Model theory and the Middle attic

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AMS regional meeting
Bloomington, IN
April 1, 2017

Cantor's Middle Attic

Uncountable cardinals whose existence can be proved in, or is at least equiconsistent with, the ZFC axioms of set theory. `http://cantorsattic.info/Middle_attic`

The lower reaches

What happens below $\beth_{\omega_1}$

To what extent is that behavior controlled by finite or at least locally finite countable structures?

What model theoretic properties happen higher up? How much higher?

Atomic Model Theory

Atomic models

Let T be a complete first order theory in a countable language.

Definition

A model M of T is atomic if every element of M realizes a principal type.

Examples

1. $(\mathbb{Q}, <)$, $(\Re, <)$
2. $(\mathbb{Z}, <)$
3. $(N, +, \times)$

Not two copies of $(\mathbb{Z}, <)$

Questions

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Does the Löwenheim-Skolem theorem hold for 'atomic' models?
down? up?

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Question

What do we know about routine properties in first order logic

existence, maximality, joint embedding, amalgamation

for atomic models?

The translation

Theorem

[Chang/Lopez-Escobar] Let ψ be a sentence in $L_{\omega_1,\omega}$ in a countable vocabulary τ. Then there is a countable vocabulary τ' extending τ, a first order τ'-theory T, and a countable collection of τ'-types Γ such that reduct is a 1-1 map from the models of T which omit Γ onto the models of ψ.

The proof is straightforward. E.g., for any formula ψ of the form $\bigwedge_{i<\omega} \phi_i$, add to the language a new predicate symbol $R_\psi(\mathbf{x})$. Add to T the axioms

$$(\forall \mathbf{x})[R_\psi(\mathbf{x}) \rightarrow \phi_i(\mathbf{x})]$$

for $i < \omega$ and omit the type $p = \{\neg R_\psi(\mathbf{x})\} \cup \{\phi_i : i < \omega\}$.

complete $L_{\omega_1,\omega}$-sentences

Definition

ϕ is $L_{\omega_1,\omega}$-complete if for every $\psi \in L_{\omega_1,\omega}$, $\phi \models \psi$ or $\phi \models \neg\psi$.

A τ-structure M is $L_{\omega_1,\omega}$-small if M realizes only countably many $L_{\omega_1,\omega}$-types (over the empty set).

Generalized Scott's theorem

A structure satisfies a complete sentence of $L_{\omega_1,\omega}$ if and only if it is small.

Reducing complete to atomic

The models of a complete sentence in $L_{\omega_1,\omega}$ can be represented as:
$\boldsymbol{K}$ is the class of atomic models (realize only principal types) of a complete first order theory (in an expanded language).

Another Direction

The techniques here construct a variety of new first order theories.

How do they fit into the (neo)-stability classification?

Can one bound the cardinality of the atomic models, when the ambient theory is stable? etc.?

Example and Background

Bounding Cardinality

Independent means with respect to subalgebra generation.

Fact

For every $k \in \omega$, if cl is a locally finite closure relation on a set X of size $\aleph_k$, then there is an independent subset of size $k+1$.

Proof. By induction on k. When $k = 0$, take any singleton not included in $\mathrm{cl}(\emptyset)$. Assuming the Fact for k, given any locally finite closure relation cl on a set X of size $\aleph_{k+1}$, fix a cl-closed subset $Y \subseteq X$ of size $\aleph_k$ and choose any $a \in X \setminus Y$. Define a locally finite closure relation cl_a on Y by $\mathrm{cl}_a(Z) = \mathrm{cl}(Z \cup \{a\}) \cap Y$. It is easily checked that if $B \subseteq Y$ is cl_a-independent, then $B \cup \{a\}$ is cl-independent.

Example: atomic models up to $\aleph_r$

Laskowski-Shelah

τ_r has infinitely many r-ary relations R_n and infinitely many $r+1$-ary functions f_n.
We consider the class $\boldsymbol{K}_0^r$ of finite τ_r-structures (including the empty structure) that satisfy the following three conditions.

- The relations $\{R_n : n \in \omega\}$ partition the $(r+1)$-tuples;
- For every $(r+1)$-tuple $\boldsymbol{a} = (a_0, \ldots, a_r)$, if $R_n(\boldsymbol{a})$ holds, then $f_m(\boldsymbol{a}) = a_0$ for every $m \geq n$;
- There is no independent subset of size $r+2$.

Conclusion and Question

Let $\hat{\boldsymbol{K}}^r$ be the collection of all structures A such that every finite substructure of A is in $\boldsymbol{K}^r_0$.
Let $\mathrm{cl}_A(X)$ be the closure relation on a model A defined by closure under functions.

Conclusion

There is no model of the class $\hat{\boldsymbol{K}}^r$ of cardinality greater than $\aleph_r$.

Question

Is there a model of the class $\hat{\boldsymbol{K}}^r$ of cardinality $\aleph_r$?

Axiomatize $\hat{K}^r$

ϕ_r

- $\bigwedge_{m \neq n}(R_n(\mathbf{x}) \rightarrow \neg R_m(\mathbf{x}))$
- $\bigvee_n R_n(x_0 \ldots x_r)$
- $(\forall \mathbf{x})R_n(\boldsymbol{a}) \rightarrow \bigwedge_{m \geq n} f_m(\mathbf{x}) = x_0$;
- There is no independent subset of size $r + 2$.

  A slightly more complicated sentence in $L_{\omega_1,\omega}$

Theorem

ϕ_r has no model of cardinality greater than $\aleph_r$

Question

Is there a completion of ϕ_r with a model of cardinality $\aleph_r$?

The general framework

Notation

Let $\boldsymbol{K}_0$ be a countable collection of *finite* structures.

We associate with $\boldsymbol{K}_0$ the collection of models such that every finite substructure of M is in $\boldsymbol{K}_0$ and call this class $\hat{\boldsymbol{K}}$; it is locally finite.

In most of this talk we deal only with substructure as the notion of 'strong' submodel.

But we do not, in general, require that $\boldsymbol{K}_0$ is closed under substructure.

Universal classes of models

Function symbols are allowed on this slide without loss of generality.

Definition

A class $(\boldsymbol{K}, \leq)$ of τ-structures and the relation $\leq$ as ordinary substructure that is closed under

1. substructure ($S(\boldsymbol{K}) = \boldsymbol{K}$)
2. unions of increasing chains ($\lim(\boldsymbol{K}) = \boldsymbol{K}$)

is called a *universal class*.

Theorem: Tarski

A class $(\boldsymbol{K}, \leq)$ of τ-structures is a universal class iff

1. $S(\boldsymbol{K}) = \boldsymbol{K}$
2. $S(A) \subset \boldsymbol{K}$ implies $A \in \boldsymbol{K}$.

Generic structures

Definition

Let $(\boldsymbol{K}_0, \leq)$ denote a class of finite (finitely generated) τ-structures and let $(\hat{\boldsymbol{K}}, \leq)$ denote the associated (closure under unions) locally finite class. (Closed under substructure if $\boldsymbol{K}$ is.)

1. A model $M \in \hat{\boldsymbol{K}}$ is finitely $\boldsymbol{K}_0$*-homogeneous* or *rich* if for all finite $A, B \in \boldsymbol{K}_0$, every embedding $f : A \rightarrow M$ extends to an embedding $g : B \rightarrow M$. We denote the class of rich models in $\hat{\boldsymbol{K}}$ as $\boldsymbol{K}^R$.
2. The model $M \in \hat{\boldsymbol{K}}$ is *generic* if M is rich and M is an increasing union of a chain of finite (finitely generated) substructures, each of which are in $\boldsymbol{K}_0$.

All rich models are (∞, ω)-equivalent.

Generalized Fraïssé

The vocabulary may be infinite and include function symbols.

Lemma

If a class $(\boldsymbol{K}_0, \leq)$ satisfies amalgamation, JEP, and has countably many elements, then there is a unique countable generic model, which is rich.

Note that we do *not* need 'closed under substructure'. $\leq$ does not have to be substructure. The new results here are about classes with $\leq$ as substructure and closed under substructure.
If the first order theory of the generic is $\aleph_0$-categorical then $\boldsymbol{K}_0, \subseteq)$ is uniformly locally finite.

The importance of Vocabulary Choice

Definition

A class $\boldsymbol{K}_0$ of finite structures in a countable vocabulary is *separable* if

1. $(\boldsymbol{K}_0, \leq)$ satisfies amalgamation;
2. For each $A \in \boldsymbol{K}_0$, there is a first order formula $\phi_A(\mathbf{x})$ such that in any $M \in \hat{\boldsymbol{K}}$, $M \models \phi_A(\mathbf{b})$ if and only if $\mathbf{b}$ enumerates a substructure of M that is isomorphic to A.

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Lemma

Suppose $\boldsymbol{K}_0$ is a class of finite τ-structures that is closed under substructure, satisfies JEP, and is separable. Then the generic M is an atomic model of $Th(M)$. Moreover, $\boldsymbol{K}^R = \mathbf{At}$, i.e., every rich model N is an atomic model of $Th(M)$.

n-disjoint amalgamation and Extending Hjorth's theorem

Hjorth's theorem

Theorem: Hjorth

There is a sequence of countable families Φ_α of complete sentences of $L_{\omega_1,\omega}$, such that some $\phi \in \Phi_\alpha$ has a model in $\aleph_\alpha$ and no bigger.

Theorem: B-Koerwien-Laskowski

There is a sequence ϕ_n of complete sentences of $L_{\omega_1,\omega}$, such that ϕ_n has a model in $\aleph_n$ and no bigger.

Question

Can the second theorem be extended to all countable α?

Excellence

Let $\boldsymbol{K}$ be an ω-stable class of atomic models.

Shelah: For Shelah the independence is the sense of ω-stability. This was the origin of the notion of otop.

Definition

A set of $\boldsymbol{K}$-structures $\overline{N} = \langle N_u : u \subsetneq k \rangle$ is a -system if it is a directed system of $\boldsymbol{K}$-structures with cardinality λ indexed by the proper subsets of k.

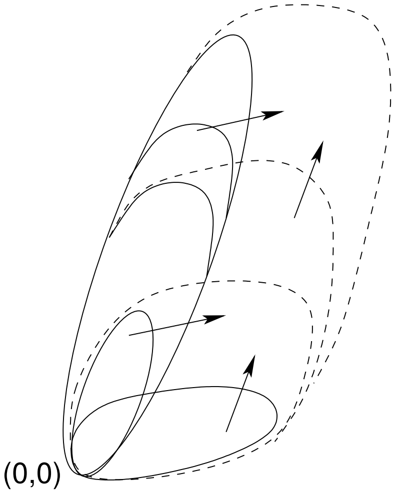
Definition

The class $\boldsymbol{K}$ has (λ, k) excellence((λ, k)-good) if every independent (λ, k)-system has a unique prime amalgam. For Shelah, excellent is $(\aleph_0, n)$-good for every n.

Consequences of Excellence

1. Shelah: An excellent class has 2-disjoint amalgamation in all cardinalities.
2. Shelah: An excellent class that is categorical up to $\aleph_\omega$ is categorical in all uncountable powers.
3. Shelah ($2^{\aleph_n} < 2^{\aleph_{n+1}}$) An atomic class that has at most $2^{\aleph_n}$ models in $\aleph_{n+1}$ for all n is excellent.

$(\aleph_0, 3)$-ap implies $(\aleph_1, 2)$-ap



k-configurations

Definition

For $k \geq 1$, a *k-configuration* is a sequence $\overline{M} = \langle M_i : i < k \rangle$ of models (not isomorphism types) from $\boldsymbol{K}$. We say $\overline{M}$ *has power* λ if $\| \bigcup_{i<k} M_i \| = \lambda$. An *extension* of $\overline{M}$ is any $N \in \boldsymbol{K}$ such that every M_i is a substructure of N.

Informal: (λ, k)*-disjoint amalgamation*

Any sequence of k models, at least one with λ elements, has a common extension, which properly extends each.

(λ, k)-disjoint amalgamation

Definition

Fix a cardinal $\lambda = \aleph_\alpha$ for $\alpha \geq -1$. We define the notion of a class $(\boldsymbol{K}, \leq)$ having (λ, k)-*disjoint amalgamation* in two steps:

1. $(\boldsymbol{K}, \leq)$ has $(\lambda, 0)$-disjoint amalgamation if there is $N \in \boldsymbol{K}$ of power λ;
2. For $k \geq 1$, $(\boldsymbol{K}, \leq)$ has $(\leq \lambda, k)$-disjoint amalgamation if it has $(\lambda, 0)$-disjoint amalgamation and every k-configuration $\overline{M}$ of cardinality $\leq \lambda$ has an extension $N \in \boldsymbol{K}$ such that every M_i is a proper substructure of N.

For $\lambda \geq \aleph_0$, we define $(< \lambda, k)$-disjoint amalgamation by: has $(\leq \mu, k)$-disjoint amalgamation for each $\mu < \lambda$.

Extending (λ, k)-disjoint amalgamation

Fix locally finite $(\boldsymbol{K}, \leq)$ with jep.

Proposition

For all cardinals $\lambda \geq \aleph_0$ and for all $k \in \omega$, if $\boldsymbol{K}$ has $(< \lambda, k+1)$-disjoint amalgamation, then it also has $(\leq \lambda, k)$-disjoint amalgamation.

{

Getting models

Theorem

[BKL] Suppose $1 \leq r < \omega$ and $\boldsymbol{K}_0$ has the $(< \aleph_0, r+1)$-disjoint amalgamation property. Then for every $0 \leq s \leq r$, $(\hat{\boldsymbol{K}}, \leq)$ has the $(\leq \aleph_s, r-s)$-disjoint amalgamation property.

In particular, $\hat{\boldsymbol{K}}$ has models of power $\aleph_r$.

Moreover, if there are only countably many isomorphism types in $\boldsymbol{K}_0$, then rich models of power $\aleph_r$ exist and the class $\boldsymbol{K}^R$ also has $(\leq \aleph_s, r-s)$-disjoint amalgamation.

Getting Rich Models

Lemma

Fix $\lambda \geq \aleph_0$. If $\boldsymbol{K}$ has $(< \lambda, 2)$-disjoint amalgamation and has at most λ isomorphism types of finite structures, then

1. every $M \in \boldsymbol{K}$ of power λ can be extended to a rich model $N \in \boldsymbol{K}$, which is also of power λ.
2. and consequently there is a rich model in λ^+.

Amalgamation Spectrum

Theorem

[BKL] For every $r \geq 1$, the class $\mathbf{At}^{\mathbf{r}}$ satisfies:

1. there is a model of size $\aleph_r$, but no larger models;
2. every model of size $\aleph_r$ is maximal, and so 2-amalgamation is trivially true in $\aleph_r$;
3. disjoint 2-amalgamation holds up to $\aleph_{r-2}$;
4. 2-ap fails in $\aleph_{r-1}$.

More technically, amalgamation for elementary submodels in $\hat{\boldsymbol{K}}^r$ also fails in $\aleph_{r-1}$.

The Amalgamation spectrum

The *finite amalgamation spectrum* of a complete sentence ϕ is the set X_ϕ of $n < \omega$ and $\operatorname{mod}(\phi)$ satisfies amalgamation in $\aleph_n$.

Many examples: X_ϕ is $\emptyset$ or ω.

This is the first example of a complete sentence an aec where the spectra was not: all, none, or just $\{0\}$.

Question

Can the amalgamation spectrum of a complete sentence of $L_{\omega_1,\omega}$ have a proper alternation?

How many models of sentences with bounded spectrum?

Fact

If $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the amalgamation property in κ then models of cardinality κ^+ can be amalgamated over models of cardinality κ.

Corollary

If all models in cardinality κ^+ are maximal and $\boldsymbol{K}$ had ap in κ then there are at most 2^κ models in κ^+.

Question

Is there any complete sentence in $L_{\omega_1,\omega}$ with such behavior?

Homogeneous Characterizability

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The complete sentence ϕ with countable model M homogenously characterizes κ if

1. P^M is a set of absolute indiscernibles.
2. ϕ has no model of cardinality greater than κ.
3. There is a model N with $|P^N| = \kappa$.

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Theorem (Gao)

If countable structure has a set of absolute indiscernibles, there is an $L_{\omega_1,\omega}$ equivalent model in $\aleph_1$.

Mergers

Mergers

1. Let θ be a complete sentence of $L_{\omega_1,\omega}$ and suppose M is the countable model of θ and $V(M)$ is a set of absolute indiscernibles in M such $M - V(M)$ projects onto $V(M)$. We will say θ is a *receptive* sentence.
2. For any sentence ψ of $L_{\omega_1,\omega}$, the *merger* of ψ and θ is the sentence $\chi = \chi_{\theta,\psi}$ obtained by conjoining with θ, $\psi \restriction N$.
3. For any model M_1 of θ and N_1 of ψ we write $(M_1, N_1) \models \chi$ if there is a model with such a reduct.

DAP: Getting receptive models

Suppose $\boldsymbol{K}_0$ is a class of finite τ structures with disjoint amalgamation and θ_0 is the Scott sentence of the generic.

Construction: B-Friedman-Koerwien-Laskowski

Add to τ unary predicates U, V and binary P.
Require that the predicates U and V partition the universe and restrict the relations of τ to hold only within the predicate V.

We set $\boldsymbol{K}_1$ as the set of finite τ_1-structures (V_0, U_0, P_0) where $V_0 \restriction \tau \in \boldsymbol{K}$ and P_0 is the graph of a partial function from V_0 into U_0.
Disjoint ap in $\boldsymbol{K}_0$ gives disjoint ap in $\boldsymbol{K}_1$.

A back and forth shows $U(\mathcal{M})$ is absolutely indiscernible.

Applying merger

Theorem: (Hjorth, B-Friedman-Koerwien-Laskowski)

There is a receptive sentence that characterizes (has only maximal models) $\aleph_1$.

Corollary: (B-Friedman-Koerwien-Laskowski)

If there is a counterexample to Vaught's conjecture there is one that has only maximal models in $\aleph_1$.

crux: Disjoint amalgamation

Homogeneous Characterization of successors

Theorem: Hjorth, B-Souldatos

If a cardinal κ is homogeneously characterizable by a complete sentence, then κ^+ is characterizable.

Construct by disjoint amalgamation an $\aleph_0$-categorical structure in which:

1. Q_0, Q_1, Q_2 partition the universe.
2. E is an equivalence relation on Q_1.
3. P_0 gives a family of maps P_a indexed by $a \in Q_1$ so that P_a maps Q_0 onto $[a]_E$, the E-equivalence class of a.
4. P_1 is a projection of Q_1 onto Q_2 such that each $P_1^{-1}(a)$ is an E-equivalence class.
5. P_2 gives an indexed family P_a for $a \in Q_2$ so that P_a maps the elements in $P_1^{-1}(a)$ onto the elements $\{b \in Q_2 : b \leq a\}$.
6. $<$ is a dense linear order of Q_2.

Maximal models

Theorem: B-Souldatos

There are complete sentences of $L_{\omega_1,\omega}$, with

1. maximal models in κ and κ^+.
2. Assume for simplicity that $2^{\aleph_0} > \aleph_\omega$. For each $n \in \omega$, there is a complete $L_{\omega_1,\omega}$-sentence ϕ'_n with maximal models in cardinalities $2^{\aleph_0}, 2^{\aleph_1}, \ldots, 2^{\aleph_n}$.
3. Assume κ is a homogeneously characterizable cardinal and for simplicity let $2^{\aleph_0} \geq \kappa$. Then there is a complete $L_{\omega_1,\omega}$-sentence ϕ_κ with maximal models in cardinalities 2^λ, for all $\lambda \leq \kappa$.

Glimpses of the upper attic

The big gap

Upper Attic

Welcome to the upper attic, the transfinite realm of large cardinals, the higher infinite, carrying us upward from the merely inaccessible and indescribable to the subtle and endlessly extendible concepts beyond, towards the calamity of inconsistency.

`http://cantorsattic.info/Upper_attic`

Theorem. B-Boney

The Hanf number for Amalgamation is at most the first strongly compact cardinal

The best known lower bound is $\aleph_\omega$.

Maximal Models and Measurable cardinals

Theorem: B-Shelah – in progress

There is a complete sentence ϕ of $L_{\omega_1,\omega}$ such that if

1. there is no measurable cardinal ρ with $\rho \leq \lambda$, $\lambda = \lambda^{<\lambda}$,
2. and there is an $S \subseteq S^{\lambda}_{\aleph_0}$, that is stationary non-reflecting, and $\diamond_S$ holds.

Then there is a maximal model M of ϕ with cardinality at most 2^{λ}

A black box is expected to remove the set theoretic hypotheses.

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Why complete sentence?

Consider a class $\boldsymbol{K}$ of 4-sorted structures:

1. P_0 is a copy of $(\omega, <)$.
2. P_1 is a set.
3. P_2 is a boolean algebra of subsets of P_1 (given by extensional binary E.
4. P_3 is a set of countable sequences from P_2. via a function $f(c, n) = b$ maps $c \in P_3, n \in P_0, b \in P_1$.)

One further axiom: If a sequence $c \in P_3$ has the finite intersection property then the intersection is non-empty.

Let $\psi \in L_{\omega_1,\omega}$ axiomatize $\boldsymbol{K}$.

Why maximal?

M is a maximal model of $\boldsymbol{K} = \mathrm{mod}(\psi)$ if

1. $\lambda <$ first measurable
2. $|P_0^M| = \lambda$.
3. $P_1^M = \mathcal{P}(P_0^M)$
4. $P_2^M = {}^\omega(P_1^M)$

M can only be extended by adding an element $a*$ to P_0^M. But then

$$\{b \in P_1^M : E(a*, b)\}$$

is a non-principal $\aleph_1$-complete ultrafilter on λ.

But ψ is not complete. $2^{\aleph_0}$ types over empty set.
c_X realizes p_X iff $|a \in P_0^M : a \in f(c, n)| \in X$.