Almost Galois ω-Stable classes

John T. Baldwin*
University of Illinois at Chicago

Paul B. Larson
Miami University
Oxford, Ohio USA

Saharon Shelah
Hebrew University of Jerusalem
Rutgers University

January 1, 2012

Abstract

Theorem. Suppose that an $\aleph_0$-presentable $\boldsymbol{K}$ is almost Galois ω-stable. If $\boldsymbol{K}$ has only countably many models in $\aleph_1$, then $\boldsymbol{K}$ is Galois ω-stable.

1 Introduction

The immediate impetus for this paper was [3] where we studied, what we called analytically presented Abstract Elementary Classes (AEC). (These classes are called by many names: $\aleph_0$-presentable classes, $PC(\aleph_0, \aleph_0)$, $PC\Gamma(\aleph_0, \aleph_0)$ or in the language of Keisler[8], PC_δ in $L_{\omega_1,\omega}$ [1]. In this paper we will most often use $\aleph_0$-presented.) 'Analytically presented' emphasizes that one can deduce from Burgess's theorem on Σ_1-equivalence relations that such a class is either ω-Galois stable, almost Galois ω-stable (no countable model has a perfect set of Galois types over it), or there is a countable model with a perfect set of Galois types over it. This topic first arose in [15]; for further background on the context see [3, 1, 14]. Our main goal is to prove that an almost Galois ω-stable $\aleph_0$-presentable AEC with only countably many models in $\aleph_1$ is Galois ω-stable. This extends earlier work by Hyttinen-Kesala [7] and Kueker [10] proving the result for sentences of $L_{\omega_1,\omega}$ with no requirement on the number of uncountable models.

Each class of models in this paper is $\aleph_0$-presented. The major tool for this investigation to expand models of set theory by predicates encoding relevant properties of the models (for some vocabulary τ) being studied). It appeared in Shelah's analysis in [12], Section VII, connecting the Hanf number for omitting families of types with well-ordering number for classes defined by omitting types.

In [11], expanding the vocabulary to describe an analysis of the syntactic types allowed the construction of a 'small' (Definition 2.2) uncountable model in an $\aleph_0$-presentable class $\boldsymbol{K}$ from an uncountable model that is small with respect to every countable fragment of $L_{\omega_1,\omega}$. In Lemma 2.8, we use this method to show that if, in addition, there are only countably many models in $\aleph_1$, then they are each small. In Section 3, we combine these two techniques to show the main theorem as stated in the abstract.

*Shelah thanks the Binational Science Foundation for partial support of this research. All authors thank Rutgers University. Currently f1179 on the Shelah Archive.

2 Small Models

{context}

Assumption 2.1. *$\boldsymbol{K} = (K, \leq_K)$ is an aec which is $\aleph_0$-presented (also known $PC_{\aleph_0}$ [15], a PC_δ over $L_{\omega_1,\omega}$ [8], $PC\Gamma_{\aleph_0,\aleph_0}$ [1]). Specifically, $\boldsymbol{K}$ is the class of reducts to τ of a class defined by a sentence $\phi \in L_{\omega_1,\omega}(\tau^+)$, where τ^+ is a countable vocabulary extending τ.*

This section deals with syntactic ($L_{\omega_1,\omega}$)-types in $\aleph_0$-presentable classes. As such the arguments are primarily syntactic and are minor variants on arguments Shelah used in [11, 13, 15]. In particular, no amalgamation assumptions are used in this section.

{small}

Definition 2.2. *1. A τ-structure M is L^*-*small *for L^* a countable fragment of $L_{\omega_1,\omega}(\tau)$ if M realizes only countably many $L^*(\tau)$-types (i.e. only countably many $L^*(\tau)$-n-types for each $n < \omega$.)*

2. *A τ-structure M is called* small *or* $L_{\omega_1,\omega}$-small *if M realizes only countably many $L_{\omega_1,\omega}(\tau)$-types.*

3. *When $X\boldsymbol{a} \subset M \in \boldsymbol{K}$, we write $\mathrm{tp}_{L^*}(\boldsymbol{a}/X, M)$ to denote the collection of L^* formulas with parameters from X that are true of $\boldsymbol{a}$ in M. (We may omit M, when the particular M is not important.)*

The following old fact plays a key role below; see page 47-48 of [1]).

{Scottsent}

Fact 2.3. *Each small model satisfies a Scott-sentence, a complete sentence of $L_{\omega_1,\omega}$.*

We quickly review the proof of this fact, as the details will be important later. For any model M over a countable vocabulary τ, we can define for each finite tuple $\overline{a}$ (of size n) from M the n-ary formulas $\phi_{\overline{a},\alpha}(\overline{x})$ ($\alpha < \omega_1$) as follows.

- $\phi_{\overline{a},0}(\overline{x})$ is the conjunction of all atomic formulas satisfied by a,
- $\phi_{\overline{a},\alpha+1}(\overline{x})$ is the conjunction of the following three formulas:
  - $\phi_{\overline{a},\alpha}(\overline{x})$
  - $\bigwedge_{c\in M} \exists x \phi_{\overline{a}c,\alpha}(\overline{x}, w)$
  - $\forall w \bigvee_{c\in M} \phi_{\overline{a}c,\alpha}(\overline{x}, w)$
- for limit $\beta < \omega_1$, $\phi_{\overline{a},\beta}(\overline{x}) = \bigwedge_{\alpha<\beta} \phi_{\overline{a},\alpha}$.

The apparent uncountability of the conjunctions in the previous definition is obviated by identifying formulas $\phi_{\overline{a}c,\alpha}$ and $\phi_{\overline{a'}c,\alpha}$ when they are equivlent in M. Working by induction on α, one gets that if M is L^*-small for each countable fragment L^* of $L_{\omega_1,\omega}(\tau)$, then the set of formulas $\phi_{\overline{a},\alpha}$ is countable for each α, letting $\overline{a}$ range over all finite tuples from M. Finally, if M is small there exists an α such that $\phi_{\overline{a},\alpha} = \phi_{\overline{a},\alpha+1}$ for all finite tuples $\overline{a}$, and then $\phi_{\langle\rangle,\alpha}$ is the Scott sentence for M. Fixing the least such α, we say that M has a Scott sentence of rank α.

We use the following fundamental result (see [8] or Theorem 5.2.5 of [1]). The notion of fragment is explained in both books. Roughly speaking, the fragment generated by a countable subset X of $L_{\omega_1,\omega}(\tau)$ is the closure of X under first order operations.

We preserve Keisler terminology in recalling the next theorem to emphasize that it deals only with the number of models and does not involve the choice of 'elementary embedding' on the class.

{keissmall}

Theorem 2.4 (Keisler). *If a PC_δ over $L_{\omega_1,\omega}$ class $\boldsymbol{K}$ has an uncountable model but less than 2^{ω_1} models of power $\aleph_1$ then for any countable fragment L^* of $L_{\omega_1,\omega}$,* every *member M of $\boldsymbol{K}$ is L^*-small. That is, each $M \in \boldsymbol{K}$ realizes only countably many L^*-types over $\emptyset$.*

{fixfrag}

Remark 2.5. Suppose a structure M is small. Then there is a countable fragment L^* of $L_{\omega_1,\omega}$ such that M is L^*-atomic. That is, for any $\boldsymbol{a} \in M$, there is $\chi_{\boldsymbol{a}}(\mathbf{x}) \in L^*$ such that for any $\lambda(\mathbf{x}) \in L_{\omega_1,\omega}$ if $M \models \lambda(\boldsymbol{a})$, then

$$M \models (\forall \mathbf{x})[\chi_{\boldsymbol{a}}(\mathbf{x}) \rightarrow \lambda(\mathbf{x})].$$

To see this, note that for any particular $\boldsymbol{a}$ there is a formula $\chi_{\boldsymbol{a}}(\mathbf{x})$ such that $M \models \chi_{\boldsymbol{a}} \rightarrow \mathrm{tp}_{L_{\omega_1,\omega}}(\boldsymbol{a}/\emptyset, M)$. Then, let L^* be the least fragment containing the $\chi_{\boldsymbol{a}}$ for an $\boldsymbol{a}$ realizing each of the countably many types realized in M.

By just changing a few words in the proof of Theorem 6.3.1 of [1], (originally in [11]) one can obtain the following result. We include the current proof for completeness; the result was implicit in [15].

{getsmall}

Theorem 2.6. *If $\boldsymbol{K}$ is an $\aleph_0$-presentable AEC and some model of cardinality $\aleph_1$ is L^*-small for every countable τ-fragment L^* of $L_{\omega_1,\omega}$, then $\boldsymbol{K}$ has a $L_{\omega_1,\omega}(\tau)$-small model M' of cardinality $\aleph_1$.*

Proof. Add to τ^+ a binary relation $<$, interpreted as a linear order of M with order type ω_1. Using that M realizes only countably many types in any τ-fragment, define a continuous increasing chain of countable fragments L_α for $\alpha < \aleph_1$ such that each type in $L_\alpha(\tau)$ that is realized in M is a formula in $L_{\alpha+1}$. Extend the similarity type further to τ' by adding new $2n+1$-ary predicates $E_n(x, \mathbf{y}, \mathbf{z})$ and $n+1$-ary functions f_n. Let M satisfy $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ if and only if $\boldsymbol{a}$ and $\mathbf{b}$ realize the same L_α-type and let f_n map M^{n+1} into the initial ω elements of the order, so that $E_n(\alpha, \boldsymbol{a}, \mathbf{b})$ if and only if $f_n(\alpha, \boldsymbol{a}) = f_n(\alpha, \mathbf{b})$. Note:

1. $E_n(\beta, \mathbf{y}, \mathbf{z})$ refines $E_n(\alpha, \mathbf{y}, \mathbf{z})$ if $\beta > \alpha$;
2. $E_n(0, \boldsymbol{a}, \mathbf{b})$ implies $\boldsymbol{a}$ and $\mathbf{b}$ satisfy the same quantifier free τ-formulas;
3. If $\beta > \alpha$ and $E_n(\beta, \boldsymbol{a}, \mathbf{b})$, then for every c_1 there exists c_2 such that $E_{n+1}(\alpha, c_1\boldsymbol{a}, c_2\mathbf{b})$ and
4. f_n witnesses that for any $a \in M$ each equivalence relation $E_n(a, \mathbf{y}, \mathbf{z})$ has only countably many classes.

All these assertions can be expressed by an $L_{\omega_1,\omega}(\tau')$ sentence χ. Let L^* be the smallest τ'-fragment containing $\chi \wedge \phi$. Now by the Lopez-Escobar bound on $L_{\omega_1,\omega}$ definable well-orderings, Theorem 5.3.8 of [1], there is a structure N of cardinality $\aleph_1$ satisfying $\chi \wedge \phi$ such that there is an infinite decreasing sequence $d_0 > d_1 > \ldots$ in N. For each n, define $E^+_n(\mathbf{x}, \mathbf{y})$ if for some i, $E_n(d_i, \mathbf{x}, \mathbf{y})$. Now using 1), 2) and 3) prove by induction on the quantifier rank of μ that $N \models E^+_n(\boldsymbol{a}, \mathbf{b})$ implies $N \models \mu(\boldsymbol{a})$ if and only if $N \models \mu(\mathbf{b})$ for every $L_{\omega_1,\omega}(\tau)$-formula μ. (Suppose the result holds for all n and all θ with quantifier rank at most γ. Suppose $\mu(\boldsymbol{a})$ is $(\exists x)\mu'(\boldsymbol{a}, x)$ with $n = \lg(\boldsymbol{a})$, μ' has quantifier rank γ, and $E^+_n(\boldsymbol{a}, \mathbf{b})$. So for some i, $E_n(d_i, \boldsymbol{a}, \mathbf{b})$ and for some a, $N \models \mu'(\boldsymbol{a}, a)$. By the conditions on the E_n, there is a b such that $E_{n+1}(d_{i+1}, \boldsymbol{a}, a, \mathbf{b}, b)$. By induction we have $N \models \mu'(\mathbf{b}, b)$ and so $N \models \mu(\mathbf{b})$.) For each n, $E_n(d_0, \mathbf{x}, \mathbf{y})$ refines $E^+_n(\mathbf{x}, \mathbf{y})$ and by 4) $E_n(d_0, \mathbf{x}, \mathbf{y})$ has only countably many classes; so N is small. $\square_{2.6}$

Definition 2.7. *We say a countable structure is* extendible *if it has an $L_{\omega_1,\omega}$-elementary extension to an uncountable model.*

{vfimpcomp}

Lemma 2.8. *If $\boldsymbol{K}$ has a model in $\boldsymbol{K}_{\aleph_1}$ that is not $L_{\omega_1,\omega}(\tau)$-small then*

1. *there are at least $\aleph_1$ complete sentences of $L_{\omega_1,\omega}(\tau)$ which are satisfied by uncountable models in $\boldsymbol{K}$;*

2. *$\boldsymbol{K}$ has uncountably many small models in $\aleph_1$ that satisfy distinct complete sentences of $L_{\omega_1,\omega}(\tau)$;*

3. *$\boldsymbol{K}$ has uncountably many extendible models in $\aleph_0$.*

Proof. Suppose that M is a model in $\boldsymbol{K}$ with cardinality $\aleph_1$ which is not $L_{\omega_1,\omega}(\tau)$-small. Let M^+ be an expansion of M to a τ^+-structure satisfying ψ. We construct a sequence of τ^+-structures $\{N^+_\alpha : \alpha < \omega_1\}$ each with cardinality $\aleph_1$ and an increasing continuous family of countable fragments $\{L'_\alpha : \alpha < \omega_1\}$ of $L_{\omega_1,\omega}(\tau)$ such that:

1. $L'_0(\tau)$ is first order logic on τ.
2. All the $N^+_\alpha \models \phi$.
3. All $N^+_\alpha \restriction \tau$ are $L_{\omega_1,\omega}(\tau)$-small.
4. χ_α is the $L_{\omega_1,\omega}(\tau)$-Scott sentence of N_α.
5. $L'_{\alpha+1}(\tau)$ is the smallest fragment of $L_{\omega_1,\omega}(\tau)$ containing $L'_\alpha(\tau) \cup \{\neg\chi_\alpha\}$.
6. For limit δ, $L'_\delta(\tau) = \bigcup_{\alpha<\delta} L'_\alpha(\tau)$.
7. For each α, $N_\alpha \equiv_{L'_\alpha(\tau)} M$.

Using the $L_{\omega_1,\omega}(\tau)$-sentence θ_α, $\bigwedge\{\mu : M \models \mu \text{ and } \mu \in L'_\alpha(\tau)\}$, the construction is straightforward. We construct N_α by applying Theorem 5.3.8 of [1] to $\phi \wedge \theta_\alpha$ (as in the proof of Theorem 2.6). Meeting the remaining conditions of the construction is easy.

Now the N_α are pairwise non-isomorphic since each satisfies a distinct complete sentence of $L_{\omega_1,\omega}(\tau)$. And each has a countable elementary submodel with respect to $L'_\alpha \cup \phi$. So there are $\geq \aleph_1$ non-isomorphic extendible models in $\aleph_0$ as well. $\square_{2.8}$

This result leads to a corollary regarding the content of the Vaught conjecture.

Corollary 2.9. *If ϕ is a counterexample to the Vaught conjecture then ϕ has $\aleph_1$* extendible *countable models.* {corvc}

Proof. If not, Lemma 2.8 implies every uncountable model of ϕ is small. But Harnik and Makkai [5] proved that a counterexample to Vaught's conjecture has a model in $\aleph_1$ that is not small. $\square_{2.9}$

Remark 2.10. Clearly, if $\boldsymbol{K}$ has only countably many models in $\aleph_1$ then $\boldsymbol{K}$ has at most $\aleph_0$ non-isomorphic extendible countable models. The converse of Lemma 2.8 asserts if that $\boldsymbol{K}$ has uncountably many extendible countable models then it has a non-small model in $\aleph_1$. The *extendibility* is essential. Example 2.1.1 of [2] is a sentence of $L_{\omega_1,\omega}$ giving rise to an AEC with a particular notion of $\prec_{\boldsymbol{K}}$, which has a.p. and j.e.p. and is $\aleph_1$-categorical. There are $2^{\aleph_0}$ countable models but only one of them is extendible. {iran1}

We pause to connect this analysis with a related but subtly distinct procedure.

Definition 2.11. *1.* **Morley's Analysis** {Manal}

(a) *Let $L^{\mathbf{K}}_0$ be the set of first order τ-sentences.*

(b) *Let $L^{\mathbf{K}}_{\alpha+1}$ be the smallest fragment generated by $L^{\mathbf{K}}_\alpha$ and the sentences of the form $(\exists \mathbf{x}) \bigwedge p(\mathbf{x})$ where p is an $L^{\mathbf{K}}_\alpha$-type realized in a model in $\boldsymbol{K}$.*

(c) *For limit δ, $L^{\mathbf{K}}_\delta = \bigcup_{\alpha<\delta} L^{\mathbf{K}}_\alpha$.*

2. $\boldsymbol{K}$ is scattered if and only if for each $\alpha < \omega_1$, $L_\alpha^{\mathbf{K}}$ is countable.

Recall Morley's theorem, which is key to his approach to Vaught's conjecture.

{getscat}

Theorem 2.12 (Morley)**.** *If $\boldsymbol{K}$ is the class of models of a sentence in $L_{\omega_1,\omega}$ that has less than $2^{\aleph_0}$ models of power $\aleph_0$ then $\boldsymbol{K}$ is scattered.*

Remark 2.13. We cannot conclude that $\boldsymbol{K}$ is scattered from just counting models in $\aleph_1$, even from the hypothesis that $\boldsymbol{K}$ is $\aleph_1$-categorical. Again, Example 2.1.1 of [2] (Example 2.10) is $\aleph_1$-categorical and has joint embedding for $\prec_{\boldsymbol{K}}$. But there are $2^{\aleph_0}$ first order types that give models that are not even first order mutually embeddible and the class $\boldsymbol{K}$ is not scattered.

Remark 2.14. The arguments of Morley and Shelah have different goals. Scattered is a condition on all models of an (in the interesting case for the Vaught conjecture) an *incomplete* sentence in $L_{\omega_1,\omega}$. The Shelah argument contracts $\boldsymbol{K}$ to a smaller class where every model is small and thus finds a $\boldsymbol{K}' \subset \boldsymbol{K}$ that is small; the hard part is to make sure $\boldsymbol{K}'$ has an uncountable model. In the most used case, $\boldsymbol{K}$ and *a fortiori* $\boldsymbol{K}'$ is $\aleph_1$-categorical.

3 Almost Galois Stability

{ags}

In this section we assume $\boldsymbol{K}$ is an $\aleph_0$-presented AEC that has amalgamation and JEP for countable models. Amalgamation and jep do not appear directly in the proofs but are used in background arguments establishing that the notion of Galois type (e.g. Definition 8.7 of [1]) is well-behaved.

Because there are two notions of weak-stability in the literature of AEC ([7, 14], we call the following notion *almost Galois ω-stability*.

Definition 3.1. *$\boldsymbol{K}$ is* almost Galois ω-stable *if for every countable model M, E_M does not have a perfect set of equivalence classes.*

Note that this condition implies for every countable model M, E_M has at most $\aleph_1$ equivalence classes. (For this, recall that we are working with an $\aleph_0$-presentable class. Thus in the terminology of [3], $\boldsymbol{K}$ is analytically presented. This implies that Galois types are defined by an analytic equivalence relation and we can apply Burgess's theorem that such an equivalence relation has $\aleph_0, \aleph_1$ or a perfect set of classes.) In the presence of CH, the converse fails, so we choose the more generally applicable definition. Amalgamation easily allows one to show.

Lemma 3.2. *If $\boldsymbol{K}$ is almost Galois ω-stable and satisfies amalgamation and jep, then there is a Galois-saturated model M in $\aleph_1$.*

{notsofast}

Example 3.3. Let $\boldsymbol{K}$ be the set of structures in the language with a single equivalence relation that have infinitely many elements in each class and exactly $\aleph_0$ classes. Let $\prec_{\boldsymbol{K}}$ be first order elementary submodel. This is an $\aleph_0$-presentable class. Note that there are $\aleph_0$ models in $\aleph_1$ (given by the number of classes that have cardinality $\aleph_0$ and $\aleph_1$ respectively). Fix a countable submodel M_0 of M, the $\aleph_1$-saturated model of $\boldsymbol{K}$. Note that either with constants naming M_0, or with a predicate for M_0, there are $2^{\aleph_0}$ non-isomorphic countable models in $\boldsymbol{K}$. First for each $f : \omega \to \omega$, let M_f have $f(n)$ elements in $E(M_f, a_n) - E(M_0, a_n)$. Let $c_f(n) = |\{a/E : a \in M_0 \wedge |E(M_f, a) - E(M_0, a)| = f(n)\}|$. Then M_f and M'_f are isomorphic isomorphisms fixing M_0 setwise iff $c_f = c_{f'}$.

Suppose $c_g(n)$ is identically 1. There are $2^{\aleph_0}$ models which are pairwise isomorphic by isomorphisms which fix M_0 setwise but are not pairwise isomorphic by isomorphisms which fix M_0 pointwise.y

But in this example, there is no pair of models N, N' with $M_0 \subset N \subset N'$,that are isomorphic by isomorphisms which fix M_0 setwise but not isomorphic by isomorphisms which fix M_0 pointwise.

This example illustrates the importance of studying truth in a particular model when we do not have a monster model that is homogenous over sets. Each of the M_f satisfy different sentences in $L_{\omega_1,\omega}(\tau')$ but if they are embedded in the saturated model M, this is not reflected in the types that elements of M_f realize *in the sense of* M.

{nice2dec}

Definition 3.4. *Suppose a model M with cardinality $\aleph_1$ is filtered by an uncountable chain of countable models $\langle M_\alpha : \alpha < \omega_1\rangle$, which are pairwise isomorphic such that $M_\alpha \prec_{\boldsymbol{K}} M$ and $M_\alpha \prec_{L_{\omega_1,\omega}(\tau)} M$. Fix automorphisms F_α of M mapping M_0 onto M_α and let $F_{\alpha,\beta} = F_\beta \circ F_\alpha^{-1}$ mapping M_α onto M_β. We say that $\langle M_\alpha, F_\alpha\rangle$ is a* nice decomposition *of M.*

{mthm}

Theorem 3.5. *Suppose that $\boldsymbol{K}$ is almost Galois ω-stable. If $\boldsymbol{K}$ has only countably many models in $\aleph_1$, then $\boldsymbol{K}$ is Galois ω-stable.*

Proof. Supposing that the conclusion of the theorem fails, let M be the Galois-saturated model of cardinality $\aleph_1$. By Lemma 2.8, M is τ-small; that is, there is countable fragment of L^* of $L_{\omega_1,\omega}$ which contains a Scott sentence ϕ for M.

By hypothesis, M realizes uncountably many Galois types over some countable substructure $M_0 \in \mathbf{K}$ satisfying ϕ.

We fix notation for the remainder of the paper. Let $\mathbf{c} = \langle c_i : i < \omega\rangle$ enumerate M_0 and call τ' the extension of τ by adding these constant symbols. Now there are three cases:

1. There are uncountably many $L^*(\tau')$-types realized in M for some countable fragment $L^*(\tau')$ of $L_{\omega_1,\omega}(\tau')$.
2. For every countable $L^*(\tau')$ of $L_{\omega_1,\omega}(\tau')$ only $\aleph_0$ $L^*(\tau')$-types are realized in M.
   (a) M is $L_{\omega_1,\omega}(\tau')$-small and so for some countable fragment $L^*(\tau')$, M has a Scott sentence in $L^*(\tau')$.
   (b) M is not $L_{\omega_1,\omega}(\tau')$-small.

Case 1: In this case there exists a perfect set of syntactic types in this fragment of the expanded language and thus a perfect set of Galois types over M_0. This contradicts the almost Galois ω-stability of $\mathbf{K}$.

Case 2a). We show $\boldsymbol{K}$ is Galois ω-stable.

{2a}

Lemma 3.6. *If M is small for $L_{\omega_1,\omega}(\tau')$ and Galois saturated then M realizes only countably many Galois types over any countable model.*

Proof. By Remark 2.5, there is a countable fragment L^* of $L_{\omega_1,\omega}(\tau')$ which contains a Scott sentence for M. Suppose some $\boldsymbol{a}, \mathbf{b} \in M^n$ realize the same $L^*(\tau')$-type over M_0 in $(M, \mathbf{c})$ (i.e. wrt to truth in M). Then this type is given by a formula in L^* (by smallness). There exists a countable $M' \in \mathbf{K}$ such that $M_0ab \subset M' \prec_{L^*} M$ so there exists an automorphism g_δ of M', fixing M_0 pointwise and with $g_\delta(\boldsymbol{a}) = \mathbf{b}$. Then, $\mathrm{gtp}(\boldsymbol{a}/M_0; M') = \mathrm{gtp}(\mathbf{b}/M_0; M')$. But every automorphism of M' extends to an automorphism of M (since M is Galois-saturated) so $\mathrm{gtp}(\boldsymbol{a}/M_0; M) = \mathrm{gtp}(\mathbf{b}/M_0; M)$. So for any $\boldsymbol{a}$ the L^*-type of $\boldsymbol{a}$ over M_0 defines the Galois type and therefore only countably many Galois types over M_0 are realized in M. $\square_{3.6}$

Case 2b). We show this case is impossible.

{2b}

Lemma 3.7. *Suppose $\boldsymbol{K}$ is $\aleph_0$-presentable, almost Galois ω-stable, has only countably many models in $\aleph_1$ and let M be the Galois-saturated model of $\boldsymbol{K}$ in $\aleph_1$. If M is small for every countable fragment of $L_{\omega_1,\omega}(\tau')$ then M is $L_{\omega_1,\omega}(\tau')$-small.*

Proof. Fix a continuous, $\subseteq$-increasing sequence $\overline{M} = \langle M_\alpha : \alpha < \omega\rangle$ of substructures of M in $\mathbf{K}$ satisfying ϕ with M_0 enumerated by $\mathbf{c}$. Then M is nicely decomposed by some set $\overline{F} = \{F_\alpha : \alpha < \omega_1\}$ such that each F_α is an automorphism of M sending M_0 to M_α.

Let $H(\theta)$, for a large enough θ be such that each of M (with the expanded language interpreted as in Lemma 2.6), $\overline{M}$ and $\overline{F}$ are elements of $H(\theta)$. Let $\mathcal{A}$ be the expansion of $H(\theta)$ to a vocabulary $\tau^+ \cup \{\epsilon\}$. The vocabulary τ^+ includes, τ', the additional symbols of Lemma 2.6 to describe types, and a new binary predicate R. The interpretation of R in the expansion of M is $R = \{\langle c, \gamma\rangle : c \in M_\gamma\}$.

Let $\langle X_\alpha : \alpha < \omega_1\rangle$ be a properly $\subseteq$-increasing continuous chain of countable elementary submodels of $\mathcal{A}$. In particular $\omega_1^{\mathcal{A}} \in X_0$ and for every $\alpha < \omega_1$ there is a countable ordinal $\beta \in X_{\alpha+1} - X_\alpha$.

For each $\alpha < \omega_1$, let P_α be the transitive collapse of X_α, and let $\rho_\alpha\colon X_\alpha \to P_\alpha$ be the corresponding collapsing mapping. Then $\rho_\alpha(\omega_1) = \omega_1^{P_\alpha}$ is the ordinal $X_\alpha \cap \omega_1$.

Either by an iteration of ultrapowers of models of set theory as in Lemma 1.5 of [3] or by iterating the construction in Theorem 2.1 of [6], we construct a family of countable models of set theory. The model theoretic argument is below.

There is a mathematical issue here. Hutchinson's theorem is for ZFC, not ZFC^o. Can one just take a model of ZFC containing $H(\theta)$?

For each $\alpha < \omega_1$, there is an elementary extension of P_α to a model P'_α (with corresponding elementary embedding $\chi_\alpha\colon P_\alpha \to P'_\alpha$) such that

1. the critical point of χ_α is $\omega_1^{P_\alpha}$, so $\omega_1^{P_\alpha}$ is an initial segment of $\omega_1^{P'_\alpha}$;
2. $\omega_1^{P'_\alpha}$ is ill-founded, {goodclub}
3. in V, there is a continuous increasing ω_1-sequence $\langle t^\alpha_\gamma : \gamma < \omega_1\rangle$ consisting of elements of $\omega_1^{P'_\alpha}$

Item (3) above implies in particular that each $\omega_1^{P'_\alpha}$ is uncountable.

To construct the P'_α, we rely on the following (paraphrase) of Theorem 2.1 of [6]. Hutchinson built on work of Keisler and Morley [9]; Enayat provides a useful source on this work in [4]. We develop an alternative technique using in iterated ultrafilters is [3].

{Hutch}

Fact 3.8. *Let $\mathcal{B}$ be a countable model of ZFC and c a regular cardinal in $\mathcal{B}$. Then there is a countable elementary extension $\mathcal{C}$ of $\mathcal{B}$ such that each a such that $\mathcal{B} \models a \in c$ is fixed but c is enlarged and there is a least new element of $\mathcal{C}$.*

Define P^α_i for $i < \omega_1$. $P^\alpha_0 = P_\alpha$; $P^\alpha_{\gamma+1}$ is obtained by applying 3.8 to P^α_γ with c as $\omega_1^{P^\gamma}$. Take unions at limits. $P'_\alpha = \bigcup_{\gamma<\omega_1} P^\alpha_\gamma$. The t^α_γ are the $\omega_1^{P^\alpha_\gamma}$.

Recall that M is the union of the continuous $\subseteq$-increasing chain $\langle M_\alpha : \alpha < \omega_1\rangle$. It follows then for each $\alpha < \omega_1$, that $M_{\omega_1^{P_\alpha}} = \rho_\alpha(M) \subset P_\alpha$, and that $M_{\omega_1^{P_\alpha}}$ has cardinality $\aleph_1$ in P_α. For each $\alpha < \omega_1$, let $N_\alpha = \chi_\alpha(M_{\omega_1^{P_\alpha}})$. Then N_α has cardinality $\aleph_1$ in P'_α.

In the argument for Lemma 2.6 replace the appeal to Theorem 5.3.8 of [1] with the observation that N_α is not well-founded by construction. The rest of the argument for Lemma 2.6 shows that, in V, each N_α is small for $L_{\omega_1,\omega}(\tau')$. Nevertheless, P'_α thinks N_α is not $L_{\omega_1,\omega}(\tau')$-small.

Since $\overline{M}$ is a sequence indexed by ω_1 in V (or in X_α), $\chi_\alpha(\rho_\alpha(\overline{M}))$ is a sequence indexed by $\omega_1^{P'_\alpha}$ in P'_α. So, in P'_α, for each element t of its ω_1, there is a t-th element of the sequence, which we denote M^α_t. Furthermore, in P'_α, $\chi_\alpha(\rho_\alpha(\overline{F}))$ is a set$\{F^\alpha_t : t \in \omega_1^{P'_\alpha}\}$ consisting of automorphisms of N_α, such that each $F^\alpha_t \in P'_\alpha$ is an automorphism of N_α sending M_0 to M^α_t. Each F^α_t is then an automorphism of N_α in V also.

Since we are assuming that there are only countably many models in $\mathbf{K}$ of cardinality $\aleph_1$, there exists a stationary set $S \subseteq \omega_1$ such that N_{α_0} and N_{α_1} are isomorphic (in V) for all α_0, α_1 in S. Fix for a moment a pair of elements α_0, α_1 of S and an isomorphism $\pi\colon N_{\alpha_0} \to N_{\alpha_1}$. Applying item (3) above and the continuity (in the sense of P'_{α_j}, for $j = 0, 1$) of the sequences $\langle M^{\alpha_0}_t : t \in \omega_1^{P'_{\alpha_0}}\rangle$ and $\langle M^{\alpha_1}_t : t \in \omega_1^{P'_{\alpha_1}}\rangle$, there must be $t_0 \in \omega_1^{P'_{\alpha_0}}$ and $t_1 \in \omega_1^{P'_{\alpha_1}}$ such that π maps $M^{\alpha_0}_{t_0}$ onto $M^{\alpha_1}_{t_1}$. To see this, start with $\gamma_0 = 0$ and, for each $n \in \omega$, let γ_{n+1} be large enough so that $M^{\alpha_1}_{t^{\alpha_1}_{\gamma_{n+1}}}$ contains $\pi[M^{\alpha_0}_{t^{\alpha_0}_{\gamma_n}}]$ and $M^{\alpha_0}_{t^{\alpha_0}_{\gamma_{n+1}}}$ contains $\pi^{-1}[M^{\alpha_1}_{t^{\alpha_1}_{\gamma_n}}]$. Then let $s_0 = t^{\alpha_0}_{\sup_{n\in\omega}\gamma_n}$ and let $s_1 = t^{\alpha_1}_{\sup_{n\in\omega}\gamma_n}$. Note that by the continuity in item (3), the s_j are in the respective P'_{α_j}, for $j = 0, 1$. So by the continuity in P'_{α_j} of $M^{\alpha_j}_t$, $M^{\alpha_j}_{s_j} = \bigcup_{n<\omega} M^{\alpha_j}_{t^{\alpha_j}_{\gamma_n}}$. Then $(F^{\alpha_1}_{s_1})^{-1} \circ \pi \circ F^{\alpha_0}_{s_0}$ is an isomorphism of N_{α_0} and N_{α_1} fixing M_0 setwise.

Finally, we show that for each α_0 such an isomorphism is impossible for arbitrarily large α_1 with $\alpha_0 < \alpha_1$.

For each α, the model P'_α thinks that N_{α_1} is small for every countable fragment of $L_{\omega_1,\omega}(\tau')$ but not $L_{\omega_1,\omega}(\tau')$-small. Thus, from the point of view of P'_α, there is no ordinal t such that $\phi_{\overline{a},t}(\overline{x}) \equiv \phi_{\overline{a},t+1}(\overline{x})$ for all finite tuples $\overline{a}$ of N_α. For each well-founded ordinal γ of P'_α (i.e $\gamma < \omega^{P'_\alpha} = \omega_1 \cap X_\alpha$ by item 1), and each finite tuple $\overline{a}$ of N_α, P'_α sees the same formula $\phi_{\overline{a},\gamma}(\overline{x})$ that the true universe V does, which means that the Scott sentence for N_α has rank at least $\omega_1 \cap X_\alpha$ (and slightly more than this, in fact).

Now choose $\alpha_0, \alpha_1 \in S$ such that α_1 is greater than the Scott rank (in V) of N_{α_0}. Since permuting the constants c_i in terms of their enumeration of M_0 has no effect on the rank of the Scott sentence for N_{α_1}, there cannot be then an isomorphism of N_{α_0} and N_{α_1} fixing M_0 setwise.

References

[1] John T. Baldwin. *Categoricity*. Number 51 in University Lecture Notes. American Mathematical Society, 2009. www.math.uic.edu/~ jbaldwin.

[2] J.T. Baldwin, T. Hyttinen, and M. Kesälä. Beyond first order logic: From number of structures to structure of numbers part II. *Bulletin of Iranian Math. Soc.*, 2011. Available online at `http://www.iranjournals.ir/ims/bulletin/` in press.

[3] J.T. Baldwin and Paul Larson. Iterated elementary embeddings and the model theory of infinitary logic. preprint.

[4] A. Enayat. On certain elementary extensions of models of set theory. *Transactions of the American Mathematical Society*, 283:705–715, 1984.

[5] V. Harnik and M. Makkai. A tree argument in infinitary model theory. *Proceeedings of the American Mathematical Society*, 67:309–314, 1977.

[6] John E. Hutchinson. Elementary extensions of countable models of set theory. *J. Symbolic Logic*, 41:139–145, 1976.

[7] T. Hyttinen and M. Kesälä. Independence in finitary abstract elementary classes. *Ann. Pure Appl. Logic*, 143(1-3):103–138, 2006.

[8] H.J Keisler. *Model theory for Infinitary Logic*. North-Holland, 1971.

[9] H.J Keisler and M. Morley. Elementary extensions of models of set theory. *Israel Journal of Mathematics*, 5:331–348, 1968.

[10] D. W. Kueker. Abstract elementary classes and infinitary logics. *Annals of Pure and Applied Logic*, 156:274–286, 2008.

[11] S. Shelah. Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$. *Israel Journal of Mathematics*, 20:127–148, 1975. paper 48.

[12] S. Shelah. *Classification Theory and the Number of Nonisomorphic Models*. North-Holland, 1978.

[13] S. Shelah. Classification theory for nonelementary classes. I. the number of uncountable models of $\psi \in L_{\omega_1\omega}$ part A. *Israel Journal of Mathematics*, 46:3:212–240, 1983. paper 87a.

[14] S. Shelah. *Classification Theory for Abstract Elementary Classes*. Studies in Logic. College Publications `www.collegepublications.co.uk`, 2009. Binds together papers 88r, 600, 705, 734 with introduction E53.

[15] Saharon Shelah. Classification of nonelementary classes II, abstract elementary classes. In J.T. Baldwin, editor, *Classification theory (Chicago, IL, 1985)*, pages 419–497. Springer, Berlin, 1987. paper 88: Proceedings of the USA–Israel Conference on Classification Theory, Chicago, December 1985; volume 1292 of *Lecture Notes in Mathematics*.