Hanf numbers for Extendibility and related phenomena

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Abstract

This paper contains portions of Baldwin's talk at the Institute for Research in Fundamental Sciences Set Theory and Model Theory Conference in October 2015 and a detailed proof that in a suitable extension of ZFC, there is a complete sentence of $L_{\omega_1,\omega}$ that has maximal models in cardinals cofinal in the first measurable cardinal and, of course, never again.

In this paper we discuss two theorems whose proofs depend on extensions of the Fraïssé method. We prove here the Hanf number for the property that every model of a (complete) sentence of $L_{\omega_1,\omega}$ with cardinality κ is extendible[1] is (modulo some mild set theoretic hypotheses that we remove in [BS18]) the first measurable cardinal. And we outline the description of an explicit $L_{\omega_1,\omega}$-sentence ϕ_n characterizing $\aleph_n$ for each n. We provide some context for these developments as outlined in the lectures at IPM[2].

The phrase 'Fraïssé construction' has taken many meanings in the over 60 years since the notion was born [Fra54] (and earlier in an unpublished thesis). There are two major streams. We focus here on variants in the original construction, which usually use the standard notion of substructure. We don't deal here directly with 'Hrushovski constructions' where a specialized notion of strong submodel varying with the case plays a central role. An annotated bibliography of developments of the Hrushovski variant until 2009 appears at [Bal].

The first variant we want to consider is the vocabulary. Fraïssé worked with a *finite, relational* vocabulary. While model theory routinely translates between functions and their graphs and there is usually little distinction between finite and countable vocabularies; in the infinite vocabulary case such extensions for the Fraïssé construction yield weaker but still very useful consequences. The second is a distinction in goal: the construction of *complete* sentences of $L_{\omega_1,\omega}$ (equivalently studying the *atomic* models of a complete first order theory) rather than constructing $\aleph_0$-categorical theories. This second shift raises new questions about the cardinality of the resulting models. The result in Section 4 pins down more precisely the existence spectra for *complete* sentences of $L_{\omega_1,\omega}$. Section 3 expresses the role of large cardinal axioms in more algebraic terms. Rephrased, it says that, consistently with the existence of a measurable cardinal, there is a

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[1]We say $\boldsymbol{K}$ is *universally extendible in* κ if $M \in \boldsymbol{K}$ with $|M| = \kappa$ has a proper $\prec_{\boldsymbol{K}}$-extension in the class. Here, this means has an ∞, ω-elementary extension.

[2]Due to our tardiness in preparing this paper it could be included in the special volume dedicated to the 2015 conference.

nicely defined (by a complete sentence of $L_{\omega_1,\omega}$) class of models that has non-extendible (maximal) models cofinally below the first measurable. The previous upper bound for such behavior was $\aleph_{\omega_1}$.

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1 Hanf numbers and Spectrum functions in infinitary logic

Recent years have brought a number of investigations of the spectrum (cardinals in which the phenomenon occurs) for various phenomena and various sorts of infinitary definable classes. Some of the relevant phenomena are existence, amalgamation, joint embedding, maximal models etc. The class might be might be defined as an abstract elementary class, the models of a (complete) sentence of $L_{\omega_1,\omega}$, etc.

Hanf observed [Han60] that for any property $P(\boldsymbol{K}, \lambda)$, where $\boldsymbol{K}$ ranges over a *set* of classes of models, there is a cardinal $\kappa = H(P)$ such that κ is the least cardinal satisfying: if $P(\boldsymbol{K}, \lambda)$ holds for some $\lambda \geq \kappa$ then $P(\boldsymbol{K}, \lambda)$ holds for arbitrarily large λ. $H(P)$ is called the Hanf number of P. e.g. $P(\boldsymbol{K}, \lambda)$ might be the property that $\boldsymbol{K}$ has a model of power λ.

Morley [Mor65] showed for an arbitrary sentence of $L_{\omega_1,\omega}(\tau)$ the Hanf number for existence is $\beth_{\omega_1}$ when τ is countable (More generally, it is $\beth_{(2^{|\tau|})^+}$. [She78]); the situation for *complete* sentences is much more complicated. Knight [Kni77] found the first complete sentence characterizing ω_1 (i.e. has a model in ω_1 but no larger) by building on the construction of many non-isomorphic $\aleph_1$-like linear orderings. Hjorth found, by a procedure generalizing the Fraïssé -construction, for each $\alpha < \omega_1$, a set S_α (finite for finite α) of complete $L_{\omega_1,\omega}$-sentences[3] such that some $\phi_\alpha \in S_\alpha$ characterizes $\aleph_\alpha$. It is conjectured [Sou13] that it may be impossible to decide in ZFC which sentence works. Baldwin, Koerwien, and Laskowski [BKL16] show a modification of the Laskowski-Shelah example (see [LS93, BFKL16]) gives a family of $L_{\omega_1,\omega}$-sentences ϕ_r, which characterize $\aleph_r$ for $r < \omega$. In Section 4 we sketch the new notion of n-disjoint amalgamation that plays a central role in [BKL16].

Further results by [BKS09, KLH16, BKS16], where the hypothesis are weakened to allow incomplete sentences of $L_{\omega_1,\omega}$ or even AEC (Abstract Elementary Classes $(\boldsymbol{K}, \leq)$ where the properties of *strong substructure*, $\leq$ are defined axiomatically) are placed in context in [BB17]. Analogous results were proved earlier for *incomplete* sentences by [BKS16] who code certain bipartite graphs in way that determine specific inequalities between the cardinalities of the two parts of the graph; in this case all models have cardinality less than $\beth_{\omega_1}$.

All the exotica mentioned here and described in more detail in [BB17] occurs below $\beth_{\omega_1}$. Baldwin and Boney [BB17] have shown that the Hanf number for amalgamation is no more than the first strongly compact cardinal. This immense gap motivated the current paper. We show that for the case of universally extendable (every model has a proper extension), there is a smaller gap. There is a *complete* sentence of $L_{\omega_1,\omega}$ which has a maximal model in cardinals cofinal in the first measurable (if such exists), but no larger maximal model. Is the same true of amalgamation? That is, can amalgamation eventually behave very differently than it does in small cardinalities? At the end of this paper we point to the only known example where amalgamation (for a complete $L_{\omega_1,\omega}$-sentence) holds on an initial segment then fails, then holds again; then there are no larger models.

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[3]Inductively, Hjorth shows at each α and each member ϕ of S_α one of two sentences, χ_ϕ, χ'_ϕ, works as $\phi_{\alpha+1}$ for $\aleph_{\alpha+1}$.

2 Disjoint Amalgamation

2.1 Classes determined by finitely generated structures

The original Fraïssé construction took place in a *finite relational* vocabulary and the resulting infinite structure was $\aleph_0$-categorical for a first order theory. We explore here several ways to construct a countable atomic model for a first order theory and thus a complete sentence in $L_{\omega_1,\omega}$.

Recall (e.g. chapter 7 of [Bal09]) that the models of a complete sentence of $L_{\omega_1,\omega}(\tau)$ are the reducts to τ of the atomic (every finite sequence realizes a principal type) models of a complete first order theory in a vocabulary τ' extending τ. We discuss classes determined by a countable set of finitely generated models. In Sections 3 and 4, we describe the examples of such classes used to prove our main results.

Definition 2.1.1. *Fix a countable vocabulary τ (possibly with function symbols). Let $(\boldsymbol{K}_0, \subseteq)$ denote a* countable *collection of* finite τ-structures and let $(\widehat{\boldsymbol{K}}, \subseteq)$ *denote the abstract elementary class containing all structures M such that every* finitely generated *substructure of M is in $\boldsymbol{K}_0$.*

These classes have syntactic characterizations.

Lemma 2.1.2.   *1. $\widehat{\boldsymbol{K}}$ is defined by an $L_{\omega_1,\omega}$-sentence ϕ.*

2. If $\boldsymbol{K}_0$ is closed under substructure then ϕ may be taken universal [Mal69].

3. $(\boldsymbol{K}_0, \subset)$ satisfies the axioms for AEC (except for unions under chains.)

While traditional Fraïssé classes are closed under substructure and produce $\aleph_0$-categorical first order structures, which are *uniformly* locally finite, the search for atomic models creates [Hjo07, BFKL16, BKL16, BS17] does always not require closure under substructure and produces structures a generic which is locally finite but not uniformly so. In Section 3, we expand the subject further by using countable collections of *finitely generated* structures as the 'Fraïssé class'.

Definition 2.1.3. *Fix a countable vocabulary τ (possibly with function symbols). Let $(\boldsymbol{K}_0, \leq)$ denote a countable collection of finite τ-structures with $(\widehat{\boldsymbol{K}}, \leq)$ as in Definition 2.1.1.*

1. A model $M \in \widehat{\boldsymbol{K}}$ is rich *or $\boldsymbol{K}_0$-homogeneous if for all A and B in $\boldsymbol{K}_0$ with $A \leq B$, every embedding $f : A \to M$ extends to an embedding $g : B \to M$. We denote the class of rich models in $\widehat{\boldsymbol{K}}$ as $\mathbf{R}$.*

2. The model $M \in \widehat{\boldsymbol{K}}$ is generic *if M is rich and M is an increasing union of a countable chain of finitely generated substructures, each of which is in $\boldsymbol{K}_0$.*

3. We let $\mathbf{R}$ denote the subclass of $\widehat{\boldsymbol{K}}$ consisting of rich models.

In the examples considered here the generic models will always be countable.

Definition 2.1.4. *An AEC $(\boldsymbol{K}, \leq)$ has $(< \lambda, 2)$-disjoint amalgamation if for any $A, B, C \in \boldsymbol{K}$ with cardinality $< \lambda$ and A strongly embedded in B, C, there is a D and strong embedding of B, C into D that agree on A and such that the intersection of their ranges is their image of A.*

$\boldsymbol{K}$ has 2-amalgamation if the ranges of the embedding are allowed to intersect outside of $f(A)$.

$\boldsymbol{K}$ has the joint embedding property *(JEP) if any two models can be embedded in some larger D.*

Fraïssé 's theorem asserted that if a class of finite models in a finite relational language is closed under substructure and satisfies AP and JEP then there is a generic model whose theory is $\aleph_0$-categorical and quantifier eliminable. The following extension of Fraïssé's theorem is well-known [Hod93] and the proof is essentially the same.

Lemma 2.1.5. *Suppose τ is countable and $\boldsymbol{K}_0$ is a countable class of finite* or countable τ-*structures that satisfies* 2- *amalgamation, in particular* $(\leq \aleph_0, 2)$-*disjoint amalgamation, and JEP, then*

1. *A $\boldsymbol{K}_0$-generic (and so rich) τ-structure M exists.*

2. *if $\boldsymbol{K}_0$ is closed under substructure, the generic is ultra-homogeneous (every isomorphism between arbitrary finitely generated substructures extends to an automorphism).*

A key distinction from the Fraïssé situation is that in the first order case $\widehat{K}$ doesn't really play a role while in the infinitary case it is an important immedidary between the finitely generated structures and $\mathbf{R}$. Fraïssé passes to the first order theory of the generic since it is $\aleph_0$-categorical in *first order logic*. In our more general situation the generic may be $\aleph_0$-categorical only in $L_{\omega_1,\omega}$. The Scott sentence of the rich model gives the $L_{\omega_1,\omega}$ sentence we study. As noted at the beginning of this section we may regard the models as reducts of atomic models of a first order theory. Thus $\widehat{K}$ may have arbitrarily large models while $\mathbf{R}$ does not; this holds of some examples in [Hjo07, BFKL16, BKL16].

Corollary 2.1.6. *Suppose $(\boldsymbol{K}_0, \leq)$ satisfies the hypotheses of Lemma 2.1.5. Fix $\lambda \geq \aleph_0$. If $\widehat{K}$ has $(\leq \lambda, 2)$-amalgamation and has at most countably many isomorphism types of countable structures, then every $M \in \widehat{K}$ of power λ can be extended to a rich model $N \in \widehat{K}$, which is also of power λ.*

Proof. Given $M \in \widehat{K}$ of power λ, construct a continuous chain $\langle M_i : i < \lambda \rangle$ of elements of $\widehat{K}$, each of size λ. At a given stage $i < \lambda$, focus on a specific finite substructure $A \subseteq M_i$ and a particular finite extension $B \in \widehat{K}$ of A. If there is an embedding of B into M_i over A, $M_{i+1} = M_i$. If not, we may assume $B \cap M_i = A$. Let M_{i+1} be the disjoint amalgamation of M_i and B over A. As there are only λ-possible extensions, we can, by iterating, organize this construction so that $N = \bigcup\{M_i : i < \lambda\}$ is rich. $\square_{2.1.6}$

Crucially, in Section 3.2.23 the class $\hat{K}$ under consideration will not satisfy two amalgamation even with finite models; but there will be amalgamation of free structures with finite.

2.2 Atomic Models of First order theories

We discuss here classes generated by finite (not finitely generated) structures. Suppose a generic τ-model M exists. When is M an atomic model of its first-order τ-theory? As remarked in Section 2 of [BKL16] this second condition has nothing to do with the choice of embeddings on the class $\boldsymbol{K}_0$, but rather with the choice of vocabulary. The following condition is needed when, for some values of n, $\boldsymbol{K}_0$ has infinitely many isomorphism types of structures of size n

We denote the class of atomic models of a complete first order theory by $\mathbf{At}$.

Definition 2.2.1. *A class $\boldsymbol{K}_0$ of finite structures in a countable vocabulary is* separable *if, for each $A \in \boldsymbol{K}_0$ and enumeration $\boldsymbol{a}$ of A, there is a quantifier-free first order formula $\phi_{\boldsymbol{a}}(\mathbf{x})$ such that:*

- $A \models \phi_{\boldsymbol{a}}(\boldsymbol{a})$ *and*

- *for all $B \in \boldsymbol{K}_0$ and all tuples $\mathbf{b}$ from B, $B \models \phi_A(\mathbf{b})$ if and only if $\mathbf{b}$ enumerates a substructure B' of B and the map $\boldsymbol{a} \mapsto \mathbf{b}$ is an isomorphism.*

In practice, we will apply the observation that if for each $A \in \boldsymbol{K}_0$ and enumeration $\boldsymbol{a}$ of A, there is a quantifier-free formula $\phi'_{\boldsymbol{a}}(\mathbf{x})$ such that there are only finitely many $B \in \boldsymbol{K}_0$ with cardinality $|A|$ that under some enumeration $\mathbf{b}$ satisfy $\phi'_{\boldsymbol{a}}(\mathbf{b})$, then $\boldsymbol{K}_0$ is separable.

Lemma 2.2.2. *[BKL16] Suppose τ is countable and $\boldsymbol{K}_0$ is a class of finite τ-structures that is closed under substructure, satisfies amalgamation, and JEP, then a $\boldsymbol{K}_0$-generic (and so rich) model M exists. Moreover, if $\boldsymbol{K}_0$ is separable, M is an atomic model of $Th(M)$. Further, $\mathbf{R} = \mathbf{At}$, i.e., every rich model N is an atomic model of $Th(M)$.*

Proof: Since the class $\boldsymbol{K}_0$ of finite structures is separable it has countably many isomorphism types, and thus a $\boldsymbol{K}_0$-generic M exists by the usual Fraïssé construction. To show that M is an atomic model of $Th(M)$, it suffices to show that any finite tuple $\boldsymbol{a}$ from M can be extended to a larger finite tuple $\mathbf{b}$ whose type is isolated by a complete formula. Coupled with the fact that M is $\boldsymbol{K}_0$-locally finite, we need only show that for any finite substructure $A \leq M$, any enumeration $\boldsymbol{a}$ of A realizes an isolated type. Since every isomorphism of finite substructures of M extends to an automorphism of M, the formula $\phi_{\boldsymbol{a}}(\mathbf{x})$ isolates $\mathrm{tp}(\boldsymbol{a})$ in M.

The final sentence follows since any two rich models are $L_{\infty,\omega}$-equivalent. $\square_{2.2.2}$

3 Hanf number for All Models Extendible

We say an abstract elementary class (the models of a complete sentence in $L_{\omega_1,\omega}$) is *universally extendible in* κ if every model of cardinality κ has a proper strong extension ($L_{\infty,\omega}$-elementary extension). In this section we prove the following theorem.

Theorem 3.0.1. *There is a complete sentence ϕ of $L_{\omega_1,\omega}$ that has arbitrarily large models. But under reasonable set theoretic conditions (specified below), we show that for arbitrarily large $\lambda < \mu$, where μ is the first measurable cardinal, and unboundedly many λ if there is no measurable, ϕ has a maximal (with respect to substructure, which in this case means $\prec_{\infty,\omega}$) model with cardinality between λ and 2^λ.*

We remove in [BS18] the set theoretic hypotheses by use of a black box as in [She] but at the cost of weakening the freeness of the P_0-maximal model; see Remark 3.3.13.

If $|M|$ is at least the first measurable μ, then for any $\aleph_1$-complete non-principal ultrafilter $\mathcal{D}$ on μ, $M^\mu/\mathcal{D}$ is a proper extension of M. This holds because we can find an $f \in M^\mu$ which hits each element $a \in M$ at most once. Thus the equivalence class of f cannot be that of any constant map on M (since $\mathcal{D}$ is non-principal). On the other hand, by the Łos theorem for $L_{\omega_1,\omega}$, since $\mathcal{D}$ is $\aleph_1$-complete, the ultrapower is an $L_{\omega_1,\omega}$-elementary extension of M. Thus, we have shown the Hanf number for extendability is at most μ:

Lemma 3.0.2. *If μ is measurable, for any $\phi \in L_{\mu,\mu}$, in particular in $L_{\omega_1,\omega}$, no model of cardinality $\geq \mu$ is maximal.*

The proof of the converse (Theorem 3.0.1) fills the remainder of this section. If we only demand the result for an arbitrary sentence of $L_{\omega_1,\omega}$ there are easy examples. An example in terms of ω-models (which is easily reinterpreted into $L_{\omega_1,\omega}$) appears in [Mag16].

Example 3.0.3. Here is a sketch of such an example. Consider a class $\boldsymbol{K}$ of 3-sorted structures where: P_0 is a set, P_1 is a boolean algebra of subsets of P_1 (given by an extensional binary E) and P_2 is just a set; $\{F_n : n < \omega\}$ is a family of unary functions which assigns to each $c \in P_2$, a sequence $F_n(c) \in P_1$. Demand: $\bigwedge_n F_n(c) = F_n(d)$ implies $d = c$. Let $\psi \in L_{\omega_1,\omega}$ axiomatize $\boldsymbol{K}$. Now M is a maximal model of

$K = \mod(\psi)$ with cardinality λ if $\lambda <$ first measurable and $\lambda^\omega = \lambda$; $|P_0^M| = \lambda$, $P_1^M = \mathcal{P}(P_0^M)$, and P_2^M codes each sequence in $^\omega(P_1^M)$ via the F_n.

If $M \not\preceq_{\omega_1,\omega} N$, there must be an element $a^* \in P_0^N - P_0^M$. But then $D = \{b \in P_1^M : E(a^*, b)\}$ is a non-principal $\aleph_1$-incomplete ultrafilter on $\lambda = P_0^M$ since λ is not measurable. Since each countable sequence of subsets of P_0^M is coded as $\langle F_n^M(c) : n < \omega \rangle$ for some $c \in P_2^M$, there is a c such $M \models \neg(\exists x) \bigwedge E(x, F_n(c))$, while $N \models \bigwedge E(a, F_n(c))$. This contradicts $M \prec_{\omega_1,\omega} N$.

However, ψ is not complete. There are $2^{\aleph_0}$ 2-types over empty set given for each $X \subset \omega$ via (c, d) realizes p_X iff $X = \{n : F_n(c) \cap F_n(d) \neq \emptyset\}$. This precludes completeness since a minor variant of Scott's characterization of countable models shows that a sentence ψ is complete if and only if only countably many $L_{\omega_1,\omega}$-types over $\emptyset$ are realized in models of ψ. In Section 3.2 we modify this example to obtain a complete sentence.

3.1 Some preliminaries on Boolean Algebras

There are a number of slightly different jargons among set theorists, model theorists, category theorists, and Boolean algebraists. In this section we will spell some of them out, indicate some translations, specify our notation, and prove some properties of Boolean algebras that will be used in the proof.

An ultrafilter *of* a Boolean algebra B is a maximal filter (i.e. a subset of B that is closed up, under intersection and contains either a or a^- – the complement of a). An ultrafilter *on* a set X is a subset of its power set and so is an ultrafilter of the Boolean algebra $\mathcal{P}(X)$.

We begin with some basic properties of independence in Boolean algebras. A key fact is an equivalence of two notions of independence on countably infinite Boolean algebras that disappears in the uncountable. That is, a countable Boolean algebra is $\aleph_0$-categorical if and only it is free on countably many generators in the sense of 3.1.1 if and only if it is generated by an independent set in the sense of 3.1.2. But this equivalence fails in the uncountable.

Definition 3.1.1.   *1. For $X \subseteq B$ and B a Boolean algebra, $\overline{X} = X_B = \langle X \rangle_B$ be the subalgebra of B generated by X.*

2. *A set Y is* independent *(or* free*) from X over an ideal $\Im$ in a Boolean algebra B if and only if for any Boolean-polynomial $p(v_0, \ldots, v_k)$ (that is not identically 0), and any $a \in \langle X \rangle_B - \Im$, and distinct $y_i \in Y$, $p(y_0, \ldots, y_k) \wedge a \notin \Im$.*

3. *Such an independent Y is called a* basis *for $\langle X \cup Y \cup I \rangle$ over $\langle X \cup I \rangle$.*

There is no requirement that $\Im$ be contained in X. Observe the following:

Observation 3.1.2.   *1. If $\Im$ is the 0 ideal, (i.e., Y is independent from X), the condition becomes: for any $a \in \langle X \rangle_B - \{0\}$, $B \models p(y_0, \ldots, y_k) \wedge a > 0$. That is, every finite Boolean combination of elements of Y meets each non-zero $a \in \langle X \rangle_B$.*

2. *Let π map B to $B/\Im$. If 'Y is independent from X over $\Im$' then the image of Y is free from the image of X (over $\emptyset$) in $B/\Im$. Conversely, if $\pi(Y)$ is independent over $\pi(X)$ in $B/\Im$, for any Y' mapping by π to $\pi(Y)$, Y' is independent from X over $\Im$.*

   So, if X is empty, the condition 'Y is independent over $\Im$' implies the image of Y is an independent subset of $B/\Im$.

3. *If a set Y is* independent *(or* free*) from X over an ideal $\Im$ in a Boolean algebra B and Y_0 is a subset of Y, then $Y - Y_0$ is* independent *(or* free*) from $X \cup Y_0$ ($\langle X \cup Y_0 \rangle_B$) over the ideal $\Im$ in the Boolean algebra B.*

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From left to right in item 2) note that if for any nontrivial term[4] $\sigma(\mathbf{v})$, and any $\mathbf{y} \in Y$ there is an a with $\sigma(\mathbf{y}) \wedge a \notin I$ then $\pi(\sigma(\mathbf{y}) \wedge a)$ is not 0 in B/I. Conversely, if some $\sigma(\pi(\mathbf{y})) \neq 0$ then if $\mathbf{y}'$ isc in $\pi^{-1}(\pi(\mathbf{y}))$, then $\sigma(\mathbf{y}') \notin I$.

The notion of independence above does *not* satisfy the axioms for a matroid (combinatorial geometry); exchange fails. It is an independence system (the empty set can be considered independent and subsets of independent sets are independent. But given X and Y independent with $|Y| > |X|$, in general there is no guarantee that some element of $Y - X$ can be added to X and maintain independence. But, see Lemma 3.1.9.

The contrast between the notion of independence above and the following is crucial for the construction here.

Definition 3.1.3. *Let X, Y be sets of elements from a Boolean algebra of sets. X is independent (free) over Y if for any infinite A that is a non-trivial finite Boolean combination of elements of X and any B which is a non-empty finite Boolean combination of elements of Y, $A \cap B$ and $A^c \cap B$ are each infinite.*

Both kinds of independence will occur in the models in Section 3.3. In $\boldsymbol{K}_0$, there is a homomorphism from P_1^M into $\mathcal{P}(P_0^M)$ that does *not* transfer from 'independence in the boolean algebra sense' to 'set independence'. In $\boldsymbol{K}_2$, there is an isomorphism from P_2^M into $\mathcal{P}(P_1^M)$ that correctly transfers 'independence'.

Definition 3.1.4. *A pushout consists of an object P along with two morphisms $i_1 : X \to P$ and $i_2 : Y \to P$ which complete a commutative square with two given morphisms f and g mapping an object Z to X and Y respectively such that any morphisms j_1, j_2 from X and Y to a Q must factor through P.*

In [FG90], it is shown by a category theoretic argument that for distributive lattices the abstract embeddings into the pushout (Notation 3.1.6) are 1-1 and if A is a Boolean algebra, the images of the embedding intersect in image of A. Thus the variety of Boolean algebras has *disjoint*[5] amalgamation.

We now connect this notion with our version of independence in Definition 3.1.1.

Lemma 3.1.5. *Let $D = A \otimes_C B$ be the Boolean algebra obtained as the pushout (Definition 3.1.4) of A and B over C. Suppose $\Im$ is an ideal of D and $I_2 \subset A - C$ such that $\langle I_2 \rangle_D \cap \Im = \emptyset$ and $B - \Im \neq \emptyset$. Then I_2 is independent from B over $\Im$.*

Proof. Fix a Boolean-polynomial $p(v_0, \ldots, v_k)$ (that is not identically 0), and suppose for contradiction there is a $d \in B - \Im$ and distinct $y_i \in I_2$ with $p(y_0, \ldots, y_k) \wedge d \in \Im$. Any morphisms f_1, f_2 from A, B to any D' must factor through D. In particular, we can extend $\Im \cap A$ and $\Im \cap B$ to maximal ideals omitting $p(y_0, \ldots y_k) \in A$ and $d \in B$; the resulting map from D that commutes with the induced f_i to the 2-element algebra sends all of $\Im$ and so $p(y_0, \ldots, y_k) \wedge d$, but not d or $p(y_0, \ldots y_k)$ to 0. But there is no such homomorphism. $\square_{3.1.5}$

There are several sets of confusing terminology arising from various perspectives in the study of Boolean algebra and misleading analogies with, for example, the study of groups. For example, consider the notion of the product of two Boolean algebras, A, B. That is, the structure on the *Cartesian (direct) product* of A and B, obtained by defining the operations coordinate-wise. Note that, while there are isomorphic copies of A and B in the product, the natural injections into $A \times \{0\}$, $\{0\} \times \{B\}$, map to ideals not sub-Boolean algebras.

A generalization of the dual of the direct product operation is often called the '*free product with amalgamation*'; we will call the free amalgamation of Boolean algebras B and A over C the one that is obtained by the pushout/free product construction of Notation 3.1.6; it is the coproduct in the category-theoretic language.

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[4]A trivial term (or polynomial) is one which is identically 0.
[5]Called strong in [FG90].

Notation 3.1.6. *Let $C \subseteq A, B$ be Boolean algebras. The disjoint amalgamation $D = A \otimes_C B$ is the Boolean algebra obtained as the pushout [AB11] of A and B over C. It is characterized internally by the following condition. For $a \in A - C, b \in B - C$: $a \leq b$ in D if and only if there is a $c \in C$ with $a < c < b$ (and symmetrically). D is generated as a Boolean algebra by $A \cup B$ where A and B are sub-Boolean algebras of D.*

We will distinguish certain subsets of our models in terms of atoms.

Notation 3.1.7. *An* atom *is an element a of a Boolean algebra such that for every c either $c \wedge a = a$ or $c \wedge a = 0$. For any Boolean algebra B, $\mathrm{At}(B)$ denotes the set of atoms of B.*

We work in a Boolean algebra P_1^M and use $P_{4,1}^M$ for $\mathrm{At}(P_1^M)$. We will denote by P_4^M the set of finite joins *of atoms and $P_{4,n}^M$ for those elements that are the join of exactly n atoms. P_4^M is always an ideal of P_1^M but it is only a Boolean algebra if it is finite, and even then it will not be a sub-Boolean algebra. A Boolean algebra is* atomic, *or in anachronistic terminology,* atomistic *if every element is an* arbitrary join *of atoms[6].*

For M in the class of finitely generated structures $\boldsymbol{K}_0$, below, the ideal P_4^M will be atomistic when viewed as a Boolean algebra (with $b_* = 1$ and complement as relative compement below b_*.) and the maximal such. For M in the class $\boldsymbol{K}_2$ the entire Boolean algebra P_1^M will be atomistic but this will be false for *all* M in $\boldsymbol{K}_0$ and for *some* M in $\boldsymbol{K}_1$ (beyond those in $\boldsymbol{K}_0$). We will use the next remark in proving Lemma 3.2.13.

Lemma 3.1.8. *Let $B_0 \subseteq B_1 \subseteq B_2$ be Boolean algebras. Suppose I_i for $i < 3$ are sequence of ideals in the respective B_i with $I_1 \cap B_0 = I_0$ and $I_2 \cap B_1 = I_1$. If, for $i = 0, 1$, $J_i \subset B_{i+1}$ is independent from B_i over I_i in B_{i+1}, then $J = J_0 \cup J_1$ is independent from B_0 over the ideal I_2.*

Proof. Let $\mathbf{b}$ be a finite sequence of distinct elements from J. Suppose $\sigma(\mathbf{y})$ is a non-zero term in the same number of variables as the length of $\mathbf{b}$. For any $d \in B_0 - I_2$, we must show $\sigma(\mathbf{b}) \wedge d \notin I_2$. Writing σ in disjunctive normal form it suffices to show some disjunct τ (which is just a conjunction of literals y_i and y_i^-) satisfies $\tau(\mathbf{b}) \wedge d \notin I_2$. Decompose $\tau(\mathbf{b})$ as $\tau_0(\mathbf{b}_0) \wedge \tau_1(\mathbf{b}_1)$ where $\mathbf{b}_i \in J_i$. Since J_0 is independent from B_0 over I_1, $\tau_0(\mathbf{b}_0) \wedge d \notin I_1$ and clearly it is some $d_1 \in B_1$. Similarly, since J_1 is independent from B_1 over I_2, $\tau_1(\mathbf{b}_1) \wedge d_1 \notin I_2$. So $\tau(\mathbf{b}) \wedge d_1 = \tau_0(\mathbf{b}_0) \wedge \tau_1(\mathbf{b}_1) \wedge d \notin I_2$ as required. $\square_{3.1.8}$

Although our notion of independence does not satisfy exchange, we are able to show that under certain conditions each suitable element is a member of a basis.

Lemma 3.1.9. *If B is a countable atomless Boolean algebra, then for any $b \neq 0, 1 \in B$, there is a basis J of B that contains b.*

Proof. Observe that by quantifier elimination all non-constant elements of B realize the same 1-type. But then if $A = \langle a_i : i < \omega \rangle$ is a basis for B, the automorphism α of B (guaranteed by $\aleph_0$-categoricity) which takes a_1 to b takes A to $\alpha(A)$ which is a basis containing b. $\square_{3.1.9}$.

The next result is used in step 2 of the proof of Claim 3.3.5.

Lemma 3.1.10. *Let $B_1 \subseteq B_2$ be Boolean algebras and suppose I_2 is an ideal of B_2 and J_1 is a countable subset of B_2 such that J_1 is independent from B_1 over I_2. If b is also independent from B_1 over I_2 and $b \in \langle J_1 \cup I_2 \rangle_{B_2}$, then there is a J_1' such that $b \in J_1'$, J_1' is independent from B_1 over I_2 and each of J_1 and J_1' generates (with I_2) the same subalgebra of B_2.*

---

[6]Equivalently for Boolean algebras, if every non-zero element is above at least one atom. The conditions are not equivalent on an arbitrary distributive lattice.

Proof. Let b^* be the image of b when π projects B_2 onto B_2/I_2 and B_3 denote the image of $\pi(\langle J_1 \cup I_2 \rangle_{B_2})$. By Lemma 3.1.9, there is a $J''_1 \subset B_2/I_2$ with $b^* \in J_1^*$ that freely generates B_3. Now choose J'_1 by choosing a preimage for each element of J''_1 and the result follows by Observation 3.1.2.2. $\square_{3.1.10}$

3.2 Defining the Complete Sentence

In this subsection we construct in an extension of ZFC a complete $L_{\omega_1,\omega}$-sentence, essentially the 'existential-completion' of Example 3.0.3, that we show in the next subsections has maximal models in λ for arbitrarily large λ less than the first measurable cardinal.

Each model is a member of the class $\boldsymbol{K}$ of Example 3.0.3; but the next definition gives a more focused notion of finitely generated models. This section is devoted to the construction of a countable generic structure for that class; the details of the construction will be essential for the main argument in the next section. Our goal is to build this generic structure as a Fraïssé-style limit of finitely generated structures; in each of these structures P_0^M and P_4^M will be finite.

Definition 3.2.1. τ *is a vocabulary with unary predicates* P_0, P_1, P_2, P_4, *binary* R, E, $\wedge$, $\vee$, *unary functions* $^-$, G_1, *constants 0,1 and unary (partial) functions* F_n, *for* $n < \omega$.

We originally introduced the properties of $\boldsymbol{K}_1$, in two stages ($\boldsymbol{K}^{-1}$ and $\boldsymbol{K}_1$) simply to allow the reader to absorb the definition more slowly. It turned out in [BS18], that the class $\boldsymbol{K}_{-1}$ plays an independent role.

We will study several classes, $\boldsymbol{K}$ with decorations. In general a class $\boldsymbol{K}^i_{<\aleph_0}$ denotes a class of finitely generated and either $\boldsymbol{K}^i$ or $\boldsymbol{K}_i$ denotes the class of all direct limits of models from $\boldsymbol{K}^i_{<\aleph_0}$.

Definition 3.2.2. $M \in \boldsymbol{K}^{-1}_{<\aleph_0}$ *is the class of finitely generated structures* M *satisfying.*

1. P_0^M, P_1^M, P_2^M *partition* M.

2. $(P_1^M, 0, 1, \wedge, \vee, <, ^-)$ *is a Boolean algebra* ($^-$ *is complement*).

3. $R \subset P_0^M \times P_1^M$ *with* $R(M,b) = \{a : R^M(a,b)\}$ *and the set of* $\{R(M,b) : b \in P_1^M\}$ *is a Boolean algebra.* $f^M : P_1^M \mapsto \mathcal{P}(P_0^M)$ *by* $f^M(b) = R(M,b)$ *is a Boolean algebra homomorphism into* $\mathbb{P}(P_0^M)$.

   Note that f *is not[7] in* τ*; it is simply a convenient abbreviation for the relation between the Boolean algebra* P_1^M *and the set algebra on* P_0 *by the map* $b \mapsto R(M,b)$.

4. $P_{4,n}^M$ *is the set containing each join of* n *distinct atoms from* M*;* P_4^M *is the union of the* $P_{4,n}^M$*. That is,* P_4^M *is the set of all finite joins of atoms. If* $b_1 \neq b_2$ *are in* P_4^M *then* $R(M,b_1) \neq R(M,b_2)$.

   There is a maximum element in P_4^M*, often denoted by* b_*.

5. G_1^M *is a bijection from* P_0^M *onto* $P_{4,1}^M$ *such that* $R(M, G_1^M(a)) = \{a\}$.

6. P_2^M *is finite (and may be empty). Further, for each* $c \in P_2^M$ *the* $F_n^M(c)$ *are functions from* P_2^M *into* P_1^M*. Note that it is allowed that for all but finitely many* n*,* $F_n^M(c) = 0_{P_1^M}$.

7. *If* $a \in P_{4,1}^M$ *and* $c \in P_2^M$ *then[8]* $(\forall^\infty n)$ $a \not\leq_M F_n^M(c)$*. This implies* $\bigcap_n \{x : (G_1(x) \in F_n^M(c)\} = \emptyset$.

---

[7]The subsets of P_0^M are *not* elements of M.

[8]Read $\forall^\infty$ as 'for all but finitely many'.

8. P_1^M is generated as a Boolean algebra by $P_4^M \cup \{F_n^M(c) : c \in P_2^M, n \in \omega\} \cup X$ where X is a finite subset of P_1^M.

We now define the class $\boldsymbol{K}^1_{<\aleph_0}$ of finitely generated structures; it will generate via direct limits a class $\boldsymbol{K}_1 = \hat{\boldsymbol{K}}^1_{<\aleph_0}$ and from it we will derive the rich class $\boldsymbol{K}_2 = \mathbf{R}$. Note that we allow P_0 and P_4 to be empty but require P_2 (and hence P_1) to have elements.

Definition 3.2.3. M is in the class of structures $\boldsymbol{K}^1_{<\aleph_0}$ if $M \in \boldsymbol{K}_{-1}$ and there is a witness $\langle n_*, \boldsymbol{B}, b_* \rangle$ such that:

1. $b_* \in P_1^M$ is the supremum of the finite joins of atoms in P_1^M. Further, for some k,
   $\bigcup_{j \leq k} P_{4,j}^M = \{c : c \leq b_*\}$ and for all $n > k$, $P_{4,n}^M = \emptyset$.

2. $\boldsymbol{B} = \langle B_n : n \geq n_* \rangle$ is an increasing sequence of finite Boolean subalgebras of P_1^M.

3. $B_{n_*} \supsetneq \{c \in P_1^M : c \leq b_*\} = P_4^M$; it is generated by the subset $P_4^M \cup \{F_n^M(c) : n < n_*, c \in P_2^M\}$.
   Moreover, the Boolean algebra B_{n_*} is free over the ideal P_4^M (equivalently, B_{n_*}/P_4^M is a free Boolean algebra[9]).

4. $\bigcup_{n \geq n_*} B_n = P_1^M$.

5. For each $c \in P_2^M$ the $F_n^M(c)$ for $n < \omega$ are distinct and independent over $\{0\}$.

6. The set $\{F_m(c) : m \geq n_*, c \in P_2^M\}$ (the enumeration is without repetition) is free from B_{n_*} over[10] $\{0\}$. $B_{n_*} \supsetneq P_4^M$ and $F_m(c) \wedge b_* = 0$ for $m \geq n_*$. (In this definition, $0 = 0^{P_1^M}$.)
   In detail, let $\sigma(\ldots x_{c_i} \ldots)$ be a Boolean algebra term in the variables x_{c_i} (where the c_i are in P_2^M) which is not identically 0. Then, for finitely many $n_i \geq n_*$ and a finite sequence of $c_i \in P_2^M$:

$$\sigma(\ldots F_{n_i}(c_i) \ldots) > 0$$

   and some $n < \omega$. Further, for any non-zero $d \in B_{n_*}$ with $d \wedge b_* = 0$, (i.e. $d \in B_n - P_M^4$),

$$\sigma(\ldots F_n(c) \ldots) \wedge d > 0.$$

7. For every $n \geq n_*$, B_n, is generated by $B_{n_*} \cup \{F_m(c) : n > m \geq n_*, c \in P_2^M\}$. Thus P_1^M and so M is generated by $B_{n_*} \cup P_2^M$.

Remark 3.2.4. The first part of Condition 6 of Definition 3.2.3 implies condition 8 of Definition 3.2.2. Note that this free generation condition is not preserved by arbitrary direct limits; in particular it will fail in the P_0-maximal model of cardinality λ. However, our construction of $\boldsymbol{K}_1$-free extensions (not in ZFC) will guarantee that the $\boldsymbol{K}_1$-freeness of submodels that are less that λ-generated. The second part of condition 6 implies, in particular, that if $b \in P_1^M - P_4^M$, there are infinitely many elements below b in P_1^M.

Note that if $\langle n_*, \boldsymbol{B}, b_* \rangle$ witnesses $M \in \boldsymbol{K}^1_{<\aleph_0}$ then for any $m \geq n_*$, so does $\langle m, \boldsymbol{B}, b_* \rangle$.

For the following example it will help to recall that the free Boolean algebra with n generators has 2^n atoms, and 2^{2^n} elements.

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[9] A further equivalence: $|Atom(B_{n_*})| - |P_{4,1}^M|$ is a power of two.
[10] As in Definition 3.1.1.2 with $X = \emptyset$.

Example 3.2.5. Here is a very basic example illustrating Definition 3.2.3. Let P_1^M be the direct product of the free algebra on one generator, a, $F_1 = \{0, a, b, 1\}$ and the atomless Boolean algebra F_∞ on generators $\langle g_0, \ldots g_n \ldots \rangle$. Then P_4^M is $F_1 \times \{0\}$ and has four elements. The Boolean subalgebra B_* of P_1^M generated by the ideal P_4^M has universe[11] $F_1 \times \{1\} \cup P_4^M$ and so eight elements. B_* has three atoms: $(\langle a, 0 \rangle, \langle b, 0 \rangle, \langle 0, 1 \rangle)$ and so is not free; note the last is *not* an atom of P_1^M. However B_* could serve as a B_{n_*} (with $n_* = 0$) to witness membership in $K_{<\aleph_0}^1$; it satisfies condition 3.2.3.3 since B_*/P_4^M has two elements and is free on 0 generators (and B^* has 2^0 new atoms). However, with $n_* = 1$, setting B_1 as the 16 element Boolean subalgebra[12] of P_1^M generated by $P_4^M \cup \{\langle 0, g_0 \rangle\}$ provides a more typical example of that definition. Note that B_1/P_4^M is free on one generator and has B_1 has two 'new' atoms, $\langle 0, g_0 \rangle$ and $\langle 1, g_0^- \rangle$; neither is an atom of P_1^M.

Note that P_1^M has two distinct representations. It was presented as the direct (Cartesian) product of the Boolean algebra[13] P_4^M (taking b_* as $\langle 1, 0 \rangle$) with F_∞. And it is also the free product[14] of B_1 with the sub-Boolean algebra generated by the $\{F_n(c) : n \geq n_* = 1\}$.

In extending this Boolean algebra P_1^M to a model in $K_{<\aleph_0}^1$; there are several further choices. We can choose any finite number as the cardinality of P_2^M; our enumeration of the generators in the current example corresponds to $P_2^M = \{c\}$ and $g_i = F_i^M(c)$. We must set P_0^M to have two elements that G_1 maps bijectively to $P_{4,1}^M = \{\langle a, 0 \rangle, \langle b, 0 \rangle\}$. But no $F_n(c)$ is above an atom. However, we could take $F_0^M(c)$ as $\langle a, g_0 \rangle$, $n_* = 1$ and B_1 generated by $P_4^M \cup \{\langle a, g_0 \rangle\}$.

Lemma 3.2.6. *Each structure in $K_{<\aleph_0}^1$ is finitely generated as a τ-structure.*

Proof. Let $M \in K_{<\aleph_0}^1$, witnessed by $\langle n_*, \boldsymbol{B}, b_* \rangle$. Then M is generated by $P_0^M \cup P_2^M$. $\square_{3.2.6}$

Lemma 3.2.7. $K_{<\aleph_0}^1$ *is countable.*

Proof. Let M be in $K_{<\aleph_0}^1$, witnessed by $\langle n_*, \boldsymbol{B}, b_* \rangle$. The isomorphism type of M is determined by the structure on P_4^M induced by the $F_n(c_i)$ and $c_i \in P_2^M$. If $m \geq n_*$, $F_m(c_i) \wedge b_* = 0$ so they leave no trace on P_4^M. Since this tail, $\{F_m(c_i) : m \geq n_*\}$ generates an atomless boolean algebra in the sense of P_1^M, that boolean algebra is $\aleph_0$ categorical. But there can be only countably many structures induced on the finite P_4^M by the countable set $F_n(c_i)$ through the formulas $x < F_n(c_i)$ which determine the values of R on P_4^M since only the $F_m(c_i)$ for $m < n_*$ are non-empty and P_2^M is finite. $\square_{3.2.7}$

The following lemma shows the prototypical models in $K_{<\aleph_0}^1$ in fact exhaust the class.

Lemma 3.2.8. *For any $M \in K_{<\aleph_0}^1$, P_1^M has a natural decomposition as a product of an atomic and an atomless Boolean algebra.*

Proof. Let $M \in K_{<\aleph_0}^1$, witnessed by $\langle n_*, \boldsymbol{B}, b_* \rangle$. Then the atomic part, P_4^M, is the collection of elements of P_1^M that are $\leq b_*$. And the independent generation by the $F_n^M(c_i)$ for $n \geq n_*$ and $c_i \in P_2^M$ shows the quotient P_1^M/P_4^M is atomless. $\square_{3.2.8}$

Definition 3.2.9. *The class $K_1 = \hat{K}$ is the collection of all direct limits of models in $K_{<\aleph_0}^1$.*

Recall that members of $K_{<\aleph_0}^1$ must have $P_2 \neq \emptyset$ but P_0 may be empty.

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[11] The sub-boolean algebra of P_1^M generated by $\langle a, 0 \rangle$ has elements $\{\langle a, 0 \rangle, \langle b, 1 \rangle, \langle 1, 1 \rangle \langle 0, 0 \rangle\}$.

[12] Freely generated by $\langle a, 0 \rangle$ and $\langle 0, g_0 \rangle$.

[13] P_4^M is an ideal but *not* sub-Boolean algebra of P_1^M.

[14] B_{n_*} is sub-Boolean algebra but *not* an ideal of P_1^M.

Lemma 3.2.10. *There is a minimal model M_{min} of $\boldsymbol{K}^1_{<\aleph_0}$, that can be embedded in any model of $\boldsymbol{K}_1$.*

Proof. Let $P_0^{M_{min}}$ be empty and so $P_4^{M_{min}} = \{0\}$, $P_2^{M_{min}}$ contain a single element c, and let the $F_n^{M_{min}}(c)$ be independent generators of $P_1^{M_{min}}$. Since $M \in \boldsymbol{K}^1_{<\aleph_0}$ implies $P_2^M \neq \emptyset$, for the embedding we can map c to any element of P_2^M and $F_n^{M_{min}}(c)$ to $F_n^M(c)$ by Definition 3.2.3.5.

Lemma 3.2.11. *If $M_0 \subseteq M_1$ are both in $\boldsymbol{K}^1_{<\aleph_0}$, witnessed by $\langle n_*^i, \boldsymbol{B}^i, b_*^i \rangle$, then for sufficiently large n, $B_n^0 = B_n^1 \cap P_1^{M_0}$.*

Proof. Recall $\boldsymbol{B}^i = \langle B_n^i : n < \omega \rangle$. Since the B_n^1 exhaust $P_1^{M_1}$, $B_{n_*}^0$ is finite, and for $c \in P_2^{M_0}$ and all r, $F_r^{M_1}(c) = F_r^{M_0}(c)$, for all sufficiently large n, B_n^1 contains the $F_r^{M_0}(c)$ for $r < n$ and thus B_n^0. But if some $b \in B_n^1 \cap P_1^{M_0}$, but is not in B_n^0 then for some k, $b \in B_{k+1}^0 - B_k^0$. But then B_{k+1}^0 is not generated by $B_{n_*}^0$ along with the $F_r^{M_0}(c)$ for $r < k$. $\square_{3.2.11}$

Note that if the conclusion of Lemma 3.2.11 hold for n, it holds for all $m \geq n$.

We now introduce some special notation for this paper by defining $\boldsymbol{K}_1$-*free over* ($\boldsymbol{K}_1$-*free extension* of) for models in $\boldsymbol{K}_1$. M_2 is a $\boldsymbol{K}_1$-free extension of M_1 if not only is the image of $P_1^{M_2}$ in the Boolean algebra $P_1^{M_2}/P_4^{M_2}$ a free extension of the image of $P_1^{M_1}$ but the $F_n(c)$ satisfy technical conditions which allow the preservation of this condition under unions of chains.

Definition 3.2.12. *When $M_1 \subseteq M_2$ are both in $\boldsymbol{K}_1$, we say M_2 is $\boldsymbol{K}_1$-free over or is a $\boldsymbol{K}_1$-free extension of M_1 and write $M_1 \subseteq_{fr} M_2$, witnessed by (I, H) when*

1. *$I \subset P_1^{M_2} - (P_1^{M_1} \cup P_4^{M_2})$ such that i) $I \cup P_1^{M_1} \cup P_4^{M_2}$ generates $P_1^{M_2}$ and ii) I is independent from $P_1^{M_1}$ over $P_4^{M_2}$ in $P_1^{M_2}$.*

2. *There is a function H from $P_2^{M_2} \setminus P_2^{M_1}$ to $\mathbb{N}$ such that the $F_n(c)$ for $n \geq H(c)$ are distinct and*

$$\{F_n^M(c) : c \in P_2^{M_2} \setminus P_2^{M_1} \text{ and } n \geq H(c)\} \subset I$$

and for every $c \neq d \in P_2^{M_2}$, $\{n : (\exists m) F_m^{M_2}(c) = F_n^{M_2}(d)\}$ is finite.

We say M is $\boldsymbol{K}_1$-free over the empty set or simply $\boldsymbol{K}_1$-free if M is a $\boldsymbol{K}_1$-free extension of M_{min}.

Lemma 3.2.13.   *1. If $M_1 \subseteq_{fr} M_2$ by (I_1, H_1) and $M_2 \subseteq_{fr} M_3$ by (I_2, H_2) then $M_1 \subseteq_{fr} M_3$ by $I_1 \cup I_2$ with the same H_i. Thus, $\subseteq_{fr}$ is a partial order.*

2. *More generally, if M_α with $\alpha < \delta$ is a continuous $\subseteq_{fr}$-increasing sequence then $M = \bigcup M_\alpha$ satisfies $M_\alpha \subseteq_{fr} M$ witnessed by $\bigcup_{\alpha < \beta < \delta} I_\beta$.*

Proof. By Lemma 3.1.8 (taking the ideals as $P_4^{M_2}$ and $P_4^{M_3}$), $I_1 \cup I_2$ is free from $P_1^{M_1}$ over $P_4^{M_3}$. But we can take $H_3(c)$ as $H_i(c)$ for $i = 1, 2$ (note the H_i are defined on disjoint sets), since the $F_n^{M_{i+1}}(c)$ for $c \in P_2^{M_{i+1}} - P_2^{M_i}$ cannot lie in $P_1^{M_i}$ by the independence. Part 2 follows by induction. Successors are similar , while limits are automatic. $\square_{3.2.13}$

Remark 3.2.14. *In an increasing chain such as that of Lemma 3.2.13.2, if some $b \in P_1^{M_{\alpha+1}}$ is free from $P_1^{M_\alpha}$ over $P_4^{M_{\alpha+1}}$ then b is also free from $P_1^{M_\alpha}$ over $P_4^{M_\beta}$ for any $\beta > \alpha$ since $P_4^{M_\beta} \cap P_1^{M_{\alpha+1}} = P_4^{M_{\alpha+1}}$.*

The next lemma uses the requirement that the B_n in the witnessing sequence are free Boolean algebras.

Lemma 3.2.15. *If $M_0 \subset M_1$ are both in $\boldsymbol{K}^1_{<\aleph_0}$ then $M_0 \subset_{fr} M_1$.*

Proof. We can assume by Lemma 3.2.11 that the n_*^i for $i < 2$ are equal and that $B_{n_*^1}^1 \cap P_1^{M_0} = B_{n_*^0}^0$. Since the B_{i,n^*} are free from $P_4^{M_i}$ over $\emptyset$, we can choose bases I_0, I_1 for B_{0,n_*^0} and B_{1,n_*^1} respectively. Now $I_0 \cup I_1 \cup \{F_n^{M_i}(c) : i < 2, n \geq n_{n_*^i}^i, c \in P_2^{M_i}\}$ is a free basis of $P_1^{M_i}$ over $P_4^{M_i}$. Hence $(I_2 \setminus I_1) \cup \{F_n^{M_i}(c) : i < 2, n \geq n_*^i, c \in P_2^{M_i}\}$ is the required I from Definition 3.2.12 with $H(c) = n^*$ for all c. $\square_{3.2.15}$

Crucially, Lemma 3.2.15 fails in general if $\boldsymbol{K}_{<\aleph_0}^1$ is replaced by $\boldsymbol{K}_1$. We can immediately deduce.

Corollary 3.2.16. *Each model N in $\boldsymbol{K}_{<\aleph_0}^1$ is $\boldsymbol{K}_1$-free over the empty set.*

To apply Lemma 3.2.13.2, we need to construct a sequence of $\boldsymbol{K}_1$-free extensions. We now show that if M_1 is $\boldsymbol{K}_1$-free, $N_1 \subseteq N_2$ and $N_1 \subseteq N_2$ with N_2 a finitely generated extension of the finitely generated substructure N_1. Then M_1 and N_2 can be amalgamated over N_1. Note that by Lemma 3.2.15, on $\boldsymbol{K}_{<\aleph_0}^1$, $\subseteq$ is the same as $\subseteq_{fr}$. There are three key ingredients: N_1 and N_2 must be finitely generated; this is reflected positively in the ability to employ the decomposition $\boldsymbol{B}^i$ in proof but also by the key role in the proof of the finite set $P_{4,1}^{N_2} - P_{4,1}^{N_1}$. Secondly, M_1 must be $\boldsymbol{K}_1$-free. This is used in the proof that $M_2 \in \boldsymbol{K}_1$. Thirdly, we must ensure that 'atomicity' is preserved in constructing extensions of Boolean algebra so the definitions of P_4 and $P_{4,1}$ are 'absolute' between models. It is this third condition which drives the complexity of steps 1 to 3 in the following proof. The free amalgam $D = A \otimes_C B$, where either of A, B has only finitely many atoms must destroy the atomicity of some elements. (If a is an atom of A and $b_1, \ldots b_n$ are the atoms of B, for at least one i, $A \otimes_C B \models 0 < a \wedge b_i < a$.) Thus we will have to construct a quotient algebra of the free amalgam in step 3 below in order to find an amalgam which does not destroy atoms.

Theorem 3.2.17. *Suppose $M_1 \in \widehat{\boldsymbol{K}} = \boldsymbol{K}_1$ is $\boldsymbol{K}_1$-free and $N_1 \subset M_1$. Let $N_1 \subset N_2$ with both in $\boldsymbol{K}_{<\aleph_0}^1$. Then there are an M_2 and an f such that:*

1. *$M_2 \in \boldsymbol{K}_1$, $M_1 \subseteq_{fr} M_2$ and so M_2 is $\boldsymbol{K}_1$-free.*

2. *f maps N_2 into M_2 over N_1. Moreover, the image in M_2 of N_2 is $\boldsymbol{K}_1$-free over N_1.*

Proof. We lay out the situation in more detail. M_1 is $\boldsymbol{K}_1$-free means that M_1 is $\boldsymbol{K}_1$-free over M_{min} by (I_1, H_1). For $i = 1, 2$, let $\langle n_*^i, \boldsymbol{B}^i, b_*^i \rangle$ witness that $N_i \in \boldsymbol{K}_{<\aleph_0}^1$. Suppose $N_1 \subseteq_{fr} N_2$ is witnessed by (I_2, H_2). Invoking Lemmas 3.2.11 and 3.2.4, we can rename n_*^i and rechoose n_* for N_2 so that $n_*^1 = n_*^2 = n_*$ and $B_n^1 = B_n^2 \cap N_1$ for $n \geq n_*$, and (since $P_2^{N_2}$ is finite) for each $c \in P_2^{N_1}$, $H_1(c) \leq n_*$. Let $J_1 \subset B_{n_*}^1$ be the pre-image of the basis of $B_{n_*}^1/P_4^{N_1}$. Then, since $J_1/P_4^{N_1}$ is a generating set of $B_{n_*}^1/P_4^{N_1}$, for each $b \in B_{n_*}^1$, there is a Boolean combination b' of elements of J_1 such that $b' \triangle b \in P_4^{N_1}$. Note also, that by our choice of n_* (Definition 3.2.3.6), if $b \in P_1^{N_1}$ is above an atom of $P_1^{N_2}$, $b \in B_{n_*}^1$. Let $k = |P_{4,1}^{N_2} - P_{4,1}^{N_1}|$, fix $a_0 \ldots a_{k-1}$ listing a new set A, and let f be 1-1 *function from* $P_{4,1}^{N_2} - P_{4,1}^{N_1}$ *onto* A.

Step 1: *Construct a Boolean algebra $\mathbb{B}_1$ that is generated by $P_1^{M_1} \cup A$ and so that the atoms of $\mathbb{B}_1$ are* $P_{4,1}^{M_1} \cup A$. For this demand, let $\mathcal{D}_\ell$, for each $\ell < k$, be an ultrafilter of the Boolean algebra $P_1^{M_1}$, disjoint from $I_1 - J_1$ such that for $b \in P_1^{N_1}$, $b \in \mathcal{D}_\ell$ if and only if $N_2 \models f^{-1}(a_\ell) \leq b$. (Such an ultrafilter exists as the set $\{b \in P_1^{N_1} : f^{-1}(a_\ell) \leq b\}$, as noted in last paragraph, contains no element of $I_1 - J_1$ and is a filter on $P_1^{N_1}$ that can be extended to an ultrafilter on the Boolean algebra $P_1^{M_1}$.)

Now let X be the union of the Stone space of $P_1^{M_1}$, denoted $S(P_1^{M_1})$ with A. For $b \in P_1^{M_1}$, let

$$X_b = \{d \in S(P_1^{M_1}) : b \in d\} \cup \{a_i : b \in \mathcal{D}_i\}.$$

Now let $\mathbb{B}_1$ be the subalgebra of $\mathcal{P}(X)$ generated by the $\{X_b : b \in P_1^{M_1}\} \cup A$. Now, generalizing the Stone representation theorem, we embed $P_1^{M_1} \cup A$ into $\mathbb{B}_1$ by a map g; let $g(b) = X_b$ for $b \in P_1^{M_1}$ and $g(a) = \{a\}$ for $a \in A$.

Since $P_{4,k}^{N_2} \cap P_1^{N_1} = P_{4,k}^{N_1}$, there can be no $b \in P_4^{N_1}$ with $N_2 \models f^{-1}(a_i) \leq b$. So the elements of A remain atoms in $\mathbb{B}_1$. Note i) that for $b \in P_1^{M_1}$, $\mathbb{B}_1 \models f^{-1}(a_i) \leq b$ iff $b \in \mathcal{D}_i$ and ii) that the atoms of $\mathbb{B}_1$ are exactly $P_{4,1}^{M_1} \cup A$. For ii) observe that e is an atom of $P_1^{M_1}$ if and only if X_e is a principal ultrafilter in $\mathbb{B}_1$.

Step 2: *Find a sub-Boolean algebra $\mathbb{B}^*$ of $\mathbb{B}_1$ that is a suitable base for amalgamating $\mathbb{B}_1$ with $P_1^{N_2}$.* For this, denote by $\mathbb{B}^*$ the sub-Boolean algebra of $\mathbb{B}_1$ generated by $g(P_1^{N_1} \cup A)$. Denote by $\check{\mathbb{B}}^*$ the sub-Boolean algebra of $P_1^{N_2}$ generated by $P_1^{N_1} \cup f^{-1}(A)$.

Compose g with the extension the union of the identity on $P_1^{N_1}$ with the map f given in the first paragraph of the proof using the operations of N_2 to give a map from $P_1^{N_1} \cup P_{4,1}^{N_2} - P_{4,1}^{N_1}$ into $\mathbb{B}_1$ that takes $\check{\mathbb{B}}^*$ to $\mathbb{B}^*$. We also denote this map by f.

To ease notation, we will suppress g and pretend that $P_1^{N_1} \cup A$ is actually[15] contained in $\mathbb{B}_1$.

Step 3: *Construct a Boolean algebra $\mathbb{B}_2$ that is an amalgam of $P_1^{M_1}$ and $P_1^{N_2}$ over $f(\check{\mathbb{B}}^*) = \mathbb{B}^*$ such that the atoms of $\mathbb{B}_2$ are $P_{4,1}^{\mathbb{B}_1} \cup A$.* $\mathbb{B}_2$ is a quotient of the pushout $\mathbb{B}_2'$ of $\mathbb{B}_1$ and $P_1^{N_2}$ over the sub-Boolean algebra $\mathbb{B}^*$ of $\mathbb{B}_1$ generated by $P_1^{N_1}$ and A. The crux of the proof is the specification of the atoms of $\mathbb{B}_2$; it allows us to extend the amalgam of Boolean algebras to an amalgam in $\boldsymbol{K}_1$.

By standard properties of the coproduct (Lemma 3.1.6), $\mathbb{B}_1$ and $P_1^{N_2}$ are disjointly embedded over $\mathbb{B}^*$ into their coproduct $\mathbb{B}_2'$. We will regard the embedding of $\mathbb{B}_1$ as the identity and denote by f the embedding of $P_1^{N_2}$ extending our earlier f mapping the sub-Boolean algebra $\check{\mathbb{B}}^*$ of $P_1^{N_2}$ into $\mathbb{B}_1$. Crucially, while $\mathbb{B}_1$ and $f(P_1^{N_2})$ are sub-Boolean algebras of $\mathbb{B}_2'$; they are *not* ideals.

The atoms of the amalgamation base $\mathbb{B}^*$ remain atoms in $\mathbb{B}_2'$ as: if a is an atom of $\mathbb{B}^*$ then every $b_1 \in \mathbb{B}_1$ satisfies $b_1 \wedge a = 0$ or $b_1 \wedge a = a$ and similarly for $b_2 \in P_1^{N_2}$ and therefore also for $b_1 \wedge b_2$; using disjunctive normal form, no element of $\mathbb{B}_2'$ contradicts the atomicity of an atom of $\mathbb{B}^*$. Recall $N_1 \subseteq_{fr} N_2$ is witnessed by (I_2, H_2). To guarantee the atoms of $\mathbb{B}_1 \setminus \mathbb{B}^*$ (i.e. $P_{4,1}^{M_1} - P_{4,1}^{N_1}$) are atoms of $\mathbb{B}_2$, we divide $\mathbb{B}_2'$ by the ideal[16], $\Im$, generated by

$$\Im_0 = \{a \wedge f(b) : a \in P_{4,1}^{\mathbb{B}_1} \setminus P_{4,1}^{N_1}, b \in I_2, a \wedge f(b) < a\}.$$

(*) Since each element of $\Im$ is *strictly* below a finite join of atoms in $\mathbb{B}_2'$ (actually in $\mathbb{B}_1$), $\Im$ is a proper ideal of $\mathbb{B}_1$ bounded by elements of $P_4^{\mathbb{B}_1}$; but $\Im \cap P_4^{\mathbb{B}_1} = \emptyset$. Indeed, by freeness of the coproduct, $\Im \cap \mathbb{B}_1 = \emptyset$. Note that the subalgebra of $\mathbb{B}_2'$ generated by $f(I_2)$ is a subset of $\mathbb{B}_1$ so it is disjoint from $\Im$.

Let π map $\mathbb{B}_2'$ onto $\mathbb{B}_2 =_{def} \mathbb{B}_2'/\Im$. By (*), no element of $\mathbb{B}_1 \cup f(I_2)$ is collapsed by the map $\pi : \mathbb{B}_2' \to \mathbb{B}_2$. Thus, π is 1-1 on $\mathbb{B}_1 \cup f(P_1^{N_2})$ and $\mathbb{B}_2$ is a disjoint amalgamation of the Boolean algebras $\pi(\mathbb{B}_1)$ and $\pi(f(P_1^{N_2}))$. Since $\mathbb{B}_2'$ is generated by $\mathbb{B}_1 \cup f(P_1^{N_2})$, without loss of generality, we can assume the preimage of a potential atom of $\mathbb{B}_2$ has the form $a \wedge f(b)$ where $a \in \mathbb{B}_1 \setminus \mathbb{B}^*$ is an atom of $\mathbb{B}_1$ and $b \in P_1^{N_2} \setminus \mathbb{B}^*$. By the freeness property of coproducts[17], $\mathbb{B}_2' \models a \wedge f(b) < a$, so $\pi(a \wedge f(b)) = 0$ and $\pi(a) = a$ is an atom.

Step 4: *The actual τ-amalgam.* Now to define the extension M_2, let $P_1^{M_2} = \mathbb{B}_2$, $P_{4,1}^{M_2} = P_{4,1}^{M_1} \cup A$; $P_4^{M_2}$ is the set of finite joins of these atoms. Then, let $P_2^{M_2} = P_2^{M_1} \cup P_2^{N_2}$ and the $F_n^{M_2}(c)$ be as in submodel in which c lies. Define $P_0^{M_2}$ to be a set in 1-1 correspondence with $P_4^{M_2}$ and call the correspondence $G_1^{M_2}$. Finally, we must define R^{M_2}: for each $b \in P_1^{M_2}$, let $R(M_2, b) = \{a \in P_0^{M_2} : G_1^{M_2}(a) \leq^{M_2} b\}$.

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[15]Clearly, this could be achieved by choosing a new copy of $\mathbb{B}_1$.

[16]Abusing notation, since $\mathbb{B}_1$ is not a τ-structure, we write $P_{4,1}^{\mathbb{B}_1}$ for the set of atoms of $\mathbb{B}_1$ and $P_4^{\mathbb{B}_1}$ for their finite joins.

[17]$\mathbb{B}_2'$ is freely generated as a Boolean algebra by (isomorphic copies of) $\mathbb{B}_1$ and $P_1^{N_2}$ over $\mathbb{B}^*$.

By Lemma 3.1.5, I_2 is independent from $\mathbb{B}_1$ over $\Im$ in $\mathbb{B}_2'$ and so, by (*), $\pi(f(I_2))$ is independent from $P_1^{M_2} = \mathbb{B}_2$ over $P_4^{M_2}$ in M_2. So $M_1 \subset_{\mathrm{fr}} M_2$ with $H_{M_2}(c) = n^*$ for $c \in P_2^{M_2}$.　$\square_{3.2.17}$

Note the M_0, M_1, M_2, M_3 in the next argument are N_0, M_1, N_2, M_2 in Lemma 3.2.17.

Corollary 3.2.18. $(\boldsymbol{K}^1_{<\aleph_0}, \subseteq)$ *has the disjoint amalgamation property.*

Proof. We know every member of $\boldsymbol{K}^1_{<\aleph_0}$ is $\boldsymbol{K}_1$-free over the empty set. So the amalgamation becomes a special case of Lemma 3.2.16 when we add a proof that the amalgam is in $\boldsymbol{K}^1_{<\aleph_0}$. We have the following situation. M_0 is $\boldsymbol{K}_1$-free over the minimal model M_{min}. That is, there are J_0, I_0, H_0 such that J_0 generates B_{0,n_*^0} and $(J_0 \cup I_0, H_0)$ witness that M_0 is a $\boldsymbol{K}_1$-free extension of the minimal model M_{min}. Similarly there are for $i = 1, 2$, J_i, I_i, H_i such that J_i generates B_{i,n_*^i} and $(J_i \cup I_i, H_i)$ that witness that M_i is a $\boldsymbol{K}_1$-free extension of the minimal model M_0.

Choose n_* as the maximum of n_*^i for $i < 3$; we can assume the n_*^i for $i < 3$ are equal and that $B_{2,n^i} \cap B_{0,n_*^0} = B_{0,n_*^0}$ for $i = 1, 2$. Rechoosing n_* by Lemma 3.2.11 we can assume for all $n \geq n_*$, $B_n^1 \cap P_1^{M_0} = B_n^0 = B_n^2 \cap P_1^{M_0}$.

Choose M_3 by Lemma 3.2.17. Let $b_{n_*}^3 = b_{n_*}^1 \wedge b_{n_*}^1$. Now let B_{3,n_*} be the subboolean algebra of M_3 generated by $J_0 \cup J_1 \cup J_3$ and for $n \geq n_*$, $B_{3,n}$ be generated by $B_{1,n} \cup B_{2,n}$. This is the required witnessing sequence.　$\square_{3.2.18}$

Since $\boldsymbol{K}^1_{<\aleph_0}$ has joint embedding, amalgamation and only countably many finitely generated models, we construct in the usual way a generic model. This construction can be rearranged in order type ω so by Theorem 3.2.17 and Lemma 3.2.13 the generic is $\boldsymbol{K}_1$-free.

Corollary 3.2.19. *There is a countable generic model M for $\boldsymbol{K}_0$. We denote its Scott sentence by ϕ_M. Moreover M is $\boldsymbol{K}_1$-free.*

Aligning our notation with earlier sections of the paper we note the models of ϕ_M are rich in the sense defined there.

Definition 3.2.20. *We say a model M in $\boldsymbol{K}_1$ is rich if for any $N_1, N_2 \in \boldsymbol{K}^1_{<\aleph_0}$ with $N_1 \subseteq N_2$ and $N_1 \subseteq M$, there is an embedding of N_2 into M over N_1. We denote the class of rich models in $\boldsymbol{K}_1$ by $\boldsymbol{K}_2 = \mathbf{R}$.*

Lemma 3.2.17 finds a $\boldsymbol{K}_1$-free extension of each $\boldsymbol{K}_1$-free model in $\boldsymbol{K}_1$; more strongly:

Corollary 3.2.21. *Let M_1 be $\boldsymbol{K}_1$-free. There exists an $M_2 \in \boldsymbol{K}_2$ which is a proper $\boldsymbol{K}_1$-free extension of M_1.*

Proof. Iterate Corollary 3.2.17 as in Corollary 2.1.6 to obtain a rich model; note that freeness is preserved at each stage.　$\square_{3.2.21}$

The crucial distinction from Corollary 2.1.6 is that here we extend only '$\boldsymbol{K}_1$-free models' in $\boldsymbol{K}_1$ to $\boldsymbol{K}_2$. While *this* construction applied to models in $\boldsymbol{K}_2$ will necessarily increase P_0 (case 2 of Construction 3.3.9), we can find extensions in $\boldsymbol{K}_1$ which do not extend P_0 or P_1 but only P_2 (case 4 of Construction 3.3.9).

For the construction in Section 3.3 we require two crucial properties of the generic model.

Lemma 3.2.22. *If M is the generic model then*

 i *if $b_1 \neq b_2$ are in $P_1^M - P_4^M$ then $R(b_1, M) \neq R(b_2, M)$, i.e. the map f from Definition 3.2.2.1.c is injective.*

ii For any $a \in P_0^M$, $b \in P_1^M$, $M \models R(a, b) \vee R(a, b^-)$. Indeed, P_1^M is an atomic Boolean algebra.

iii For each $b \in P_1^M - P_4^M$, $R^M(M, b)$ is infinite and coinfinite.

Proof. For i) fix a finitely generated model M_0 containing b_1, b_2; there is a finitely generated extension M_1 in $\boldsymbol{K}_{<\aleph_0}^1$ by adding $a \in P_0^{M_1}$ with $R^{M_1}(a, b_1) \wedge \neg R^{M_1}(a, b_2)$. This shows the injectivity; the other conditions are similar. $\quad \square_{3.2.22}$

Lemma 3.2.23. *If $M, N \in \boldsymbol{K}_2$, $M \equiv_{\infty, \omega} N$ so they satisfy the Scott sentence Φ_M. Moreover, if $M \subset N$ and are both in $\boldsymbol{K}_2$, $M \prec_{\infty, \omega} N$.*

Proof: Suppose M and N are in $\boldsymbol{K}_2$. We define a back-and-forth between M and N for $\boldsymbol{a} \in M^n$, $\mathbf{b} \in N^n$ by $\boldsymbol{a} \equiv \mathbf{b}$ if they realize the same first order type over the $\emptyset$ with respect to T. Fix such $\boldsymbol{a} \equiv \mathbf{b}$ and choose $c \in M$. The interest is when c is not in $A = \operatorname{acl}(\boldsymbol{a})$, If $c \in P_1^M - A$, let $A_1 = \langle A \cup \{c\} \rangle_M$. Since $M \in \boldsymbol{K}_1$, $A_1 \in \boldsymbol{K}_1$. Now let $B = \langle \mathbf{b} \rangle_N$ that is equivalent to A. By richness there exists B_1 isomorphic to A_1 with $B \subset B_1 \subset N$.

If $M \subset N$ and both are in $\boldsymbol{K}_2$, then $\operatorname{acl}_M(\boldsymbol{a}) = \operatorname{acl}_N(\boldsymbol{a})$ for $\boldsymbol{a} \in M$; this yields the moreover. $\quad \square_{3.2.23}$

This completes our description of the class $\boldsymbol{K}_2$ of rich models and its Scott sentence. At this point we show any $\boldsymbol{K}_1$-free-member of $\boldsymbol{K}_2$ has a proper $\boldsymbol{K}_1$-free-extension in $\boldsymbol{K}_1$. In case 2 of Construction 3.3.9, we apply Corollary 3.2.21 to regain a member of $\boldsymbol{K}_2$.

Lemma 3.2.24. *If $M \in \boldsymbol{K}_2$, there is an N such that $M \subset_{fr} N$, both are in $\boldsymbol{K}_1$, $P_2^N = P_2^M$, $P_0^N = P_0^M$, and P_1^N is generated by $P_1^M \cup \{b\}$ and $b \in N'$ with $N \prec N'$. Moreover given $u \subseteq P_0^M$, we can require $R(N, b) = u$ and b is free from P_1^M over P_4^N. Moreover, if M is $\boldsymbol{K}_1$-free then so is N.*

Proof: Let $p(x)$ be the type of an element satisfying $P_1(x) \wedge \neg P_4(x)$:

$$\{x \geq G_1(a) \colon a \in u\} \cup \{G_1(a) \wedge x = 0 \colon a \in P_4^N \setminus u\} \cup \{b \wedge \sigma(x) \neq a \colon b \in P_1^M \setminus P_4^M, a \in P_4^M\},$$

where $\sigma(x)$ ranges over nontrivial Boolean polynomials. Each finite subset q of p is satisfied in M because $M \in \boldsymbol{K}_2$ so p is finitely satisfied in M. Thus there is an elementary extension N' of M where p is realized by some b. Let $\mathbb{B}$ be the boolean subalgebra of $P_1^{N'}$ generated by $P_1^M \cup \{b\}$. Since N' satisfies the first order properties of $\boldsymbol{K}_2$, the atoms of M are atoms of $\mathbb{B}$.

Define a τ-structure N with $P_1^N = \mathbb{B}$. Interpret P_2 and the F_n in N as in M. Extend G_1^M and P_0^M so that $P_0^N = (G_1^N)^{-1}(Y)$. The structure N is well-defined; we must prove it is in $\boldsymbol{K}_1$.

Let $\langle (M_i, Z_i) \colon i < |M|, Z_i \subset_\omega Y \rangle$ list the pairs of finitely generated $M_i \subset M$ in $\boldsymbol{K}_{<\aleph_0}^1$ and finite subsets Z_i of Y. (The M_i will be repeated.) Let $N_i \subset N$ with $P_0^{N_i} = P_0^{M_1} \cup Z_i$, $P_2^{N_i} = P_2^{M_1}$, and $P_1^{N_i}$ be the universe of the Boolean subalgebra of N τ-generated by $P_2^{N_1} \cup \{b\} \cup Z_i$. It is easy to check each $N_i \in \boldsymbol{K}_{<\aleph_0}^1$. Now N is the direct limit of the finitely generated $\{N_i \colon i < |M|\}$ so it is in $\boldsymbol{K}_1$.

Finally b is free from from P_1^M over P_4^N since no nontrivial unary polynomial σ satisfies maps $\sigma(b) \wedge a \in P_4^N$ with $a \in P_1^M - P_4^M$. The moreover follows by Definition 3.2.12 from the independence of b. $\quad \square_{3.2.24}$

3.3 Constructing maximal models in an extension of ZFC

We show that for arbitrarily large cardinals below a measurable cardinal, assuming a mild set theoretic hypothesis described below, $\boldsymbol{K}_2$ has maximal models. We begin by defining a pair of set theoretic notions and some specific notions of maximal model.

Definition 3.3.1 ($\diamond_S$). *Given a cardinal κ and a stationary set $S \subseteq \kappa$, $\diamond_S$ is the statement that there is a sequence $\langle A_\alpha : \alpha \in S \rangle$ such that*

1. *each $A_\alpha \subseteq \alpha$;*

2. *for every $A \subseteq \kappa$, $\{\alpha \in S : A \cap \alpha = A_\alpha\}$ is stationary in κ.*

Definition 3.3.2 (S reflects). *Let κ be a regular uncountable cardinal and let S be a stationary subset of κ. For $\alpha < \kappa$ with uncountable cofinality, S reflects at α if $S \cap \alpha$ is stationary in α. S reflects if it reflects at some $\alpha < \kappa$.*

Definition 3.3.3.   1. *A model $M \in \boldsymbol{K}_2 = \mathbf{R}$ is P_0-maximal (for $\boldsymbol{K}_1$) if $M \subseteq N$ and $N \in \boldsymbol{K}_2$ ($\in \boldsymbol{K}_1$) implies $P_0^M = P_0^N$.*

2. *A model $M \in \boldsymbol{K}_2$ is maximal for $\boldsymbol{K}_2$ if $M \subseteq N$ and $N \in \boldsymbol{K}_2$ implies $M = N$.*

Let $S_{\aleph_0}^\lambda$ denote the stationary set $\{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0, \delta \text{ is divisible by } |\delta|\}$.

We now define a crucial notion and prove a lemma about it; it is applied to perform case 5 of the construction. Any proper P_0-extension of a model M induces a non-principal ultrafilter A on P_1^M. The following lemma is instrumental in constructing, for the particular M under consideration , an ostensibly $\aleph_1$-complete ultrafilter on $\mathcal{P}(P_0^M)$ which contradicts that λ is not measurable. See 3.3.11.

Definition 3.3.4 (A-good defined). *Suppose that $N_n \subset_{\mathrm{fr}} N_{n+1}$ for $n < \omega$, is sequence of models, $\overline{N}$, in $\boldsymbol{K}_1$. We say a sequence $\mathbf{b} = \langle b_n : n < \omega \rangle$ is*

1. good for $\overline{N}$ if

    (a) *$P_2^{N_{n+1}} - P_2^{N_n}$ is infinite;*

    (b) *for each n, $b_n \in P_1^{N_{n+1}}$ and $\{b_n\}$ is free from $P_1^{N_n}$ over $P_4^{N_{n+1}}$;*

    (c) *if $a \in P_0^{N_i}$, then for all but finitely many $n \geq i$, $a \notin R(N_{n+1}, b_n)$.*

2. *for $A \subset \bigcup \overline{N}$, $\mathbf{b}$ is A-good if each $b_n \in A$.*

3. *and labeled if there is a pair $(N^{\mathbf{b}}, c^{\mathbf{b}}) =$ with $N^{\mathbf{b}} \in \boldsymbol{K}_1$ and $N^{\mathbf{b}} \supseteq N_\omega = \bigcup N_n$ such that for each n, $F_n^N(c^{\mathbf{b}}) = b_n$. By the definition of $\boldsymbol{K}_1$, $\bigcap_n R(N^{\mathbf{b}}, F_n^N(c^{\mathbf{b}})) = \emptyset$.*

Note that for every $c \in N_m \subsetneq N_\omega$, at most finitely many of any good sequence $\langle b_k : k < \omega \rangle$ occur in the sequence $F_n^{N_m}(c)$ for $n < \omega$ (as $F_n^{N_m}(c) \in N_m$ and for $k > m$, $b_k \notin N_m$.)

Claim 3.3.5. *Suppose that for $n < \omega$, $\overline{N} = \langle N_n \subset_{\mathrm{fr}} N_{n+1} \rangle$ are in $\boldsymbol{K}_1$. For $A \subseteq N_\omega$, if Condition A) holds then so does condition B).*

  A) *There is an A-good sequence for $\overline{N}$.*

  B) *There is a labeled A-good sequence for $\overline{N}$.*

Proof. The following construction is for the fixed A-good sequence $\mathbf{b}$. Let $N = N_\omega \bigcup_{n<\omega} N_n$. Note that each $P_1^{N^{\mathbf{b}}} = P_1^N$; the extension $N^{\mathbf{b}}$ only adds an element c to $P_2^{N_\omega}$ and interprets the $F_m^{N^{\mathbf{b}}}(c)$.

The difficulty is that while we know each N_{n+1} is $\boldsymbol{K}_1$-free over N_n, witnessed by some (I_n, H_n), we don't know $b_n \in I_n$. We need to find I'_n which witnesses both $N_n \subset_{\mathrm{fr}} N_{n+1}$ and $b_n \in I'_n$. After this construction we will choose an $N^{\mathbf{b}}$ extending N witnessing goodness.

To find I'_n, we first find (X_n, J_n) such that:

1. $X_n \subseteq P_1^{N_n}$ is finite.

2. $J_n \subset I_n$ is countable.

3. If $c \in P_2^{N_{n+1}} - P_2^{N_n}$ then for sufficiently large m, $F_m^{N_{n+1}}(c) \notin J_n$.

4. $b_n \in BA(X_n \cup J_n)$, the *Boolean algebra generated* by $X_n \cup J_n$ in $P_1^{M_{n+1}}$.

First step: First, we construct such an (X_n, J_n). Note that b_n is in a subalgebra generated by a finite subset X_n of $P_1^{N_n}$ and a finite subset J_n' of I_n.

Now, by 1a) of Definition 3.3.4 , fix a sequence $\langle c_i \colon i < \omega \rangle$ of distinct elements of $P_2^{N_{n+1}} - P_2^{N_n}$. Note that for $i, j < \omega$ if $n_i > H_n(c_i)$ and $n_j > H_n(c_j)$ then $F_{n_i}^{N_{n+1}}(c_i) \neq F_{n_i}^{N_{n+1}}(c_i)$. Now we can construct a $J_n'' = \{d_{n,k} : k < \omega\}$ from $I_n - J_n'$ by $d_{n,k} = F_m^{N_{n+1}}(c_k)$ for some $m > H_n(c_k)$. We now have a countably infinite J_n'' contained in $I_n - J_n'$ such that for each $c \in P_2^{N_{n+1}} - P_2^{N_n}$ all but finitely many of the $F_m^{N_{n+1}}(c)$ are in $I_n - (J_n' \cup J_n'')$. Set $J_n = J_n' \cup J_n''$.

Second step: Now apply Lemma 3.1.10[18] to find J_n^* with J_n^* independent from $P_1^{M_n}$ over $P_4^{M_{n+1}}$ such that $\langle J_n^* \cup P_4^{M_{n+1}} \rangle_{P_1^{M_{n+1}}} = \langle J_n \cup P_4^{M_{n+1}} \rangle_{P_1^{M_{n+1}}}$ but $b_n \in J_n^*$. Now, I_n' can be taken as $(I_n - J_n) \cup J_n^*$. To ensure that $N_n \subseteq_{fr} N_{n+1}$ with basis I_n', replace $H_N(c_n)$ by $H_{N_n}(c_n) + r_n$ where (by Definition 3.2.3) some r_n bounds the number of m such that $F_m^{N_n}(c_n) \in \langle J_n \rangle_{P_1^{M_{n+1}}}$.

Having found an appropriate basis for $N = \bigcup N_n$, we extend N to $N^{\mathbf{b}}$ by adding an element $c^{\mathbf{b}}$ to $P_2^{N_2}$ and defining $F_n^{N^{\mathbf{b}}}(c^{\mathbf{b}}) = b_n$. The sentence immediately before Claim 3.3.5 guarantees that $N^{\mathbf{b}}$ is $\mathbf{K}_1$-free; set $H^{N^{\mathbf{b}}}(c^*) = 0$; thus, $N^{\mathbf{b}} \in \mathbf{K}_1$. Since the same b_n were used, it is clear the labeled sequence is A-good. (Note that there is no requirement that $m, n < \omega$, $c \in P_2^{M_0}$, $d \in P_2^{M_1}$ imply $F_n^{M_1}(c) \neq F_m^{M_1}(d)$; we only require that there be only finitely many such conflicts.) $\quad \square_{3.3.5}$

We now state precisely the main theorem.

Theorem 3.3.6. *Fix $\mathbf{K}_{<\aleph_0}^1$, $\mathbf{K}_1 = \hat{\mathbf{K}}_{<\aleph_0}^1$, and $\mathbf{K}_2 = \mathbf{R}$ as in Definitions 3.2.3, 3.2.9 and 3.2.20. There is a P_0-maximal for $\mathbf{K}_2$ model $M \in \mathbf{K}_2$ of card λ if there is no measurable cardinal ρ with $\rho \leq \lambda$, $\lambda = \lambda^{<\lambda}$, and there is an $S \subseteq S_{\aleph_0}^\lambda$, that is stationary non-reflecting, and $\diamond_S$ holds.*

Under $V = L$, the hypotheses are clearly consistent and imply there are arbitrarily large maximal models of $\mathbf{R}$ in L. When a measurable cardinal exists, the consistency of the conditions can be established by forcing; see the article by Cummings in the Handbook of Set Theory [Cum08] or by considering the inner model of a measurable $L[D]$ where is D is a normal ultrafilter on μ.

The argument for Theorem 3.3.6 will have three parts. First, we describe the requirements on a construction of a model; then we carry out the construction. Finally, we show the model constructed is P_0-maximal when λ is below the first measurable.

Construction 3.3.7 (Requirements). Let $\langle U_\alpha : \alpha < \lambda \rangle$ list $[\lambda]^{<\lambda}$ so that each subset is enumerated λ times and $U_\alpha \subseteq \alpha$. Since the set of ordinals $\alpha < \kappa$ such that $|\alpha|$ divides α is a cub for any κ, without loss of generality, each $\alpha \in S$ is a limit ordinal and is divided by $|\alpha|$. Let $\overline{A}^* = \langle A_\delta^* : \delta \in S \rangle$ be a $\diamond_S$-sequence.

We will choose M_α for $\alpha < \lambda$ by induction to satisfy the following conditions. (Since the universe of M is a subset of λ, its elements are ordinals so we may talk about their order although the order relation is not in τ.)

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[18]This is the crucial application of Lemma 3.1.10 which stengthened our notion of independence by getting a standard consequence of exchange, even though exchange fails here.

1. M_0 is isomorphic to the minimal model of $\boldsymbol{K}_1$. For $\alpha > 1$, $M_\alpha \in \boldsymbol{K}_2$ has universe an ordinal between α and λ.

2. $\langle M_\beta : \beta < \alpha \rangle$ is $\subseteq$- continuous.

3. If $\beta \in \alpha - S$ then M_α is $\boldsymbol{K}_1$-free over M_β, *and $M_\alpha \in \boldsymbol{K}_2 = \mathbf{R}$.*

4. If $\alpha = \beta + 2$ and $U_\beta \subseteq P_0^{M_\beta}$ then there is a $b_\beta \in P_1^{M_\alpha}$ such that $R(M_\alpha, b_\beta) \cap M_{\beta+1} = U_\beta$ and in the Boolean algebra $P_1^{M_\alpha}$, $\{b_\beta\}$ is free from $P_1^{M_{\beta+1}}$ over $P_4^{M_\alpha}$. Moreover $P_2^{M_\alpha} - P_2^{M_\beta}$ is infinite.

5. If $\delta \in S$ and $\alpha = \delta + 1$ then A) implies B), where:

    A) there is an A-good sequence $\overline{\gamma} = \langle \gamma_{\delta,n}, b_{\delta,n} : n < \omega \rangle$, where the $\gamma_{\delta,n}$ are increasing with n and not in S such that the $\langle b_{\delta,n} : n < \omega \rangle$ are good for the $M_{\gamma_{\delta,n}}$.

    B) there is a labeled A-good sequence $\hat{\overline{\gamma}} = \langle \hat{\gamma}_{\delta,n}, \hat{b}_{\delta,n} : n < \omega \rangle$, for $\langle M_{\gamma_{\delta,n}} : n < \omega \rangle$ with $c \in M_{\delta+1}$.

Remark 3.3.8. Condition 5 asserts that for any $A \subseteq \bigcup_{n < \omega} M_{\gamma_{\delta,n}}$ if there is an A-good sequence then there is a labeled A-good sequence. In the proof of Claim 3.3.5 we, in fact, took the same sequence so the 'A' is preserved automatically. But for each δ we construct only one pair of a c labeling a sequence $b_{\delta,n}$. We fix the relevant A in the first paragraph of 3.3.11; it will be the ultrafilter on P_1^M induced by a proper extension.

We now carry out the inductive construction.

Construction 3.3.9. *Details*

Case 1: $\alpha = 0$. Let M_0 be the minimal model from Lemma 3.2.10. The generic can be taken as M_1.

Case 2: $\alpha = \beta + 1$ and $\beta \notin S$. If β is a limit we only have to choose, by Lemma 3.2.21, M_α to be a $\boldsymbol{K}_1$-free extension of M_β in $\boldsymbol{K}_2$. If β is a successor, there is an additional difficulty. If $U_\beta \subset P_0^{M_\beta}$; we must choose b_β to satisfy condition 4) and with $M_\alpha \in \boldsymbol{K}_2$. For this, apply Lemma 3.2.24 with U_β as U and $M_{\beta+1}$ as M to construct N and b_β. Now iterate Corollary 3.2.21 $|M_{\beta+1}|$ times to obtain $M_\alpha \in \boldsymbol{K}_2$. This iteration also ensures $P_2^{M_\alpha} - P_2^{M_\beta}$ is infinite.

Case 3: $\alpha = \delta$, a limit ordinal that is not in S. Set $M_\delta = \bigcup_{\gamma < \delta} M_\gamma$. We must prove that if $\beta \in \delta \setminus S$ then M_δ is $\boldsymbol{K}_1$-free over M_β. Since S does not reflect there exists an increasing continuous sequence $\langle \alpha_i : i < \mathrm{cf}(\delta) \rangle$ of ordinals less than δ, which are not in S and with $\alpha_0 = \beta$. By the induction hypothesis, since $\alpha_j \notin S$, for each $i < j < \mathrm{cf}(\delta)$, M_{α_j} is $\boldsymbol{K}_1$-free over M_{α_i}. And by Lemma 3.2.13, M_δ is $\boldsymbol{K}_1$-free over M_β as required.

Case 4a: $\alpha = \delta + 1$, $\delta \in S$, and clause 5A fails. This is just as in case 2.

Case 4b: $\alpha = \delta + 1$, $\delta \in S$, but clause 5A holds.

So, suppose $\langle M_\beta, b_\beta \rangle$ for $\beta < \delta$ have been defined. If there exists $\overline{\gamma}$ as in condition 5A) of Construction 3.3.7 we must construct $\hat{\overline{\gamma}} = \langle \hat{\gamma}_{\delta,n}, \hat{b}_n^\delta : n < \omega \rangle$ and $\hat{c}^\delta$ to satisfy condition 5B). Take any $\langle \gamma_{\delta,n}, b_{\delta,n} : n < \omega \rangle$ satisfying 5A. Let the M_{γ_n} be the N_n from Claim 3.3.5 and by that claim, choose $M_{\delta+1}, \hat{c}_\delta \in P_2^{M_{\delta+1}}$ such that for each n, $F_n^{M_{\delta+1}}(\hat{c}_\delta) = \hat{b}_{\delta,n}$.

Case 5: Recall that δ is divisible by $|\delta|$ so we can choose the γ_n so that $\gamma_{n+1} \geq \gamma_n + \omega$ and each γ_n is not in S. So, by iterating as in Corollary 3.2.21, $P_2^{M_{\gamma_{n+1}}} - P_2^{M_{\gamma_{n+1}}}$ is infinite. Moreover, again since each γ_n is not in S, $M_{\gamma_{n+1}}$ is $\boldsymbol{K}_1$-free over M_{γ_n} so by Lemma 3.2.13, M_δ is $\boldsymbol{K}_1$-free.

This completes the construction.

Claim 3.3.10. *The structure $M = \bigcup_{i < \lambda} M_i \in \boldsymbol{K}_2$.*

Proof. Since we required the extension to be in $\boldsymbol{K}_2 = \boldsymbol{R}$ in requirement 3 of Construction 3.3.7, for cofinally many i, $M_i \in \boldsymbol{K}_2$. By Lemma 3.2.23, they are ∞, ω-elementary extensions. Hence $M \in \boldsymbol{K}_2$. $\square_{3.3.10}$

Construction 3.3.11. *Verification that the construction suffices*

Now we now show that M is P_0-maximal for $\boldsymbol{K}_2$. Suppose for contradiction there exists N in $\boldsymbol{K}_1$ extending M such that $P_0^N \supsetneq P_0^M$. Choose $a^* \in P_0^N - P_0^M$. Let

$$A = \{b \in P_1^M : R^N(a^*, b)\}.$$

Then, by Lemma 3.2.22.ii, for every $a \in P_0^N$, in particular a^* and every $b \in P_1^N$ (and so every $b \in P_1^M$) either $R^N(a^*, b)$ or $R^N(a^*, b^-)$. Thus, the subset A of P_1^M is a non-principal ultrafilter of the Boolean algebra P_1^M. For, if A is principal, it is generated by some atom $b_0 \in P_4^{M_1}$. Then b_0 must be in $P_{4,1}$ and so $\neg R^N(a^*, b_0)$, contrary to hypothesis. We will show that A induces an $\aleph_1$-complete ultrafilter on $\mathcal{P}(P_0^{M_{\alpha^*}})$ for some $\alpha^* < \lambda$. But this contradicts that λ is below the first measurable.

Recall that the A_δ are the diamond sequence fixed in requirement 3.3.7. Note

$$S_A = \{\delta \in S : M_\delta \text{ has universe } \delta \ \& \ A_\delta = A \cap \delta\}$$

is a stationary subset of λ. In the construction, we chose b_α for $\alpha < \lambda$ which satisfied requirement 4 of Construction 3.3.7. Note

$$C = \{\delta < \lambda : \delta \text{ limit } \& \ \alpha < \delta \to b_\alpha < \delta\}$$

is a club on λ.

There are two cases. We will show the first is impossible and the second implies λ is measurable, contrary to hypothesis. So the construction yields a P_0-maximal model in $\boldsymbol{K}_2$.

Case i): For every $\alpha < \lambda$ there is a $b \in P_1^M \cap A$ such that $R(M, b)$ is disjoint from α and $\{b\}$ is independent from $P_1^{M_\alpha}$ over P_4^M.

Choose $\delta^* \in S_A \cap C$. Since δ^* has cofinality ω we can choose a sequence $\hat{\gamma}^{\delta^*}_n$ such that each is a successor (so not in S), and, as we are in case i), with $b_{\gamma^{\delta^*}} < \gamma^{\delta^*}_{n+1}$. Since condition 5B) holds there are $\hat{c}_\delta \in P_2^{M_{\delta+1}}$ such that for each n, $F_n^{M_{\delta+1}}(\hat{c}_\delta) = \hat{b}_{\delta,n}$. Since $M_{\delta+1} \in \boldsymbol{K}_1$, by clause 8 of Definition 3.2.2, $M_{\delta+1} \models \neg(\exists x) \bigwedge_n R(x, F_n(c_\delta^*))$. This contradicts that we chose $b_{\gamma^{\delta^*}_n} \in A$, since by the definition of A, for each $n < \omega$, $R^N(a^*, b_{\gamma^{\delta^*}_n})$ holds.

case ii) For some α^*, there is no such b. That is, if $b \in P_1^M$ is independent from $P_1^{M_{\alpha^*}}$ over P_4^M and $R(M, b)$ is disjoint from α^* then $\neg R(a^*, b)$. From the list of elements of $[\lambda]^{<\lambda}$ at the beginning of Construction 3.3.7, we consider the subsequence $\langle v_\gamma : \gamma < \lambda \rangle$ enumerating $\mathcal{P}(P_0^{M_{\alpha^*}})$; recall each element appears λ times in the list.

We now choose inductively by requirement 4 of Construction 3.3.7 and Lemma 3.2.24 a subsequence[19] b_γ of the $b_\alpha \in P_1^M$ and M_γ such that $b_\gamma \in P_1^{M_{\gamma+1}}$ and $R(M_\gamma, b_\gamma) \cap P_0^{M_{\alpha^*}} = v_\gamma = R(M.b_\gamma)$ and $\langle b_\beta : \beta \le \gamma \rangle$ is independent from $P_1^{M_{\alpha^*}}$ over P_4^M. In particular, b_β is independent from $P_1^{M_\beta}$ over $P_4^{M_{\beta+1}}$ and so by Remark 3.2.14 over P_4^M.

---

[19] For local intelligibility (and at the risk of global confusion) we use indices b_γ and M_γ rather than b_{α_γ} and M_{α_γ} that would keep more precise track of the subsequence fact.

We claim that if $\gamma_1 < \gamma_2 \wedge v_{\gamma_1} = v_{\gamma_2}$ then $R^N(a^*, b_{\gamma_1}) \leftrightarrow R^N(a^*, b_{\gamma_2})$. For this, let $b' = b_{\gamma_1} \triangle b_{\gamma_2}$. Then $R(M, b') \cap P_0^{M_{\alpha^*}} = \emptyset$ so by the case choice, $\neg R(a^*, b')$. But $\neg R(a^*, b')$ implies $R^N(a^*, b_{\gamma_1}) \leftrightarrow R^N(a^*, b_{\gamma_2})$, as required.

Continuing the proof of case ii) we define an ultrafilter $\mathcal{D}$ on $\mathcal{P}(P_0^{M_{\alpha^*}})$ by $v \in \mathcal{D}$ if for some (and hence any) b_γ *from our chosen subsequence* with $R(M, b_\gamma) \cap P_0^{M_{\alpha^*}} = v$, $R^N(a^*, b_\gamma)$. (This is an ultrafilter as each $u \subset P_0^{M_{\alpha^*}}$ is $R(M, b_\gamma) \cap P_0^{M_{\alpha^*}}$ for some γ by requirement 4 of Construction 3.3.7.)

Now we show the coding of the elements of $\mathcal{D}$ extends to the entire original sequence.

Claim 3.3.12. *For any $b \in P_1^M$, which is one of the original sequence of independent b_α, if $v = R(M, b) \cap P_0^{M_{\alpha^*}}$ and $v \in \mathcal{D}$ then $N \models R(a^*, b)$.*

Proof. We can choose β, β_1 so that $\alpha^* < \beta < \lambda$, $b \in P_1^{M_\beta}$ and $\beta_1 > \beta$ such that $v_{\beta_1} = v$. Now $\check{b} = b \triangle b_{\beta_1} \in P_1^M$ and $R(M, \check{b}) \cap P_0^{M_{\alpha^*}} = \emptyset$. Note that since $\langle b_\beta : \beta < \lambda \rangle$ is independent from $P_1^{M_{\alpha^*}}$ over P_4^M in P_1^M, in particular b and b_{β_1} are independent so the singleton $b \triangle b_{\beta_1}$ is independent from $P_1^{M_{\alpha^*}}$ over P_4^M in P_1^M. So by the choice of α_*, $N \models \neg R(a^*, \check{b})$. So, $N \models \neg R(a^*, b)$ if and only $N \models \neg R(a_*, b_{\beta_1})$. But, we have $v \in \mathcal{D}$ and $R(M, b_{\beta_1}) \cap P_0^{M_{\alpha^*}} = v$, so $N \models R(a^*, b_{\beta_1})$ and thus $N \models R(a_*, b)$ as required. $\square_{3.3.12}$

There is no $\aleph_1$-complete ultrafilter on $\mathcal{P}(P_0^{M_{\alpha^*}})$ since $|P_0^{M_{\alpha^*}}| < \lambda$ is not measurable. So there are $\langle w_n \subseteq P_0^{M_{\alpha^*}} : n < \omega \rangle$, each in $\mathcal{D}$, that are decreasing and intersect in $\emptyset$. Now we can find $\delta^* > \alpha^*$ such that $\delta^* \in S_A \cap C$, the universe of M_{δ^*} is δ^*, $A_{\delta^*} \cap \delta^* = A \cap \delta^*$, and there is an increasing sequence $\langle \gamma_n^{\delta^*} : n < \omega \rangle$ with limit at most δ^* and each $\gamma_n^{\delta^*} \notin S$. Further, by requirement 4 on the construction, we can choose $\gamma_n^{\delta^*}$ so that $b_{\gamma_n^{\delta^*}}$ (another subsequence of the original sequence) satisfies $a \in R(M_{\gamma_n}, b_{\gamma_n})$ if and only if $a \in w_n$, $b_{\gamma_n^{\delta^*}} \in M_{\gamma_{n+1}^{\delta^*}}$, and the sequence $\{b_{\gamma_n^{\delta^*}}\}$ is independent from $P_1^{M_{\delta^*}}$ over P_4^M. Since the w_n are decreasing with empty intersection, no $a \in M_{\alpha^*}$ is in more than finitely many of the w_n. Thus, Definition 3.3.4 1c is satisfied.

So by clause 5) of the construction, there is a *labeled A-good sequence* $\hat{b}_{\delta^*, n}$ for M_{δ^*+1}, $\hat{c}_\delta^* \in P_2^{M_{\delta^*+1}}$ such that for each n, $F_n^{M_{\delta^*+1}}(\hat{c}_\delta^*) = \hat{b}_{\delta^*, n}$. And by clause 8 of Definition 3.2.2, this contradicts Claim 3.3.12; the intersection of $R(N, F_n^N(c))$ for $n < \omega$ must be empty but it contains a^*. So we finish case ii) and thus Lemma 3.3.6. $\square_{3.3.6}$

Remark 3.3.13. In the construction we showed for limit δ that M_δ is $\boldsymbol{K}_1$-free using S does not reflect if $\delta \notin S$ and that $\operatorname{cf}(\delta) = \omega$ for $\delta \in S$. We have no such tools to show the P_0-maximal model, $M = M_\lambda$ built in Theorem 3.3.14 is $\boldsymbol{K}_1$-free. In fact, by the contrapositive of Corollary 3.2.21 the final P_0-maximal model, which might be M, is *not $\boldsymbol{K}_1$-free*.

Note that every subset of M with cardinality $< \lambda$ is contained in a $\boldsymbol{K}_1$-free substructure; this fails in the ZFC proof [BS18] of maximal models of $\boldsymbol{K}_2$ cofinal in a measurable.

We have constructed a P_0-maximal model in $\boldsymbol{K}_2$; one step remains.

Corollary 3.3.14. *Under the hypotheses of Theorem 3.3.6, there is a maximal model of $\boldsymbol{K}_2$ of cardinality at most 2^λ.*

Proof. Fix a P_0-maximal model N_0 of cardinality λ from Theorem 3.3.6. Build for as long as possible a continuous $\subseteq$-increasing chain of $N_\alpha \in \boldsymbol{K}_2$ such that each $N_\alpha \neq N_{\alpha+1}$. Recall that by Lemma 3.2.22.1 the relation R is injective. So, each $|P_1^{N_\alpha}| \leq 2^{|P_0^{N_0}|} = 2^\lambda$. So this construction must stop and the final, maximal in $\boldsymbol{K}_2$, model has cardinality at most 2^λ. $\square_{3.3.14}$

4 Hanf Number for Existence

As mentioned in the introduction, we improved in [BKL16] Hjorth's result [Hjo02] by exhibiting for each $n < \omega$ a complete sentence ψ_n such that ψ_n characterizes $\aleph_n$. This improvement is achieved by combining the combinatorial idea of Laskowski-Shelah in [LS93] with a new notion of n-dimensional amalgamation. We explain the main definition and theorem here (as in the Tehran lectures) and refer to [BKL16] for the proofs. The combinatorial fact is:

Fact 4.0.1. *For every $k \in \omega$, if* cl *is a locally finite closure relation on a set X of size $\aleph_k$, then there is an independent subset of size $k + 1$.*

Fix a vocabulary τ_r with infinitely many r-ary relations R_n and infinitely many $r + 1$-ary functions f_n. We consider the class $\boldsymbol{K}_0^r$ of finite τ_r-structures (including the empty structure) that satisfy the following three conditions; closure just means subalgebra closure with respect to the functions.

- The relations $\{R_n : n \in \omega\}$ partition the $(r + 1)$-tuples;

- For every $(r + 1)$-tuple $\boldsymbol{a} = (a_0, \ldots, a_r)$, if $R_n(\boldsymbol{a})$ holds, then $f_m(\boldsymbol{a}) = a_0$ for every $m \geq n$;

- There is no independent subset of size $r + 2$.

It is easy to see from Fact 4.0.1 that every model in $\aleph_r$ is maximal. The main effort is to show there is a complete sentence ϕ_r satisfying those conditions which has model in $\aleph_r$. For this we introduce a notion patterned on excellence[20] but weaker. We pass from a class $\boldsymbol{K}_0^r$ of, now, locally finite structures to the associated class $\widehat{\boldsymbol{K}}$ as in Definition 2.1.1.

Definition 4.0.2. *For $k \geq 1$, a k-configuration is a sequence $\overline{M} = \langle M_i : i < k \rangle$ of models (not isomorphism types) from $\boldsymbol{K}$. We say $\overline{M}$ has power λ if $\|\bigcup_{i<k} M_i\| = \lambda$. An extension of $\overline{M}$ is any $N \in \boldsymbol{K}$ such that every M_i is a substructure of N.*

Informally a (λ, k)-disjoint amalgamation holds when for any sequence of k models, at least one with λ elements, there is common extension, which properly extends each model in the sequence. Crucially, there is assumption of a universal model. Here is the precise formulation.

Definition 4.0.3. *Fix a cardinal $\lambda = \aleph_\alpha$ for $\alpha \geq -1$. We define the notion of a class $(\boldsymbol{K}, \leq)$ having (λ, k)-disjoint amalgamation in two steps:*

1. *$(\boldsymbol{K}, \leq)$ has $(\lambda, 0)$-disjoint amalgamation if there is $N \in \boldsymbol{K}$ of power λ;*

2. *For $k \geq 1$, $(\boldsymbol{K}, \leq)$ has $(\leq \lambda, k)$-disjoint amalgamation if it has $(\lambda, 0)$-disjoint amalgamation and every k-configuration $\overline{M}$ of cardinality $\leq \lambda$ has an extension $N \in \boldsymbol{K}$ such that every M_i is a proper substructure of N.*

For $\lambda \geq \aleph_0$, we define $(< \lambda, k)$-disjoint amalgamation by: has $(\leq \mu, k)$-disjoint amalgamation for each $\mu < \lambda$.

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[20]Shelah's theory of excellence concerns unique free disjoint amalgamations of infinite structures in ω-stable classes of models of complete sentences in $L_{\omega_1, \omega}$.

Whether or not a given k-configuration $\overline{M}$ has an extension depends on more than the sequence of isomorphism types of the constituent M_i's, as the pattern of intersections is relevant as well. For example, when (as here) strong substructure is just substructure), a 2-configuration $\langle M_0, M_1 \rangle$ with neither contained in the other has an extension if and only if the triple of structures $\langle M_0 \cap M_1, M_0, M_1 \rangle$ has an extension amalgamating them disjointly. Thus we abuse notation a bit and write $(< \lambda, 2)$ amalgamation for both the notion defined here and the one in Definition 2.1.4. But there is no existing analog of our disjoint $(< \lambda, k)$-amalgamation for $k > 2$.

Now we modify a theme familiar from the theory of excellence. If the cardinality increases by one the the number of models that can be amalgamated drops by one. In Shelah's context [She09] (chapter 21 of [Bal09]) there is a reliance on Fodor's lemma to obtain compatible filtrations of the models in κ^+ to prove the version of Proposition 4.0.4. A very different approach was needed to go from the finite to the countable. Instead of the kth level concerning finding an embedding into an upper corner for a given 2^{k-1} vertices of a k-cube, we consider actual containment for k-models and do not worry about their intersections.

Lemma 4.0.4 (Proposition 2.20 of [BKL16]). *Fix locally finite $(\boldsymbol{K}, \leq)$ with JEP. For all cardinals $\lambda \geq \aleph_0$ and for all $k \in \omega$, if $\boldsymbol{K}$ has $(< \lambda, k+1)$-disjoint amalgamation, then it also has $(\leq \lambda, k)$-disjoint amalgamation.*

Together, these propositions yield 1)-3) of the next result. Recall from Definition 2.1.4, that by 2-amalgamation, we mean the usual notion that allows identifications. We say 2-amalgamation is *trivially* true in a cardinal κ in all models in κ are maximal.

Theorem 4.0.5 (Theorem 3.2.4 of [BKL16]). *For every $r \geq 1$, the class $\mathbf{At^r}$ satisfies:*

1. *there is a model of size $\aleph_r$, but no larger models;*

2. *every model of size $\aleph_r$ is maximal, and so 2-amalgamation is trivially true in $\aleph_r$;*

3. *disjoint 2-amalgamation holds up to $\aleph_{r-2}$;*

4. *2-amalgamation fails in $\aleph_{r-1}$.*

5. *Each of the classes $\hat{\boldsymbol{K}}^r$ and $\mathbf{At}^r$ have $2^{\aleph_s}$ models in $\aleph_s$ for $1 \leq s \leq r$. In addition, $\hat{\boldsymbol{K}}^r$ has $2^{\aleph_0}$ models in $\aleph_0$.*

Parts 4) and 5) require a further refinement of the notion of disjoint amalgamation.

Definition 4.0.6. *Given a cardinal λ and $k \in \omega$, we say that $\boldsymbol{K}$ has* frugal $(\leq \lambda, k)$-disjoint amalgamation *if it has $(\leq \lambda, k)$-disjoint amalgamation and, when $k \geq 2$, every k-configuration $\langle M_i : i < k \rangle$ of cardinality $\leq \lambda$ has an extension $N \in \boldsymbol{K}$ with universe $\bigcup_{i<k} M_i$.*

Thus the domain of a frugal amalgamation is just the union of the models amalgamated. It is easy to see that this property holds for the example in [BKL16]. It is essential for the intricate constructions to verify the last two parts of Theorem 4.0.5 and for the work in [BKS16, BS17].

The *finite amalgamation spectrum* of an abstract elementary class $\boldsymbol{K}$ with $LS(\boldsymbol{K}) = \aleph_0$ is the set $X_{\boldsymbol{K}}$ of $n < \omega$ and $\boldsymbol{K}$ satisfies amalgamation[21] in $\aleph_n$. There are many examples[22] where the finite amalgamation spectrum of a complete sentence of $L_{\omega_1, \omega}$ is either $\emptyset$ or ω.

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[21]We say amalgamation holds in κ in the trivial special case when all models in κ are maximal. We say amalgamation fails in κ if there are no models to amalgamate.

[22]Kueker, as reported in [Mal68], gave the first example of a complete sentence failing amalgamation in $\aleph_0$.

Theorem 4.0.5 gave the first example of such a sentence with a non-trivial spectrum: for each $1 \leq r < \omega$ amalagmation holds up to $\aleph_{r-2}$, but fails in $\aleph_{r-1}$. It holds (trivially) in $\aleph_r$ (since all models are maximal); there is no model in $\aleph_{r+1}$.

This result leaves open whether the property, AP in λ, can be true or false in various patterns as λ increases? Is there even an AEC (and more interestingly a complete sentence of $L_{\omega_1,\omega}$) and cardinals $\kappa < \lambda$ such that amalgamation holds non-trivially in both κ and λ but fails at some cardinal between them?

Relying on the construction in [BKL16], Baldwin and Souldatos [BS17] show there exist *complete* sentences of $L_{\omega_1,\omega}$ that variously have maximal models a) in two successive cardinals, b) in κ and κ^ω and c) in countably many cardinals. In each case all maximal models of the sentence have cardinality less than $\aleph_{\omega_1}$. That proof includes an intricate construction of a complete sentence that has a model in each successor cardinal κ^+ with a definable subset of power κ. The [BS17] result is distinguished from the one here in several ways. It constructs maximal models in designated cardinals rather than an initial segment. The crucial amalgamation properties are quite different. The example in [BKL16] satisfies $(< \lambda, 2)$ amalgamation in all cardinals.

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