Does set theoretic pluralism entail model theoretic pluralism?

III. The entanglement of infinitary logic and set theory

Aberdeen

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July 14, 2016

[1] Thanks to J. Kennedy and A. Villeveces

The panorama of mathematics

Observation

Most theorems of mathematics are either

1. about structures of cardinality at most the continuum
   All, *finite* division rings are commutative, thus all finite Desarguesian planes are Pappian.
   or
2. do not depend on cardinality.
   All Pappian planes are Desarguean, but not conversely.

Exception

Of course, many set theoretic facts are highly dependent on the particular cardinal; but these are Combinatorial.

Today's questions

1. Does this observation hold only because we have only explored the border of Cantor's paradise?
2. Is there a classification of the kind of problems that are cardinal dependent?
3. How does it depend on the logic in which the problems are expressed?
4. Does the existence of cardinal dependent problems support Set Theoretic Pluralism?

Much of the talk is simply laying out the mathematical data – Cardinal dependent properties that are not *a priori* 'simply combinatorial'.

Epistemic Goals

Reliability

Usual position: We can't be sure mathematics is correct if there is more than one theory of sets.
But this doesn't affect results provable in ZFC?

But is reliability the only goal?

Reliability and Clarity I

Coffa places the relationship between ‘reliability and clarity’ in historical perspective:

> *[We consider] the sense and purpose of foundationalist or reductionist projects such as the reduction of mathematics to arithmetic or arithmetic to logic.*
>
> *It is widely thought that the principle inspiring such reconstructive efforts were basically a search for certainty.*

Reliability and Clarity: II

Coffa continues.

> *This is a serious error.*
> *It is true, of course, that most of those engaging in these projects believed in the possibility of achieving something in the neighborhood of Cartesian certainty for principles of logic or arithmetic on which a priori knowledge was to based.*
>
> *But it would be a gross misunderstanding to see in this belief the basic aim of the enterprise. A no less important purpose was the clarification of what was being said. . . .*
> *The search for rigor might be, and often was, a search for certainty, for an unshakable "Grund". But it was also a search for a clear account of the basic notions of a discipline.*

Clarity as a goal of model theory

The model theoretic view takes the basic notions of mathematics not to be found by coding into a universal theory such as set theory but by analyzing the actual notions of mathematics.

Example

Barwise and Eklof describe the issues around formalizing the Lefschetz principle as follows.

> *What we call Lefschetz principle has been stated by Weil as follows:*
> *“for a given value of the characteristic p, every result, involving only a finite number of points and varieties, which has been proved for some choice of universal domain remains valid without restriction; there is but one algebraic geometry of characteristic p; not one algebraic geometry for each choice of universal domain.”*
> *Weil says that a formal proof of this principle would require a ‘formal metamathematical’ characterization of the type of proposition’ to which it applies; “this would have to depend upon the ‘metamathematical’ i.e. logical analysis of all our definitions.*

Example:Seidenberg's formulation

Theorem: Minor Principle of Lefschetz

Let ϕ be a sentence in the language $\mathcal{L}_r = \{0, 1, +, -, \cdot\}$ for rings, where $0, 1$ are constants and $+, -, \cdot$ are binary functions. The following are equivalent:

1. ϕ is true in every algebraically closed field of characteristic 0.
2. ϕ is true in some algebraically closed field of characteristic 0.
3. ϕ is true in algebraically closed fields of characteristic p for arbitrarily large primes p.
4. ϕ is true in algebraically closed fields of characteristic p for sufficiently large primes p.

Barwise-Eklof formulation

Seidenberg argues that this formulation does not really reflect mathematical practice and conjectures that the Lefschetz principle really needs to be formulated in a fragment of $L_{\omega_1,\omega}$.

Barwise Eklof

“Thus, in contrast to previous mathematical formulations of the Lefschetz principle which arose from general logical considerations, [...] our starting point has been an analysis of the definitions of algebraic geometry.” They extend Seidenberg to a transfer principle in an infinitary version of finite type theory to encompass such notions as integers, affine and abstract varieties, polynomial ideals, and finitely generated extensions of the prime field.

Eklof builds on work of Feferman to construct a simpler logic than Barwise-Eklof, a many-sorted language for $L_{\infty,\omega}$.

Transition: What is model theory?

Macyintyre wrote:

> *It seems to me now uncontroversial to see the fine structure of definitions as becoming the central concern of model theory, to the extent that one could easily imagine the subject being called definability theory in the near future.*
> *While it simply true that most 'structures' of ordinary mathematics can be construed as Tarskian structures, few model theorists can have failed to notice how unappealing the formulation is to other mathematicians. . . . However, in those parts of model theory with more relevance for algebra and geometry, the set-theoretical, "rigourous" foundation seems to me to have given practically nothing, and arguably to be currently inhibiting.*

Transition:

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A simple example of this is a special case of the contrapositive of the main gap theorem:
if a theory has fewer than the maximal number of models in some uncountable cardinality, it does *not* interpret either an infinite linear order or the group of integers.
It is not the mere counting of models but study of the ordering given by elementary embedding that is crucial.

Basic Definition

Definition

T is *categorical* or *monomorphic* or *univalent* if it has exactly one model (up to isomorphism).

T is *categorical in power* κ if it has exactly one model in cardinality κ.

T is *totally categorical* if it is categorical in every infinite power.

A structure M is $\mathcal{L}$-categorical for a logic $\mathcal{L}$, if $\mathrm{Th}_{\mathcal{L}}(M) = \{\phi \in \mathcal{L}(\tau) : M \models \phi\}$ is categorical.

Kinds of discrete mathematical properties

discrete vrs continuous

1. combinatorial
2. geometric
3. algebraic

Can one make this more precise along the lines of Zilbers: trivial, modular, field-like.
Note this is more methodological question that a mathematical one.

Some 'algebraic' properties of classes of models

1. (disjoint) amalgamation
2. joint embedding
3. existence of maximal models
4. tameness and locality
5. categoricity in power
6. (dependence relations – combinatorial geometry)

4 kinds of behavior

1. The Lower Infinite
2. Eventual Behavior

4 kinds of behavior

1. The Lower Infinite
2. Eventual Behavior

Four possibilities a class might exhibit

1. good behavior on the Lower Infinite but then chaos
2. eventually good behavior
3. alternate good and bad behavior
4. always bad

Shelah thesis: God makes up his mind

Shelah's Conjecture

There is a κ such that if an AEC is categorical in one cardinal greater than κ then it is categorical in all cardinals greater than κ.

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Does God use large cardinals?

Philosophical question:

Are the properties described in this talk,
some of which depend on large cardinals,
fundamentally 'algebraic'

Shelah infinitary categoricity theorem: $L_{\omega_1,\omega}$

No Assumption of arbitrarily large models

Theorem [Shelah]

1. (For $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$) A complete $L_{\omega_1,\omega}$-sentence which has very few models in $\aleph_n$ for each $n < \omega$ is excellent.
2. (ZFC) An excellent class has models in every cardinality.
3. (ZFC) Suppose that ϕ is an excellent $L_{\omega_1,\omega}$-sentence. If ϕ is categorical in one uncountable cardinal κ then it is categorical in all uncountable cardinals.

Thus, under VWGCH (For $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$)
Categoricity below $\aleph_\omega$ implies total categoricity.

Shelah infinitary categoricity theorem: Cardinal dependence

Good up to $\aleph_{k-1}$; then chaos but models exist and have amalgamation.

Theorem (Hart-Shelah, B-Kolesnikov)

For each $2 \leq k < \omega$ there is an $L_{\omega_1,\omega}$-sentence ϕ_k such that:

1. ϕ_k is categorical in μ if $\mu \leq \aleph_{k-2}$;
2. ϕ_k is not $\aleph_{k-2}$-Galois stable;
3. ϕ_k is not categorical in any μ with $\mu > \aleph_{k-2}$;
4. ϕ_k has the disjoint amalgamation property;
5. For $k > 2$,
    1. ϕ_k is $(\aleph_0, \aleph_{k-3})$-tame; indeed, syntactic first-order types determine Galois types over models of cardinality at most $\aleph_{k-3}$;
    2. ϕ_k is $\aleph_m$-Galois stable for $m \leq k-3$;
    3. ϕ_k is not $(\aleph_{k-3}, \aleph_{k-2})$-tame.

ABSTRACT ELEMENTARY CLASSES

A class of L-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an *abstract elementary class: AEC* if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$;

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Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

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2. Closure under direct limits of $\prec_{\boldsymbol{K}}$-chains;
3. Downward Löwenheim-Skolem.

Examples

First order and $L_{\omega_1,\omega}$-classes
$L(Q)$ classes have Löwenheim-Skolem number $\aleph_1$.

Eventual behavior is determined at the Löwenheim-Skolem number

If $(\boldsymbol{K}_{\leq\kappa}, \prec_{\boldsymbol{K}})$ is an AEC, there is a unique maximal AEC that restricts to $(\boldsymbol{K}_{\leq\kappa}, \prec_{\boldsymbol{K}})$.

Close under unions.

One Completely General Result

Theorem: ($2^\lambda < 2^{\lambda^+}$) (Shelah)

Suppose $\lambda \geq \mathrm{LS}(\boldsymbol{K})$ and $\boldsymbol{K}$ is λ-categorical. For any Abstract Elementary class, if amalgamation fails in λ there are 2^{λ^+} models in $\boldsymbol{K}$ of cardinality λ^+.

Is $2^\lambda < 2^{\lambda^+}$ needed?

Is $2^\lambda < 2^{\lambda^+}$ needed?

Let $\lambda = \aleph_0$:

a Definitely not provable in ZFC: There are $L(Q)$-axiomatizable examples

   i Shelah: many models with CH, $\aleph_1$-categorical under MA

   ii Koerwien-Todorcevic: consistent to have many models under MA, $\aleph_1$-categorical from PFA.

b Independence Open for $L_{\omega_1,\omega}$

Bottom up

Two Directions in AEC

1. **Work from the bottom up**
2. **Eventual Behavior** Assume there are arbitrarily large models (and often ap and jep)

Work from the bottom up

1. **Frames**: Place very strong (superstability) conditions in a fixed cardinal and bootstrap your way up. So ap and jep are assumed (with more) in a single cardinal.
   Uses weak diamond and sometimes large cardinals
2. **Explore** Can we fill in the white spaces on the map that are nearby?
   ZFC but recently large cardinals
   1. What are the spectra of existence, jep, ap ?
   2. Are syntactic hypotheses such as 'complete sentence in $L_{\omega_1,\omega}$ significantly stronger than abstract AEC hypotheses?

This section concerns examples of ‘exotic’ behavior in small cardinalities as opposed to behavior that happens unboundedly often or even eventually. We discuss known work on the spectra of existence, amalgamation of various sorts, tameness, and categoricity.

Hanf's principle

If a certain property can hold for only set-many objects then it is eventually false.
Hanf refines this twice.

1. If $\mathcal{K}$ a *set* of collections of structures $\boldsymbol{K}$ and $\phi_P(X, y)$ is a formula of set theory such $\phi(\boldsymbol{K}, \lambda)$ means some member of $\boldsymbol{K}$ with cardinality λ satisfies P then there is a cardinal κ_P such that for any $\boldsymbol{K} \in \mathcal{K}$, if $\phi(\boldsymbol{K}, \kappa')$ holds for some $\kappa' \geq \kappa_P$, then $\phi(\boldsymbol{K}, \lambda)$ holds for arbitrarily large λ.
2. If the property P is closed down for sufficiently large members of each $\boldsymbol{K}$, then 'arbitrarily large' can be replaced by 'on a tail' (i.e. eventually).

Existence:

Any AEC in a countable vocabulary with countable Löwenheim-Skolem number with models up to $\beth_{\omega_1}$ has arbitrarily large models.

Lower bounds

1. (Morley) This bound is tight for arbitrary sentences of $L_{\omega_1,\omega}$.
2. (Hjorth) The bound is also tight for *complete*-sentences of $L_{\omega_1,\omega}$.

Hjorth's showed only one of (countably many if α is infinite) sentences worked at each $\aleph_\alpha$;
it is conjectured that it may be impossible to decide in ZFC which sentence works.
But a single sentence has been found for each $\aleph_n$
(B-Koerwien-Laskowski)

Amalgamation:upper bound on Hanf number

Theorem (B-Boney)

Let κ be strongly compact and ***K*** be an AEC with Löwenheim-Skolem number less than κ.

- If ***K*** satisfies *AP*$(< \kappa)$ then ***K*** satisfies *AP*.
- If ***K*** satisfies *JEP*$(< \kappa)$ then ***K*** satisfies *JEP*.
- If ***K*** satisfies *DAP*$(< \kappa)$ then ***K*** satisfies *DAP*.

Amalgamation: lower bound

The best lower bound for the disjoint amalgamation property is $\beth_{\omega_1}$.

1. Incomplete Sentences
   1. (B-Kolesnikov-Shelah) disjoint embedding up to $\aleph_\alpha$ for every countable α but did not have arbitrarily large models.
   2. (Kolesnikov & Lambie-Hansen) disjoint embedding up to $\aleph_\alpha$ for every countable α and arbitrarily large models.
2. (Complete Sentences) Baldwin-Koerwein-Laskowski) At least trivially the amalgamation spectrum does not have to be an interval.
   Disjoint amalgamation and even amalgamation fail in $\aleph_{r-1}$ but holds (trivially) in $\aleph_r$; there is no model in $\aleph_{r+1}$.

Joint embedding: lower bound

B-Koerwien-Souldatos

If $\langle \lambda_i : i \leq \alpha < \aleph_1 \rangle$ is a strictly increasing sequence of characterizable cardinals whose models satisfy JEP($< \lambda_0$), there is an $L_{\omega_1,\omega}$-sentence ψ such that

1. The models of ψ satisfy JEP($< \lambda_0$), while JEP fails for all larger cardinals and AP fails in all infinite cardinals.
2. There exist $2^{\lambda_i^+}$ non-isomorphic maximal models of ψ in λ_i^+, for all $i \leq \alpha$, but no maximal models in any other cardinality; and
3. ψ has arbitrarily large models.

Joint embedding: lower bound

Theorem (B-Shelah: in preparation)

There is a complete sentence of $L_{\omega_1,\omega}$ that has maximal models cofinally below the first measurable. Every larger model is extendible.

Souldatos notes that this pushes the lower bound for the Hanf number of jep up to the first measurable (and perhaps higher).

caveat

The current set theoretic hypothesis is expected to be eliminated. There is a P_0-maximal model $M \in \widehat{\boldsymbol{K}}$ of card λ if there is no measurable cardinal ρ with $\rho \leq \lambda$, $\lambda = \lambda^{<\lambda}$, and there is an $S \subseteq S^{\lambda}_{\aleph_0}$, that is stationary non-reflecting, and $\diamond_S$ holds.

GALOIS TYPES: Algebraic Form

Suppose $\boldsymbol{K}$ has the amalgamation property.

Definition

Let $M \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} \mathbb{M}$ and $a \in \mathbb{M}$. The Galois type of a over M is the orbit of a under the automorphisms of $\mathbb{M}$ which fix M.

We say a Galois type p over M is realized in N with $M \prec_{\boldsymbol{K}} N \prec_{\boldsymbol{K}} \mathbb{M}$ if $p \cap N \neq \emptyset$.

Galois vrs Syntactic Types

Syntactic types have certain natural locality properties.

- **locality** Any increasing chain of types has at most one upper bound;
- **tameness** two distinct types differ on a finite set;
- **compactness** an increasing chain of types has a realization.

The translations of these conditions to Galois types do not hold in general.

Tameness

Grossberg and VanDieren focused on the idea of studying ‘tame’ abstract elementary classes:

Definition

We say $\boldsymbol{K}$ is (χ, μ)-*tame* if for any $N \in \boldsymbol{K}$ with $|N| = \mu$ if $p, q, \in \mathcal{S}(N)$ and for every $N_0 \leq N$ with $|N_0| \leq \chi$, $p \restriction N_0 = q \restriction N_0$ then $q = p$.

Tameness-Algebraic form

Suppose ***K*** has the amalgamation property.

K is (χ, μ)-tame if for any model *M* of cardinality μ and any $a, b \in \mathcal{M}$:

Tameness-Algebraic form

Suppose $\boldsymbol{K}$ has the amalgamation property.

$\boldsymbol{K}$ is (χ, μ)-tame if for any model M of cardinality μ and any $a, b \in \mathcal{M}$:

If for every $N \prec_{\boldsymbol{K}} M$ with $|N| \leq \chi$ there exists $\alpha \in \mathrm{aut_N}(\mathcal{M})$ with $\alpha(a) = b$,

then there exists $\alpha \in \mathrm{aut_M}(\mathcal{M})$ with $\alpha(a) = b$.

Hanf number for locality

Definition

- κ is δ-measurable if there is a uniform, δ-complete ultrafilter on κ.
- κ is almost measurable if it is δ-measurable for all $\delta < \kappa$.

Theorem (Shelah)

If every AEC with Löwenheim-Skolem number less than κ is κ-local, then κ is almost measurable.

Hanf numbers of tameness

Boney 

Definition

- κ is (δ, λ)-strongly compact for $\delta \leq \kappa \leq \lambda$ if there is a δ-complete, fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$.
- κ is (δ, ∞)-strongly compact if it is (δ, λ)-strongly compact for all δ with $\delta < \kappa$.
- κ is almost strongly compact if it is (δ, ∞)-strongly compact for all $\delta < \kappa$.

Theorem (Boney-Unger)

Let κ be uncountable such that $\mu^\omega < \kappa$ for every $\mu < \kappa$. If every AEC with Löwenheim-Skolem number less than κ is κ-tame, then κ is almost strongly compact.

Consequences of Tameness

Suppose $\boldsymbol{K}$ has arbitrarily large models, amalgamation and joint embedding.

Theorem

[Grossberg-Vandieren/Lessman] If $\boldsymbol{K}$ with $\mathrm{LS}(\boldsymbol{K}) = \aleph_0$ is $\aleph_1$-categorical and $(\aleph_0, \infty)$-tame then $\boldsymbol{K}$ is categorical in all uncountable cardinals

Categoricity I

Definition/Theorem Shelah 1999

1. A Jónsson AEC is one with arbitrarily large models satisfying joint embeddding and amalgamation.
   Let $\mathcal{K}_\kappa$ with be class of Jónsson AEC's with $LS_{\boldsymbol{K}}$ bounded by a cardinal κ.
   Let H_2 be the 'second Hanf number' of $\mathcal{K}_\kappa$.

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   Let H_2 be the 'second Hanf number' of $\mathcal{K}_\kappa$.

Theorem Shelah 1999

1. If $\boldsymbol{K} \in \mathcal{K}$ is categorical in a sufficiently large successor cardinal κ^+, then it is categorical on $[H_2, \kappa^+]$.
2. Consequently, there is a cardinal μ such that if a $\boldsymbol{K} \in \mathcal{K}$ is categorical in a cardinal $\kappa^+ > \mu$ then it is categorical in all larger cardinals.

Categoricity I: Assessment

Assessment

1. in ZFC
2. assumes Jónsson class
3. only for successors
4. consequently μ is not calculable.

Categoricity II

Theorem Boney 2015

The Hanf number for categoricity in a successor is at most the first strongly compact above κ.

Boney's result depended on the results on tameness discussed in the previous slides.

Assessment

1. proves the eventual Jónsson conditions from categoricity
2. still only for successors
3. not in ZFC

Building on work of Shelah in his classification theory book; Vasey unites the frame approach with the eventual behavior approach.

Categoricity III

Theorem(Vasey) (under review)

For universal class in a countable vocabulary, the Hanf number for categoricity is at most $\beth_{\beth_{\omega_1}}$.

Assessment

1. in ZFC
2. calculates the Hanf number
3. Universal class is a strong assumption.
4. Note that the lower bound for the Hanf number for categoricity is $\aleph_\omega$.

Set Theoretic and Model Theoretic Pluralism

Eventual behavior in set theory: supercompact

Supercompactness has the flavor of ‘smoothing out the universe:

1. Singular Cardinal Hypothesis holds above a supercompact;
2. if κ is supercompact then V_κ is Σ_2-elementary in V,
3. Magidor’s characterization of supercompacts: those κ so that for all $\theta > \kappa$, there is $j : V_\eta \rightarrow V_\theta$ with $j(\mathrm{crit}(j)) = \kappa$ for some $\eta < \kappa$.

thanks to Sherwood Hachtman

Eventual behavior in set theory: measurable and supercompact

As our examples show,

The existence of ultraproducts preserving $L_{\omega_1,\omega}$ sentences enforce a uniformity above a measurable.

As does the compactness theorem for $L_{\kappa,\kappa}$ above a strong compact.

Eventual behavior in set theory affects model theory

We have given a number of examples of the entanglement of model theoretic properties of AEC with large cardinal axioms.
Can these examples be extended to find an interaction with definability in a model that shows the van den Dries remark has cardinal dependent examples?