa, \lambda)$. Combining these two with earlier results of Souldatos [Sou13] one obtains several different ways to show the lower bound on the Hanf number for a complete $L_{\omega_1,\omega}$-sentence having maximal models is $\beth_{\omega_1}$. In contrast to [BKS16], all of these examples have no models beyond $\beth_{\omega_1}$.

No maximal models: Baldwin and Shelah [BS15a] have announced that the exact Hanf number for the non-existence of maximal models is the first measurable cardinal. Souldatos observed that this implies the lower bound on the Hanf number for $\boldsymbol{K}$ has joint embedding of models at least μ is the first measurable.

Tameness: Note that the definition of a Hanf number for tameness is more complicated as tameness is fundamentally a property of two variables: $\boldsymbol{K}$ is $(< \chi, \mu)$-tame if for any $N \in \boldsymbol{K}_\mu$, if the Galois types p and q over N are distinct, there is an $M \prec N$ with $|M| < \chi$ and $p \restriction M \neq q \restriction M$.

Thus, we define the *Hanf number for* $<$ *κ-tameness* to be the minimal λ such that the following holds:

if $\boldsymbol{K}$ is an AEC with $LS(\boldsymbol{K}) < \kappa$ that is $(< \kappa, \mu)$-tame for *some* $\mu \geq \lambda$, then it is $(< \kappa, \mu)$-tame for arbitrarily largeμ.

[8]Note that, under joint embedding, the existence of a maximal model is equivalent to the non-existence of arbitrarily large models

The results of [Bon] show that Hanf number for $< \kappa$-tameness is κ when κ is strongly compact[9]. However, this is done by showing a much stronger "global tameness" result that ignores the hypothesis: *every* AEC $\boldsymbol{K}$ with $LS(\boldsymbol{K}) < \kappa$ is $(< \kappa, \mu)$-tame for all $\mu \geq \kappa$. Boney and Unger [BU], building on earlier work of Shelah [She], have shown that this global tameness result is actually an equivalence (in the almost strongly compact form). Also, due to monotonicity results for tameness, the Boney results show that the Hanf number for $< \lambda$-tameness is at most the first almost strongly compact above λ (if such a thing exists). The results [BU, Theorem 4.9] put a large restriction on the structure of the tameness spectrum for any ZFC Hanf number. In particular, the following

Fact 4.0.1. *Let $\sigma = \sigma^\omega < \kappa \leq \lambda$. Every AEC $\boldsymbol{K}$ with $LS(\boldsymbol{K}) = \sigma$ is $\left(< \kappa, \sigma^{(\lambda^{<\kappa})}\right)$-tame iff κ is (σ^+, λ)-strongly compact.*

This means that a ZFC (i. e., not a large cardinal) Hanf number for $< \kappa$-tameness would consistently have to avoid cardinals of the form $\sigma^{(\lambda^{<\kappa})}$ (under GCH, all cardinals are of this form except for singular cardinals and successors of singulars of cofinality less than κ).

One could also consider a variation of a Hanf number for $< \kappa$ that requires $(< \kappa, \mu)$-tameness *on a tail* of μ, rather than for arbitrarily large μ. The argument above shows that that is exactly the first strongly compact above κ.

Categoricity: Another significant instance of Hanf's observation is Shelah's proof in [She99a] that if $\mathcal{K}$ is taken as all AEC's $\boldsymbol{K}$ with $LS_{\boldsymbol{K}}$ bounded by a cardinal κ, then there is such an eventual Hanf number for categoricity in a successor. Boney [Bon] places an upper bound on this Hanf number as the first strongly compact above κ. This depended on the results on tameness discussed in the previous paragraphs.

Building on work of Shelah [She09, She10], Vasey [Vasb] proves that if a universal class (see [She87]) is categorical in a λ at least the Hanf number for existence, then it has amalgamation in all $\mu \geq \kappa$. The he shows that for universal class in a countable vocabulary, *that satisifies amalgamation*, the Hanf number for categoricity is at most $\beth_{\beth_{(2^\omega)^+}}$. Note that the lower bound for the Hanf number for categoricity is $\aleph_\omega$, ([HS90, BK09]).

Question 4.0.2. *1. Can one calculate in ZFC an upper bound on these Hanf numbers for 'amalgamation'? Can[10] the gaps in the upper and lower bounds of the Hanf numbers reported here be closed in ZFC? Will smaller large cardinal axioms suffice for some of the upper bounds? Does categoricity help?*

2. (Vasey) Are there any techniques for downward transfer of amalgamation[11]?

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[9] This can be weakened to almost strongly compact; see Brooke-Taylor and Rosický [BTR15] or Boney and Unger [BU].

[10] Grossberg initiated this general line of research.

[11] Note that there is an easy example in [BKS09] of a sentence in $L_{\omega_1,\omega}$ that is categorical and has amalgamation in every uncountable cardinal but it fails both in $\aleph_0$.

3. *Does every AEC have a functional expansion to a $PC\Gamma$ class. Is there a natural class of AEC's with this property — e.g. solvable groups?*

4. *Can*[12] *one define in ZFC a sequence of sentences ϕ_α for $\alpha < \omega_1$, such that ϕ_α characterizes $\aleph_\alpha$?*

5. *(Shelah) If $\aleph_{\omega_1} < 2^{\aleph_0}$ $L_{\omega_1,\omega}$-sentence has models up to $\aleph_{\omega_1}$, must it have a model in $2^{\aleph_0}$? (He proves this statement is consistent in [She99b]).*

6. *(Souldatos) Is any cardinal except $\aleph_0$ characterized by a complete sentence of $L_{\omega_1,\omega}$ but not homogeneously?*

References

[Bal09] John T. Baldwin. *Categoricity*. Number 51 in University Lecture Notes. American Mathematical Society, Providence, USA, 2009. www.math.uic.edu/~ jbaldwin.

[Bal15a] J.T. Baldwin. The entanglement of model theory and set theory. lecture slides and audio `http://homepages.math.uic.edu/~jbaldwin/pub/osuapr15.pdf`, 2015.

[Bal15b] J.T. Baldwin. Formalization without foundationalism; model theory and the philosophy of mathematics practice. book manuscript available on request, 2015.

[Bau74] J. Baumgartner. The Hanf number for complete $L_{\omega_1,\omega}$-sentences (without GCH). *Journal of Symbolic Logic*, 39:575–578, 1974.

[BFKL16] J.T. Baldwin, Sy Friedman, M. Koerwien, and C. Laskowski. Three red herrings around Vaught's conjecture. *Transactions of the American Math Society*, 368:22, 2016. Published electronically: November 6, 2015.

[BK09] John T. Baldwin and Alexei Kolesnikov. Categoricity, amalgamation, and tameness. *Israel Journal of Mathematics*, 170:411–443, 2009. also at `www.math.uic.edu/~\jbaldwin`.

[BKL15] J.T. Baldwin, M. Koerwien, and C. Laskowski. Amalgamation, characterizing cardinals, and locally finite aec. *Journal of Symbolic Logic*, page 11, 2015. to appear, preprint:`http://homepages.math.uic.edu/~jbaldwin/pub/LS-2014-03-10jb.pdf`.

[BKS09] J.T. Baldwin, A. Kolesnikov, and S. Shelah. The amalgamation spectrum. *Journal of Symbolic Logic*, 74:914–928, 2009.

[12]This question seems to have originated from discussions of Baldwin, Souldatos, Laskowski, and Koerwien.

[BKS16] J.T. Baldwin, M. Koerwien, and I. Souldatos. The joint embedding property and maximal models. *Archive for Mathematical Logic*, 55:545–565, 2016.

[Bon] W. Boney. Tameness from large cardinal axioms. preprint, to appear: Journal of Symbolic Logic.

[Boo76] Boos. Infinitary compactness without strong inaccessibility. *The Journal of Symbolic Logic*, 41:33–38, 1976.

[Bou50] Nicholas Bourbaki. The architecture of mathematics. *The American Mathematical Monthly*, 57:221–232, 1950.

[BS15a] J.T. Baldwin and S. Shelah. Hanf numbers for maximality. preprint Shelah F1533, 2015.

[BS15b] J.T. Baldwin and I. Souldatos. Complete $L_{\omega_1,\omega}$-sentences with maximal models in multiple cardinalities. submitted, 2015.

[BTR15] Andrew Brooke-Taylor and Jiří Rosický. Accessible images revisited. Submitted, arXiv:1506.01986, 2015.

[BU] W. Boney and S. Unger. Large cardinal axioms from tameness in AECs. In preparation: arXiv:1509.01191.

[BV] W. Boney and S. Vasey. Categoricity and infinitary logics. preprint.

[CCHJ] B. Cody, S. Cox, J. Hamkins, and T. Johnston. The weakly compact embedding property. In preparation.

[CHL85] G.L. Cherlin, L. Harrington, and A.H. Lachlan. $\aleph_0$-categorical, $\aleph_0$-stable structures. *Annals of Pure and Applied Logic*, 28:103–135, 1985.

[CK73] C.C. Chang and H.J Keisler. *Model theory*. North-Holland, 1973. 3rd edition 1990.

[Cor92] Leo Corry. Nicolas bourbaki and the concept of mathematical structure. *Synthese*, 92:315–348, 1992.

[Cor04] Leo Corry. *Modern Algebra and the Rise of Mathematical Structures*. Birkhuser Verlag, 2004.

[Fra54] R. Fraïssé. Sur quelques classifications des systèmes de relations. *Publ. Sci. Univ. Algeria Sèr. A*, 1:35–182, 1954.

[Gro02] R. Grossberg. Classification theory for non-elementary classes. In Yi Zhang, editor, *Logic and Algebra*, pages 165–204. AMS, 2002. Contemporary Mathematics 302.

[GV06] R. Grossberg and M. VanDieren. Galois stability for tame abstract elementary classes. *Journal of Mathematical Logic*, 6:1–24, 2006.

[Hjo02] Greg Hjorth. Knight's model, its automorphism group, and characterizing the uncountable cardinals. *Journal of Mathematical Logic*, pages 113–144, 2002.

[Hjo07] Greg Hjorth. A note on counterexamples to Vaught's conjecture. *Notre Dame Journal of Formal Logic*, 2007.

[HK06] T. Hyttinen and M. Kesälä. Independence in finitary abstract elementary classes. *Ann. Pure Appl. Logic*, 143(1-3):103–138, 2006.

[HS90] B. Hart and S. Shelah. Categoricity over P for first order T or categoricity for $\phi \in \mathrm{l}_{\omega_1\omega}$ can stop at $\aleph_k$ while holding for $\aleph_0, \cdots, \aleph_{k-1}$. *Israel Journal of Mathematics*, 70:219–235, 1990.

[Jec06] T. Jech. *Set Theory: 3rd Edition*. Springer Monographs in Mathematics. Springer, 2006.

[Jón56] B. Jónsson. Universal relational systems. *Mathematica Scandinavica*, 4:193–208, 1956.

[Jón60] B. Jónsson. Homogeneous universal relational systems. *Mathematica Scandinavica*, 8:137–142, 1960.

[Ken13] Juliette Kennedy. On formalism freeness: Implementing gödel's 1946 princeton bicentennial lecture. *Bulletin of Symbolic Logic*, 19:351–393, 2013.

[KLH14] Alexei Kolesnikov and Christopher Lambie-Hanson. Hanf numbers for amalgamation of coloring classes. preprint, 2014.

[Kue08] D. W. Kueker. Abstract elementary classes and infinitary logics. *Annals of Pure and Applied Logic*, 156:274–286, 2008.

[LR] M. Lieberman and Jirí Rosický. Classification theory of accessible categories. to appear JSL.

[LS93] Michael C. Laskowski and Saharon Shelah. On the existence of atomic models. *J. Symbolic Logic*, 58:1189–1194, 1993.

[Mal68] J. Malitz. The Hanf number for complete $L_{\omega_1,\omega}$ sentences. In J. Barwise, editor, *The syntax and semantics of infinitary languages*, LNM 72, pages 166–181. Springer-Verlag, 1968.

[Mor65] M. Morley. Omitting classes of elements. In Addison, Henkin, and Tarski, editors, *The Theory of Models*, pages 265–273. North-Holland, Amsterdam, 1965.

[MS90] Michael Makkai and Saharon Shelah. Categoricity of theories in $l_{\kappa,\omega}$, with κ a compact cardinal. *Annals of Pure and Applied Logic*, 47:41–97, 1990.

[She] S. Shelah. Maximal failures of sequence locality in a.e.c. preprint on archive.

[She87] Saharon Shelah. Universal classes, part I. In J.T. Baldwin, editor, *Classification theory (Chicago, IL, 1985)*, pages 264–419. Springer, Berlin, 1987. paper 300:Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; volume 1292 of *Lecture Notes in Mathematics*.

[She99a] S. Shelah. Categoricity for abstract classes with amalgamation. *Annals of Pure and Applied Logic*, 98:261–294, 1999. paper 394. Consult Shelah for post-publication revisions.

[She99b] Saharon Shelah. Borel sets with large squares. *Fundamenta Mathematica*, 159:1–50, 1999.

[She09] S. Shelah. *Classification Theory for Abstract Elementary Classes*. Studies in Logic. College Publications `www.collegepublications.co.uk`, 2009. Binds together papers 88r, 600, 705, 734 with introduction E53.

[She10] S. Shelah. *Classification Theory for Abstract Elementary Classes: II*. Studies in Logic. College Publications `<www.collegepublications.co.uk>`, 2010. Binds together papers 300 A-G, E46, 838.

[Sou13] Ioannis Souldatos. Characterizing the powerset by a complete (Scott) sentence. *Fundamenta Mathematica*, 222:131–154, 2013.

[Vasa] Sebastien Vasey. Infinitary stability theory. xarchive: `http://arxiv.org/abs/1412.3313`.

[Vasb] Sebastien Vasey. Shelah's eventual categoricity conjecture in universal classes. preprint.