Model Theoretic Perspectives on the Philosophy of Mathematics

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> And some infinites are larger than other infinites and some are smaller.
>
> *Robert Grosseteste* 13th century [Fre54]

We approach the 'practice based philosophy of logic' by examining the practice in one specific area of logic, model theory, over the last century. From this we try to draw lessons not for the philosophy of logic but for the philosophy of mathematics. We argue in fact that the philosophical impact of the developments in mathematical logic during the last half of the twentieth century were obscured by their mathematical depth and by the intertwining with mathematics. That is, that concepts which are normally regarded by both mathematicians and philosophers as 'simply mathematics' have philosophical importance. We make two claims. First is that the mere fact that logical methods have had mathematical impact is important for any investigation of mathematical methodology. Twentieth century logic introduced techniques that were important not just for the problems they were originally designed to solve (arising out of Hilbert's program) but across broad areas of mathematics. But, from a philosophical standpoint, there is a further impact. These methods actually provide tools for the analysis of mathematical methodology.

We view the practice-based philosophy of 'Subject X' as a broad inquiry into and critical analysis of the conceptual foundations of actual work in subject X[1]. The topic of this workshop was the (practice-based) philosophy of logic *and* mathematics. These are clearly different areas[2]; there is a substantial overlap in mathematical logic. By mathematical logic we mean the use of formalization of mathematical arguments and concepts to investigate both mathematics and philosophical questions about mathematics. Emphasizing 20th century model theory, we describe below some aspects of the development of mathematical logic since its foundation by such as Boole, Frege, Pierce, and Shröder in the late 19th century. Thus we restrict to the study of formalization of declarative sentences about mathematical objects[3]. Other areas of logic, which study more complex natural language sentences (tense, deontic, modal, etc.) and often use mathematical tools, are not discussed here.

The longtime standard definition of logic is 'the analysis of methods of reasoning'. This does not describe the perspective of a contemporary model theorist. A model theorist is a self-conscious mathematician. A model theorist uses various formal languages and semantics to prove mathematical theorems. But there is an inherently

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[1] This is a paraphrase of part of Dutilh-Novaes presentation.

[2] Even if some form of the logicism program had succeeded, as we argue below the logical foundations of mathematics do not exhaust the philosophy of mathematics.

[3] The ontological status of these objects is not relevant to our analysis.

metamathematical aspect. The very notion of model theory involves seeking common patterns across distinct areas of mathematical investigation. One of our goals below is to make this notion of 'distinct area' more precise.

In particular, we view the practice-based philosophy of mathematics as a broad inquiry into and critical analysis of the conceptual foundations of actual mathematical work. This investigation also includes a study of the basic methodologies and proof techniques of the subject[4]. The foundationalist goal of justifying mathematics is a part of this study. But the study we envision cannot be carried out by interpreting the theory into an über theory such as ZFC; too much information is lost. The coding does not reflect the ethos of the particular subject area of mathematics. The intuition behind fundamental ideas such as homomorphism or manifold disappears when looking at a complicated definition of the notion in a language whose only symbol is ϵ. Tools must be developed for the analysis and comparison of distinct areas of mathematics in a way that maintains meaning; a simple truth preserving transformation into statements of set theory is inherently inadequate. The traditional foundationalist approach sacrifices explanation on the altar of justification.

The discussion below has both a sociological and a philosophical aspect. Sociologically, in this introduction, we describe recent examples illustrating model theoretic practice. Philosophically, in the two main sections of the paper we propose some tools for studying the methodology of mathematics. We outline a program for using model theoretic concepts for a) formalizing a notion of 'area of mathematics' and b) analyzing basic concepts of mathematics. In Section 1) we sketch the history of model theory in the twentieth century and in particular the development of the notion of a complete theory. We argue for the notion of a first order complete theory as a useful unit of analysis for describing 'an area of mathematics'. We conclude the historical discussion with an introduction to the sophisticated model theoretic methods developed in the last 40 years. In Section 2) we discuss how these methods can provide insight into the way fundamental notions are specified in different areas of mathematics. We glimpse the use of model theoretic notions to analyze one particular mathematical notion: dimension.

In both cases a) and b) , our main point is that model theoretic tools can be brought to bear. We are simply giving introductory sketches of illustrations of that thesis.

In the sociological mode, we now describe some papers at the Mid-Atlantic Model Theory conference held in the Fall of 2008 at Rutgers. In the late 70's a large gap was seen between EC 'East Coast' and WC 'West Coast' model theory: a) EC used model theory in various parts of mathematics and b) WC developed an independent subject area of 'model theory'. We elaborate this contrast below. But the current situation is best summed up by Pillay's affirmation at the 2000 ASL panel on the future of logic – 'there is only one model theory.'[5] At least seven of the ten papers at the Rutgers conference, even those focused on algebraic notions such as non-archimedean geometry, semi-abelian varieties, or difference fields integrate the deep concepts developed in the pure theory since 1970. For example, one paper concerned a conjecture of Cherlin which uses model theoretic concepts to lift the program of the classification of finite simple groups to the classification of simple groups of finite Morley rank. Even the statement of the problem is posed in model theoretic terms that provide a way to organize topics that were already in the mathematical air. The investigation involves significant techniques from model theory, finite group theory, and algebraic groups. Even the relatively few papers at this conference that were 'pure' developed concepts central to current research in e.g. the model theory of valued fields[6]. Our summary of this conference has focused on 'main-stream' contemporary model theory[7] which develops logical

[4]For a broad investigation in the philosophy of mathematics including a study of leading contemporary mathematicians (e.g. Grothendieck, Langlands, Shelah, Zilber) see [Zal09]

[5]This is my recollection of Pillay's oral statement. He describes the 'unification' at considerable length in his contribution to [BKPS01].

[6]Shelah's concept of theories without the independence property (nip or dependent depending on the author) were expounded in the least-applied talk. Hrushovski's paper 'Stable groups and approximate group theory'[Hru09], which uses the model theoretic analysis of these theories as a tool for the study of groups, was the subject of semester-long seminars at UCLA, Berkeley, Urbana, and Leeds in the Fall of 2009. Fields medalist Terrence Tao discusses the progress of the UCLA seminar in the blog at `http://terrytao.wordpress.com/`.

[7]It does not encompass a number of other areas of model theory such as models of arithmetic, finite model theory, model theory in computer science, higher order and other extensions of first order logic, and universal algebra.

techniques of model theory and integrates them in the investigation of problems across mathematics.

1 Historical Survey of model theory

The integral connections of model theory with modern mathematics as described in the introduction are often (and often correctly) seen as a falling away from philosophical concerns. But as we'll see below, many of these interactions do stem from concerns about explanation and coherence of mathematical ideas that have a philosophical basis. The divorce is from narrow concern with the formal justification of results. Moreover, natural philosophical issues arise from some technical results. For example, there are vast differences between the role of $\aleph_0$ and any uncountable cardinal in the study of categoricity. What is so different about countability?

After a survey of the history of model theory I expound the use of model theoretic concepts as a tool for such an analysis of the foundations of mathematics. We review this history from a standpoint similar to this paper but with an emphasis on the mathematical applications in [Bal10]. For lack of space, I concentrate on first order logic; recent work in infinitary logic [Bal09, She09, Zil04] contributes new problems and methods to our analysis.

We distinguish three types of analysis in first order model theory:

1. Properties of first order logic (1930-1965)
2. Properties of complete theories (1950-present)
3. Properties of classes of theories (1970-present)

1.1 Properties of first order logic (1930-1965)

The essence of model theory is a clear distinction between syntax and semantics. Sentences in a formal language for a vocabulary τ are true or false in structures for τ. While the full formal treatment of this notion first appears in [Tar35], the distinction was evident to Frege and Hilbert[8] and the basic idea is essential for Gödel's completeness theorem [Göd29] (see below) to even make sense. While the completeness theorem plays a fundamental role in first order model theory, a formal proof system is *not* essential to formulating many of the crucial concepts. Thus the compactness theorem requires the completeness theorem for neither its statement nor its proof although proofs using completeness are typical.

While this notion is familiar to both philosophers and mathematicians, my experience in philosophy and mathematics departments presented the connection in two entirely different lights. Beginning my logic studies in a philosophy department, the syntactic aspect was primary. Models were created from the syntactical base to, for example, establish the completeness theorem, but had no independent interest. In contrast, in the mathematics department during graduate school, the two aspects had more equal status. Semantic structures exist in all the rest of the mathematics courses. Syntax provides a formalism for reasoning about and describing the structures and (necessarily in non-trivial cases[9]) classes of structures.

The prehistory of model theory includes the work before 1950 of Löwenheim, Skolem, Gödel, Malcev, and Tarski. They isolated fundamental properties of first order logic such as completeness, compactness, and the Lowenheim-Skolem-Tarski theorem. The prehistorical aspect is illustrated by references in logic courses to the 'Lowenheim-Skolem-Tarski theorem' and its proof by Malcev and Gödel.

[8] It was less clear in Pierce, the algebraic school and Russell.

[9] Only finite structures are uniquely describable by (sets of) sentences in first-order logic.

The term model theory was popularised in the early 1950's, especially by Tarski and Robinson. Work in that decade provided syntactic characterizations of preservation properties. E.g., The models of a first order theory are closed under unions of chains if and only the theory is axiomatized by 'for all, there exist' sentences. But what we might now call 'syntactic' and 'semantic' formulations are described as more of a contrast between 'logical' and 'mathematical'. In [Tar54], Tarski writes 'universal classes can be characterized in a purely mathematical terms'. The compactness theorem is given both 'logical' proofs from the completeness theorem and 'mathematical' proofs via ultraproducts. Tarski and Vaught [TV56] define the notion of elementary extension and prove the union of elementary chain is an elementary extension of each member of the chain; this both refines the original theory and helps to develop the correct category for model theory.

Further general properties of first order logic developed in the 50's included interpolation theorems and the Robinson Consistency theorem. Much model theoretic work in the 60's and 70's extended these kinds of notions to logic with infinite conjunctions or with generalized quantifiers of various sorts. We focus on a crucial idea that crystalized in the 1950's: a complete theory.

Before proceeding to complete theories we discuss a different notion with the same name: completeness of a logic. By a logic, we mean as in [BF85] a syntactical notion of a collection of sentences $\mathcal{L}(\tau)$ for a vocabulary τ and a satisfaction relation $\models_{\mathcal{L}}$ between sentences $\phi \in \mathcal{L}(\tau)$ and τ-structures M. The logic is complete if there is some proof system $\vdash_{\mathcal{L}}$ of $\mathcal{L}$ such that:

$$\vdash_{\mathcal{L}} \phi \text{ if and only if for every } M \models_{\mathcal{L}} \phi.$$

A theory T is a satisfiable set of sentences in a logic $\mathcal{L}$. (We will consider first order, second order, $L_{\omega_1,\omega}$ and $L_{\omega_1,\omega}(Q)$.)

Our discussion of prehistoric times is not complete without mentioning the American Postulate Theorists [AR02a, AR02b]. Already in 1902, Huntington introduced the notion of an axiom system having exactly one model. By 1904 [Veb04], this notion had been christened 'categoricity' and Veblen proves the categoricity of a set of (second order) axioms for geometry. Following the terminology of [AR02a], we say.

Definition 1 1. *A theory T is* semantically $\mathcal{L}$-complete *if for each $\mathcal{L}$-sentence ϕ and any pair of models M, N of T,*

$$M \models_{\mathcal{L}} \phi \text{ if and only if } N \models_{\mathcal{L}} \phi.$$

2. *A theory T is* deductively (or syntactically) $\mathcal{L}$-complete *if for each $\mathcal{L}$-sentence ϕ either $T \vdash_{\mathcal{L}} \phi$ or $T \vdash_{\mathcal{L}} \neg\phi$*

If $\mathcal{L}$ satisfies the (extended) completeness theorem then these notions are equivalent. As reported in [AR02a], Fraenkel [Fra28] had distinguished these notions without establishing that they are really distinct. In [Ken], Kennedy discusses the significance of the first paragraph of Gödel's thesis. She points out this distinction becomes clear only with Gödel's proof of the completeness theorem. Kennedy further notes that Gödel argues that categoricity and an effective proof theory implies syntactic completeness. Thus Gödel foreshadows the incompleteness theorem in his argument that a proof is needed for completeness (contrary to the view that 'consistency implies existence' is tautological). There is a categorical axiomatization of the real numbers including arithmetic in second order logic; this yields semantic, but not syntactic completeness of the second order theory. Vaught's proof [Vau54] of the Łos-Vaught test (a first order theory with no finite models categorical in some infinite power is complete) writes the argument in modern terms[10]: Categoricity plus upward and downward Löwenheim-Skolem implies semantic completeness; syntactic completeness follows by Gödel. What now seem obvious compactness arguments for the existence of non-standard models were clearly *not* in the air in 1930 [Ken, Vau86].

[10]There is no indication of a connection with Gödel's argument cited above.

Note that for any structure M, $\mathrm{Th}(M) = \{\phi : M \models \phi\}$ is a *semantically* complete theory for every logic $\mathcal{L}$ is under consideration. This method of obtaining complete theories is fundamental.

1.2 Properties of (complete) theories (1950-present)

The mathematical significance of the fundamental notion of a first order complete theory was stressed by Abraham Robinson [Rob56]. He provides a number of mathematically interesting examples of complete first order theories and shows common model theoretic characteristics involving the form of the axiomatization or quantifier elimination for a number of them.

Axiomatic theories arise from two distinct motivations. One is to understand a single significant structure such as $(N, +, \cdot)$ or $(R, +, \cdot)$. The other is to find the common characteristics of a number of structures; theories of the second sort include groups, rings, fields etc. There are a number of *second order* theories of the first sort that are categorical.

Both of these motivations aim at studying fundamental properties (formulated in a specified logic) which determine all or at least many such properties of a structure or a group of structures. But the axiomatizations have quite different impact. The (usually) second order axioms characterizing a single important structure delineate exactly what makes that structure unique. These axioms illuminate a key feature of the structure: i.e., the reals are the unique complete ordered field with a countable dense subset. But this light is shed on the particular structure. Bourbaki represents a triumph of axiomatization for the second reason. Large parts of mathematics were organized into coherent topics by providing informative axiomatizations.

Consider the relation with categoricity. To avoid trivialities, we deal only with infinite models. T is categorical if it has exactly one model (up to isomorphism). T is categorical in power κ if it has exactly one model in cardinality κ.

Note that under these definitions, every categorical first order theory is semantically complete. Further every theory in a logic which admits the upward and downward Löwenheim-Skolem theorem for *theories* that is categorical in some infinite cardinality is semantically complete. First order logic is the only one of our examples that satisfies this condition. Semantic (and indeed syntactic completeness) can be deduced from $\aleph_1$-categoricity for sentences of $L_{\omega_1,\omega}$ [She83a, She83b, Bal09].

Most people have an intuition for only a few infinite structures: arithmetic on the natural numbers[11], the rationals, and perhaps on the reals. Most mathematicians extend this to the complex numbers and then to a deeper understanding of various structures depending on their own specialization: ($SL_2(\Re)$, $\boldsymbol{P}_1$, initial segments of the ordinals [12]). But all these structures have cardinality at most the continuum. There are few strong intuitions of structures with cardinality greater than the continuum. However, there is a crucial exception to this remark. It is rather easy to visualize a model that consist of copies of single countable or finite object. Consider a vocabulary with a unary function f. Assert that $f(x)$ never equals x but $f^2(x) = x$. Then any model is a collection of 2-cycles. On the one hand we have the notion that there are models of arbitrarily large cardinality but I really have no really different vision distinguishing among the models of different large cardinality. This situation generalizes when the number of disjoint copies of the same structure is replaced by the dimension of a vector space or field. Thus we might consider the class of structures A_κ, a direct sum of κ copies of Z_2. The isomorphism type of the model depends solely on the number κ of copies (and not at all on the internal structure of the cardinal κ).

Many such visualizable structures arise in a standard way; they are the class of models of a first order theory that is categorical in all uncountable cardinalities. Categoricity is not a necessary condition for such a clear

[11] Roman Kossak points out that this intuition is really only for the structure with successor and addition; with multiplication th structure is extremely complex.

[12] Recall Paul Cohen's intelligence test: for what ordinals can you visualize the descending chains witnessing well-foundedness?

visualization: consider an equivalence relation with two infinite classes, fix a totally categorical theory and make each class a model of the given theory. Each model is determined by two cardinals– the cardinality (or more precisely the dimension) of each equivalence class. More sophisticated investigation and slightly relaxing the notion of 'visualize' shows that categoricity does provide a sufficient condition for such a visualization. And then interpreting 'visualise' as: admitting a structure theorem (in the sense of Theorem **??**), we can obtain exact conditions for being able to 'see' all models of a first order theory.

It can I think be easily argued that I am taking the notion of visualization too broadly. But this sort of generalization frequently takes place in mathematics and one 'sees' collections of objects (subspaces with dimensions) and remembers that one can 'see' each member of the collection by focusing on it. Certainly common model theoretic diagrams represent this kind of idea. (A diagram might show a tree of little clouds with a straight line in each cloud to represent its basis.)

There is little general information to be discovered about a structure, just by the observation that it is the unique model of a second order sentence[13]. However, the situation for *categoricity in power* of a first order sentence is quite different. First order theories that are categorical in an uncountable power share a number of attributes that flow from that fact. Further their study stimulated a powerful unifying technique for the study of first order theories. Again, we contrast the two perspectives of investigating a particular structure and investigating a class of structures.

Starting with a single prototypical structure, such as the complex field, categoricity in power is the best approximation that first order logic can make to categoricity. But, it turns out to have far more profound implications for studying the original structure than second order categoricity does. If the axioms are universal existential then the theory is model complete (and under slightly more technical conditions admits elimination of quantifiers). Thus the complexity of definable sets is determined by global properties of the class of models. This general structural condition replaces proofs of quantifier elimination by induction on quantifiers that depend on the specific theory; these inductive proofs can be very technical. Every model of an $\aleph_1$-categorical theory is 'determined' by a definable *strongly minimal* set which admits a dimension theory similar to that of vector spaces[BL71].

Model theory is often characterized as the study of definability. Consider two variants on this remark: study definable subsets in a given structure or definable classes of structures[14]. While analysis of the definable subsets of, say, the real field is certainly an important topic it turns out that in many cases the important theorems revolve around definability in *every* model of a theory. Thus a theory T *admits elimination of quantifiers* if every first order formula $\phi(\mathbf{x})$ is equivalent in T to a $\phi'(\mathbf{x})$ which has no quantifiers. This means that that full understanding of the (in general much more complicated) cylindric algebra of all definable subsets under the operations of intersection, union, negation, and projection (existential quantification) is obtained by studying the Boolean algebra of quantifier free definable sets. This is a property not of a particular structure but of every model of a theory and the map from ϕ to ϕ' is uniform across the models. Completeness is not essential here but completeness is a guarantee of very close similarity of the various models of a theory.

We use the word 'tractable' informally to mean that it is possible to develop a structure theory i) for the definable subsets of models or ii) for the class of all models of a theory. Nontrivially, i) and ii) are closely related.

Crucially, important results about a single structure are deduced by studying non-standard models of its theory. I argue below that the model theorists analyze *complete* theories. Observationally, this is largely true. But whether

[13] Jouko Vaananen has pointed out: If $V = L$, then a structure is characterized up to isomorphism by second order categorical sentence if and only if it has a second order complete characterization by a single sentence. When $V \neq L$, it is possible that some structure has a second order complete characterization by a single sentence but no second order sentence characterizes the structure up to isomorphism. In contrast to the first order case, there are second order sentences with a unique model which has 'no' geometry.

[14] Note that technically they are the same process; studying uniform definability of n-ary relations on all models of a theory is equivalent to studying classes of models in the same vocabulary augmented by n individual constants.

this is a technical convenience or a fundamental distinction is a subject for (philosophical?) investigation.

Work of e.g. Robinson, Tarski, Vaught, Łoś, Ehrenfeucht, Mostowski, Keisler, Morley, Shelah led to the understanding that complete first order theories admitting elimination of quantifiers provided the most fruitful field of study. Elimination of quantifiers can arise in two radically different ways. By fiat: Morley noticed that there is an extension by explicit definition of any complete first order theory to one which has elimination of quantifiers. Most studies in pure model theory adopt the convention that this extension has taken place. But this extension requires a large price; the vocabulary is no longer tied to the basic concepts of the area of mathematics. Thus for applications model theorists work very hard to find the minimal extension by definitions that must be made to obtain quantifier elimination (or the weaker model completeness). But there is a clear understanding in either the pure or applied case that it is desirable to have a limited number (of alternations) of quantifiers available so that definable sets can be analyzed.

This emphasis on definability underlies much of mathematical logic. Definability conditions are fundamental to descriptive set theory and underlie much work in axiomatic set theory; the characterization of computatibility in terms of definability in arithmetic is central to recursion theory.

Starting from a class of structures, there is little gained simply from knowing a class is axiomatized by first order sentences. In general, the various completions of the theory simply provide too many alternatives. But for complete theories, the models are sufficiently similar so information can be transferred from one to another. One example is transfer from an analytic proof of the classification of finite dimensional algebras over the reals to classification of finite dimensional algebras over an arbitrary real closed field. The Lefschetz principle in algebraic geometry provides an interesting application by considering different completions of the theory of algebraically closed fields. Each completion is determined by specifying a characteristic and the informal Lefschetz principle of algebraic geometry can be formalized as any sentence true in an algebraically closed field of characteristic 0 is true in algebraically closed fields of characteristic p for almost all p.

Beeson [Bee] notes that the theory of 'constructible geometry' (i.e. the geometry of ruler and compass) is undecidable. This result is an application of Ziegler's proof [Zie82] that any *finitely axiomatizable theory* in the vocabulary $(+, \cdot, 0, 1)$ of which the real field is a model is undecidable. Thus the complete theory is tractable while none of its finitely axiomatized subtheories are.

The importance, even in applications, of uniform definability over all the models of a complete theory is evidence for our first thesis.

Thesis I: Studying the models of different (complete first order) theories provides a framework for understanding the foundations of *specific areas* of mathematics.

The study of complete theories has become the basic framework for model theoretic investigations. We discuss in Section 2 the classification of theories according to structural properties. Over the twentieth century there has been an important shift in the choice of which logic to use for formalization and in choice of which mathematical topic to investigate. Early work focused on 'foundational theories' and the line between the various logics had not yet been clarified. The introduction to Gödel's thesis [Ken] implicitly assumed that any system (at least of the real numbers) will include an axiomatization of arithmetic. But work of Gödel and Tarski shortly after that thesis established that arithmetic is undecidable while the theory of the real field is decidable. Most current model theoretic research into specific theories focuses on theories are both mathematically important and tractable. Model theory has given tools for discovering which theories are tractable. The gain is that many theories of general mathematical interest are tractable. But the cost is that tractable theories are *not* foundational in the traditional sense; both ZFC and PA suffer from the Gödel phenomena (the existence of a pairing function and sufficient strength to encode syntax) and are not susceptible to the general model theoretic techniques discussed here. A theory displaying the Gödel phenomena will be undecidable for intrinsic reasons. (It is perfectly possible to code undecidability into the axioms of extremely well-behaved theories.) Because of the foundational significance the interplay between PA and true arithmetic is an important theme in rather distinct subject: Model theory of arithmetic. Some techniques overlap those discussed here but the viewpoint is different[KS06].

As one example of the use of complete theories to provide a foundation for a specific area of mathematics, we consider algebraic geometry. A long standing model theoretic aphorism asserts: Algebraic geometry is the study of definable subsets of algebraically closed fields[15]. There is much truth in this. Algebraic geometry studies the solution in fields of systems of equations. And the requisite unity of studying solution sets in different fields is provided by using the complex numbers as a universal domain and interpreting the same equation in different subfields. This is a fundamental model theoretic view point but one that took decades to develop in algebraic geometry. Even more, the notion of a generic point on a variety [Lan64], is made more specific under the Morley analysis: a generic point of a variety is a realization in an extension field of a type of maximum Morley rank in the variety. The Weil-Hrushovski theorem, every constructible group is definably isomorphic to an algebraic group (Theorem 4.13 of [Poi87]), is a further example of definability providing a different conceptual foundation for a fundamental mathematical idea.

But the aphorism fails in two ways. The most obvious is that algebraists are concerned with systems of equations. This seems to be a great deal more restrictive than arbitrary first order definability. After all neither logical connectives nor quantifiers are involved. But the quantifiers are illusory. A fundamental result goes by two names with rather different connotations: Chevalley-Tarski Theorem:

1. Chevalley: The projection of a constructible set is constructible.
2. Tarski: The theory of algebraically closed fields admits elimination of quantifiers.

The connection between the two versions is the observation that projection of the set defined by $\phi(x_1, \ldots x_n)$ in n-space to $n-1$-space is the solution set of $\exists(x_n)\phi(x_1, \ldots x_n)$. This theorem shows that any first order definable subset in an algebraically closed field is definable by a Boolean combination of equations. But the algebraic geometer really distinguishes the case where there are no negations (a conjunction of equations - a trick makes disjunctions disappear.) From a general model theoretic standpoint $p(\mathbf{x}) = 0$ and $p(\mathbf{x}) \neq 0$ are taken to be at the same level. In the early 90's Hrushovski and Zilber [HZ93] introduced the notion of a Zariski geometry, which via the use of a topology (setting solutions of equations as closed) provides a model theoretic basis for being able to distinguish sets definable by positive formulas.

The second drawback is that, more precisely, this approach describes 'Weil' style algebraic geometry of the 1950's and does not directly interpret the more modern 'Grothendieck' style. There is disagreement about the significance of this alleged weakness in the usual model theoretic approach [Mac03, Hru02].

There are a number of important theorems that invoke model theoretic ideas to attain more traditional mathematical results.

1. Artin-Schreier theorem (A. Robinson)
2. Decidability and quantifier eliminability of the real field and complex fields (Tarski/Robinson)
3. Decidability and model completeness of valued fields (Ax-Kochen-Ershov)
4. Quantifier elimination for p-adic fields (Macintyre)
5. O-minimality of the real exponential field (Wilkie)

[15] Actually, algebraically closed fields do not form a complete theory; the characteristic has to be specified. But in fact the standard mathematical analysis also specifies the characteristic for most of the work.

Although many of these results seem 'logical' in nature, some have purely mathematical statements and most have significant consequences in core mathematics.

We have argued that the notion of a complete theory provides an appropriate unit of analysis for distinguishing an area of mathematics. In the examples before 1980, the main model theoretic idea was definability and the main tools were compactness and elimination of quantifiers. We have already indicated that more sophisticated model theoretic tools have played an essential role in the mathematical applications. These techniques also illustrate the more important philosophical contribution of model theory: providing tools for understanding the connections across areas.

1.3 Properties of classes of theories (1970-present)

The development of Shelah's stability theory could be (and indeed was) misperceived as mere technical mathematics concerned with abstruse cardinalities. As we'll see it provides both a mathematically powerful classification of areas of mathematics and tools for methodological investigations.

Thesis II: Studying classes of theories provides an even more informative framework for the understanding of the methodology of specific areas of mathematics.

The second current of model theory revolves around properties of classes of theories. The key to this analysis is Shelah's concept of the Stability Hierarchy.

Theorem 2 (Shelah) *Every complete first order theory T falls into one of the following 4 classes.*

1. *ω-stable*
2. *superstable but not ω-stable*
3. *stable but not superstable*
4. *unstable*

Moving down this list in general reflects decreasing structure of the models of T. Note that the hierarchy provides an organization of various areas of mathematics that illuminates connections that are not apparent from the usual mathematical standpoints. We list a number of different algebraic examples at various levels in the hierarchy. Some ω-stable theories are: algebraically closed fields (of any fixed characteristic) and algebraic groups over algebraically closed fields , differentially closed fields (of characteristic 0), compact complex manifolds. Some strictly superstable theories are: $(\mathbb{Z}, +)$, $(Z_2^\omega, H_i)_{i<\omega}$ (where H_i is a subgroup of finite index). Some strictly stable theories are: $(\mathbb{Z}, +)^\omega$ and separably closed fields of characteristic p. Unstable theories include Arithmetic, Real closed fields, complex exponentiation, and the theory of the random graph. Recent model theoretic work in two directions (dependent theories) (NIP)[HPP08, She] and infinitary logic [Zil04]) provide systematic tools to distinguish and analyze theories with intractable Gödel phenomena from those more susceptible to model theoretic analysis.

This classification provides a *totally new* way of organizing mathematical discourse. The underlying invariant is the cardinality of the Stone space of the Boolean algebra of formulas over a model M of T ($S(M)$). We say T[16] is stable in λ if for every M with $|M| = \lambda$, $|S(M)| = \lambda$. Then, ω-stable implies stable in all λ; superstable means stable above the continuum; stable means stable in some λ and unstable means stable in no λ. But this purely model theoretic and apparently combinatorial notion imposes important structural conditions on

[16] We restrict to countable theories for simplicity.

the models of the theory that we discuss in Section 2. Crucially, stability can be described in purely syntactic terms.

Shelah's techniques for analysis of models of stable theories and his more complex notions such as: orthogonality, canonical bases, regular types, etc. have many applications. In particular, Hrushovski combined these methods and those of 'geometric stability theory' with a deep understanding of Diophantine geometry to provide fundamental advances related to the Mordell-Lang conjecture [Bou99, Hru96]. Notably, although the application is to an ω-stable theory of algebraically closed fields; the analysis (for the characteristic p-case) involves strictly stable theories of separably closed fields. We have noted first that both basic model theoretic ideas of definability and compactness and later the more sophisticated model theoretic methods have been used to solve problems of core mathematics. Just this fact is important from the standpoint of any analysis of mathematical methodology. But these model theoretic tools themselves provide tools for analysis. On their face they illustrate distinctions and similarities across different areas. In the next section of the paper we sketch how these tools allow us to analyze some mathematical notions as they span areas of mathematics.

2 Concept Analysis: Dimension

Our general claim is that the techniques and concepts developed in stability theory can be useful for a philosophical investigation of the methodology of mathematics. In this section we outline the development of two themes; space is short so even a sketch of the argument is deferred. The notion of dimension is a basic mathematical idea and model theory provides a unifying approach among several avatars of this notion[17]. Moreover, the stability hierarchy provides a way to compare different areas of mathematics in terms of the strength of their dimension notion.

The article on Dimension in [Gow08] suggests five notions of dimension that occur in such fields as real or complex geometry, differential geometry, topology and algebra. Shelah [She78] defines a fruitful and far-reaching generalization of the notion of dimension in vector spaces (and in algebraic geometry). Any stable theory admits such a (family of) dimensions on each model. Using this notion, [Bal01] analyzes two of Gower's notions using model theoretic notions arising out of stability theory, distinguishing 'algebraic' and 'geometric' intuitions.

Investigating the dimensions and connections between (type)-definable subsets of models leads to a fundamental theorem: the Main Gap [She91].

Theorem 3 (Shelah's Main Gap) *For every first order theory T, either*

1. *Every model of T is decomposed into a tree of countable models with uniform bound on the depth of the tree, or*
2. *The theory T has the maximal number of models in all uncountable cardinalities.*

The impact of this theorem is to divide first order theories into two classes. The models (of any cardinality) of a classifiable theory can be decomposed in a uniform way from countable models. The models of unclassifiable theories are creative; new patterns continually emerge as models of larger cardinality are considered.

A fundamental idea, that appears only technical, is to decompose into trees of models. This decomposition was a tool for counting the number of models in each cardinality of a theory. But *systematic representation of a model as prime over a tree of (independent) submodels is a fundamentally new mathematical notion.*

[17] Other notions that could be given a similar analysis include: chain conditions, notions of finiteness, 'genericity', group actions (E.g., what are sufficient conditions for the development of Galois Theory [MTB10]?).

Arithmetic is the paradigmatic example where no notion of dimension makes sense. Model theory is able to make clearer distinctions of when a well-behaved dimension is possible.

Theorem 4 *1. If a model admits a pairing function, it has no well-behaved notion of dimension.*

2. If T admits a pairing function then T is not superstable.

The first of these observations is folklore. Much more well-behaved (from a model-theoretic standpoint) theories can have pairing functions. Lachlan (reported in [BM82]) showed the second result in Theorem 4, which shows that pairing does force a theory to the non-structure side. In particular, any theory with a pairing function has many models in all uncountable cardinalities. Thus the stability hierarchy becomes a tool for determining which theories admit good notions of dimension. Note that the coding of these tame theories into foundational theories such as ZFC or arithmetic completely destroys these salient tame properties of mathematical notion under study. But there are also important structures, most notably the real field, which do not have pairing functions, which do admit rank functions, but are not stable. These led to the study of *o*-minimality[dD99].

3 Conclusion

We have discussed three issues concerning the relationship of contemporary model theory, mathematics, and philosophy. The first observation is that model theory is a vigorous part of mathematics that uses tools that were invented for 'logical analysis' to solve problems arising in more traditional mathematics. In this respect model theory differs only in degree from logic in general. Ideas stemming from computability and relative computability permeate computer science and model theoretic ideas arise in many aspects of computer science. Such notions as the Curry-Howard isomorphism and the analysis of weak theories of arithmetic to study computational complexity show the influence of proof theory across mathematical disciplines. Set theory has a similar interaction with mathematics both by the discovery that certain classical mathematical problems depend on set theoretic principlesn

and e.g. by the integration of set theoretic methods with those from dynamical systems in studying the Borel classification of problems [KM04].

The identifying characteristic of *logic* in these mathematical examples is not an 'analysis of reasoning' but an explicit attention to *means of definability.* The intricate history of the relationship between 'core mathematics' and 'logic' is certainly a fit topic for study in the practice-based philosophy of logic.

Secondly we made the argument that the notion of a complete theory provides a unit of analysis for examining different areas of mathematics. We both examined the abstract reasons that it is a suitable unit of analysis and examined one case, algebraic geometry, in a bit more detail.

And thirdly, recall Thesis II: studying classes of theories provides an even more informative framework for the understanding of the methodology of specific areas of mathematics. We cursorily described the connection of model theoretic dimension with such notions in algebraic geometry, complex exponentiation and tame topology.

But model theory also provides entirely new areas of mathematics for study. It provides two new general notions of how mathematical properties might change as the cardinality of the structures involved change. Eventual behavior: what happens on all sufficiently large cardinals. Initial behavior: what can we say about the 'lower infinite', cardinals below say, $\beth_{\omega_1}$. Much of core mathematics is much coarser: it studies either properties of particular structures of size at most the continuum or makes assertions that are totally cardinal independent. E.g., if every element of a group has order two then the group is abelian. Model theory *of infinitary logic* allows a more sophisticated analysis in two directions; determination of properties that hold only eventually rather than everywhere and study of classes that are well-behaved on small cardinals to determine whether this behavior propagates to the entire universe. Certain properties allow us to chart the infinite. Some properties (e.g. categoricity for certain classes of models of infinitary sentences) are now known to be eventual; but major questions remain about from what level they propagate. But other properties (amalgamation, tameness) may not propagate; there is a real difference between large and small models for such properties. Still other properties, e.g. saturation, occur cofinally but not eventually for interesting classes of models. Thus, model theory begins to explore the paradise of the infinite, conceived by Grosseteste and delivered by Cantor. But with Shelah's classification theory the study of infinity moves into adolescence– it moves beyond combinatorial analysis into structural and algebraic investigations.

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