Categoricity for the inferential ω-logic and $L_{\omega_1,\omega}$

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Abstract

This paper provides two extensions of first order logic by 'ω-rules'. In each case we characterize the countable structures whose theory in the logic is categorical (has a unique model). In the one-sorted inferential ω-logic, both Robinson's system Q and Peano Arithmetic become categorical. In the two-sorted generalized ω-logic we show each complete $L_{\omega_1,\omega}$ sentence defines the same class of structures as a first-order theory with the appropriate $G-\omega$-rule. These logics are much weaker than second order logic and we argue that they do not appeal to the arithmetical concepts that the categoricity theorems themselves aim to secure. The results depend on proving that the inferential rules for the logics are categorical, i.e. they uniquely determine certain truth-conditions for the logical connectives and quantifiers. We provide an extensive answer to the doxological challenge (on referential determinacy) proposed in [**?**] and we develop a philosophical view of mathematics -which we call *cognitive modelism*- according to which classical mathematics is best understood as a complex process of constructing and developing a distinctive class of concepts, rather than merely describing a fixed pre-existing realm of structures.

KEYWORDS: categoricity, inferentialism, first-order logic, first-order theories, ω-rules, $L_{\omega_1,\omega}$.

Contents

1 Introduction

{intro}

This paper exploits diverse meanings of 'categoricity' in disparate areas of logic. On one hand, there is the conviction that the theory of canonical structures like arithmetic or the real numbers ought to be axiomatized by a categorical theory (in some logic), one that has a unique model (up to isomorphism). On the other, logical inferentialists argue that the truth conditions of the logical operators should be determined by the rules of inference, that is logical calculi should be categorical in a different sense. However, as early as [**?**], Carnap gave conflicting assignments of truth values for propositional formulas that each satisfied the same rules of inference.

Since at least Benacerraf and Putnam, the notion of determinacy of reference for canonical mathematical structures has been a central topic in the philosophy of mathematics. One road has been the structuralist attempt to avoid the entire (even naive) set theoretic notion of model. Most attention has focused on choosing a logic whose theory of that intuitively well-conceived structure has a unique model. Naturally, Dedekind's second order axiomatization of arithmetic [**?**] is the standard to measure against. Our discussion builds on [**?**, Part B] which critiques both second-order and weaker logics abilities for this task. First-order logic is properly dismissed immediately because of the upward Löwenheim-Skolem Tarski theorem.

The major innovations of this paper provide modified inferential definitions of the ω-rule that extend $L_{\omega,\omega}$ and categorically characterize countable structures as in $L_{\omega_1,\omega}$.

The Inferential ω-rule (the $I - \omega$-rule; Definition 4.1) allows (putative) non-standard parameters to appear in the formula in its hypothesis. Adding this rule yields a categorical interpretation of the universal quantifier. Further, we show Theorem 4.9: Peano arithmetic (even more, Robinson's system **Q**) is categorical with our $I - \omega$-rule.

Working in two-sorted $G - \omega$-logic (G for generalized) we have Theorem 5.14: Each $L_{\omega_1,\omega}(\tau)$-complete (Scott; Definition 5.3.1) sentence φ is 'structurally equivalent with' (same τ-models as) a pseudo-elementary class (the relativized τ-reducts of an associated first-order theory $\hat{T}^{\varphi}$ in the $G - \omega$-logic). Thus, the philosophical issue of the existence of infinite formulas is reduced to the issue of the reliability of infinitary proofs in ordinary first order logic. From this we conclude Corollary 5.16: A complete $L_{\omega_1,\omega}$ sentence φ is categorical if and only if an associated pseudo-elementary class (reducts of a theory $\hat{T}^{\varphi}$ studied in Section 5) is categorical in $G - \omega$-logic. For each of our rules, we characterize those countable structures that admit an $L_{\omega_1,\omega}$-categorical description.

We expect that both philosophers and mathematicians will need to consult the beginning of Section 2 for a bilingual dictionary, while the latter portion of it and Section 3 will be clear to those familiar with *inferential logic*. Sections 4 and 5 are technical but 4 differs from standard treatments of arithmetic only in its new rule. Sections 6 and 7 present the philosophical arguments.

We structure the paper as follows: in Section 2 we set up the terminological and conceptual settings of the paper and introduce the main ideas of an inferentialist approach to logic. Section 3.1 briefly discusses Tarski's use of the ω-rule in arguing that a proof-theoretic account of logical consequence fails to capture the ordinary notion of consequence. We argue that this criticism depends on the assumed soundness of the ω-rule, which in turn presupposes a substitutional interpretation of quantification. Consequently, Section 3.2 discusses Carnap's substitutional ω-logic (i.e. $S - \omega$-logic). In Section 4 we introduce a much stronger ω-logic, $I - \omega$-logic, and prove that its rules of inference are categorical. We then prove that PA plus the $I - \omega$-rule, PA^{ω}, is categorical. In section 5, relying on certain results of Scott, Chang, Vaught, and Morley we obtain the characterization of categoricity in $L_{\omega_1,\omega}$ described above.

Finally, we discuss two philosophical problems raised by Button and Walsh [**?**] in connection to the identification of a unique isomorphism class, i.e. the doxological challenge (section 6) and the circularity argument (section 7). The doxological challenge asks a philosopher: 'How can we pick out an isomorphism type'. They argue that various species of philosopher (modelist of different varieties) cannot answer this question. The circularity argument asserts that any proof of a categoricity theorem invokes mathematical concepts that were supposed to be justified by that very categoricity theorem. We argue that the purported difficulty of the doxological challenge is generated by the confusion between the notions of *structure for a vocabulary (signature)* and *model of a theory* and we clarify it by means of four (Carnapian) frameworks (i.e. structure framework, theoretical framework of set theory, metatheoretical framework of formalized theories in systems of logic, and the philosophical framework). The challenge proves meaningful and easily answered in the second framework, but it lacks cognitive content, and it is thus meaningless in the first one. In Section 6.3, building on such as [**?**, **?**, **?**, **?**] we develop *cognitive modelism* as a response to their dismissal of 'concepts modelism' [**?**, p 149].

We address the circularity argument in Section 7 by showing that our inferential approach (§ 4 and § 5) requires only the notions of finite and countably infinite and so does not presuppose the notion of an ω-sequence or arithmetical notions.

Here is a general overview of the main steps of our approach. We study four first-order logical calculi (i.e. calculi with quantification on individual variables) in a multiple conclusion format. We emphasize that all our logics, as extensions of first order logic, continue to have the downward Löwenheim-Skolem property.

1. The first one (A) is a standard first-order logical calculus, but it contains in addition two multiple conclusion rules (Definition 2.10) needed for the categoricity of the propositional calculus. Specifically, we add Gentzen style rules for the quantifiers (Definition 2.11) and identity rules (Definition 2.12) to a classical propositional calculus (the one with the rules from Examples 2.7, 2.8, 2.9 and Definition 2.10).

2. The second (B1) is the substitutional ω-logic is obtained by adding the (standard from substitutional logic) quantifier rules (Definition 3.2) to the system A. This system is usually used by the inferential logicians to show the match between the substitutional semantics for the universal quantifier in a fixed countable vocabulary (i.e. every constant is assigned in every valuation) and the $S-\omega$-rule (Proposition 3.3), but it is inadequate to obtain the categoricity results for first-order theories.

3. The third system (B2) is obtained by adding to A the new inferential ω-rule and the other inferential rules from Definition 4.1. These inferential rules are also categorical (Proposition 4.2). But now, we obtain the categoricity of such theories as PA^{ω}.

4. The fourth system (C) provides the $G-\omega$-rule for the two-sorted ω-logic of [**?**, **?**]. It extends a 2-sorted version of A with the rules in Definition 5.12, by our $G-\omega$-rule. Theorem 5.14 shows that a complete (Definition 5.3.1) $L_{\omega_1,\omega}$ sentence φ and $\hat{T}^{\varphi}+G-\omega$-rule are structurally equivalent (have the same models). This yields that non-generative (not isomorphic to a proper extension) structures are exactly those that are categorical in $L_{\omega_1,\omega}$

The collaboration producing this paper has revealed vast differences in the meanings of terms between model theory and inferential logic. Accordingly, many more terms are defined carefully and contrasted and, in an attempt to reduce confusion, we have included many definitions/distinctions that would be omitted in articles for either of our two audiences.

2 Rules, Valuations and Logical Inferentialism

{rules valuat

From a logical inferentialist point of view, the meanings (i.e. the truth-conditions) of the logical connectives and quantifiers from a language (i.e. vocabulary τ) should be uniquely determined by the formal rules of inference that govern their use in a certain deductive system defined over that vocabulary. The main idea is that the soundness of the rules of inference works as a constraint on the class of possible valuations such that only those valuations that preserve the truth of the logical theorems and the soundness of the rules of inference will count as admissible (see Notation 2.4.3). The truth-conditions of the logical connectives and quantifiers, specified in terms of Robinsonian interpretations, are intended to be uniquely determined by the class of admissible valuations (see Example 2.7). But this requires care in choosing the inference rules.

This inferentialist perspective takes a logic to comprise a syntactic vocabulary τ that contains non-logical and logical expressions (i.e. connectives and quantifiers), syntactical rules of formation that define the notion of well-formed formula in the vocabulary τ, and a set of rules of inference for the logical expressions. Since our goal is to approach categoricity from both the model theoretic and proof theoretic standpoint, we follow the general definitions from model-theory but make explicit and use the syntactic and proof theoretic apparatus that is only a very special case in [**?**].

A crucial insight of [**?**] views a logic $\mathcal{L}$ as an operator that takes vocabularies (signatures) τ to collections of $\mathcal{L}[\tau]$-sentences and a satisfaction relation between τ-structures and $\mathcal{L}[\tau]$-sentences. We employ this view for extensions of first order logic in Definition 2.4.2 below. We now specify these notions and their connection with traditional notions of interpretations and valuations. The restriction to a specific τ is now universal in model theory while an inferentialist approach might assume that one can read off the τ-structures over which the logical connectives and quantifiers have only their intended truth-conditions.

{str}

Definition 2.1 (Vocabularies and structures). *1. By a vocabulary τ we mean a set of relation, and function symbols with prescribed arity. A 0-ary function is called a constant. These are not logical symbols, but vary with the topic being formalized.*

2. *For any vocabulary τ, a τ-structure M[1] consists of a domain M equipped with for each n-ary relation symbol in τ R a subset $R^M \subseteq M^n$, the sequences that satisfy R, and for each n-function symbol f a function f^M mapping M^n into M.*

3. *Our theories are all formulated starting from a countable base vocabulary τ. To define truth on uncountable structures, we allow uncountably many auxiliary constants*

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[1] More pedantically one should denote the structure by a different font, e.g. $\mathfrak{M}$. Following current model theoretic notation, we simply overuse M.

Definition 2.6 giving expanded vocabularies τ_M. By the downward Löwenheim-Skolem theorem our arguments can take place in countable models so inference rules invoke only countable sets of constants.

Note that this definition of structure is the same[2] as [**?**, Definition 1.2]. They suggest this notion is given in naive set theory writing 'as usual in set theory' [**?**, p. 8]. But they fail to note that the doxological challenge, 'How can we pick out isomorphism types' [**?**, p.143] is immediately answered in the same naive set theory. The isomorphism type of M is the class of τ-structures that are isomorphic to M. There is a crucial distinction here. The isomorphism type is defined externally in the sense of [**?**]. That is, one uses the resources of set theory to define the isomorphism type. In contrast, the internal aim of categoricity of a *theory* is to choose (or given) a vocabulary τ and a τ-structure, M, find a logic $\mathcal{L}$ and an $\mathcal{L}$-theory T whose unique model up to isomorphism in M. Our goal here is to find substantially weaker choices of $\mathcal{L}$ that can give categorical axiomatizations of countable structures.

{`trv`}

Definition 2.2 (Permissible Valuations)**.**

1. A *valuation* is a function from a set Σ of sentences to truth values.
2. A valuation is *permissible* if it takes values on all quantifier free sentences in a given vocabulary, the resulting set of sentences is consistent[3], and it preserves the soundness of the inference rules for identity (Definition 2.12).

Permissibility significantly narrows Garson's general notion of valuation as any function from sentences to truth values[4]. In fact, since each permissible valuation v gives a structure M if and only $\overline{a} \in R^M$ if and only if $v(R(\overline{a})) = true$[5] and similarly for functions, we have:

{`modval`}

Lemma 2.3. *Each permissible valuation v gives a τ-structure for the vocabulary τ whose symbols appear in the domain of v. All permissible valuations on a τ-structure M give the same values to the quantifier free sentences of τ.*

Note that the extension from 'same values on quantifier free sentences' to 'on all sentences' is not automatic. It will follow for our inferential logics only after we prove the

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[2] They use the word language rather than vocabulary and $\mathcal{L}$ rather τ. We distinguish because language is often taken to imply a specific logic. Vocabulary is essentially synonymous with the older 'similarity type'

[3] In particular, this prevents three sentences $f(a) = b, f(a) = c, b \neq c$ all being given the value true.

[4] [**?**, pp. 12, 214] requires in the propositional case that at least one sentence is false and in the first order case he assigns truth values to formulas with free variables.

[5] $\overline{a}$ is an abbreviation for $\langle a_0, \ldots a_n \rangle$.

quantifiers are inferentially determined. For any vocabulary τ in either the one or two-sorted case the collection for τ-structures permissible for either of our logics (and the collection of sentences) is exactly the same as in first order. However, there are fewer admissible (Notation 2.4.3.a) than permissible valuations; this restriction fuels the categoricity argument.

The following notations make our approach explicit. They come with explanations because many have related but distinct meanings in the two cultures.

`{langdef}`

Notation 2.4. Logics:

A logic $\mathcal{L}$ consists of the lexical and semantic items defined below and two maps delineating the true (in a model) sentences (2) and the provable ones (3).

1. Syntax:
    (a) In this paper, a logic $\mathcal{L}$ specifies logical operators including propositional connectives $\&$, $\vee$, $\sim$ and the first order quantifiers[6] ($\forall$, $\exists$) with a list of symbols for variables.

    (b) For τ as in Definition 2.1, $\mathcal{L}(\tau)$-formulas are defined by induction as in [**?**, §1.2][7].

    (c) An $\mathcal{L}(\tau)$ theory is a consistent set of τ-sentences. A *pseudo-elementary* class[8] is titled the number of models in a pseudo-elementary class. of τ structures is the collection of τ reducts of a theory in larger vocabulary τ'.

2. First order semantics: We use the normal model theoretic definition of truth, E.g. [**?**, **?**, **?**]. We call this the $\mathbf{R}$ (Robinson) semantics for reasons explained in Remark 2.5. For a given vocabulary τ, a τ-structure[9] is a domain M, a relation or function (in τ) (e.g. R^M, f^M) of appropriate arity on M for each relation or function symbol (e.g R or f) of τ. A constant is a 0-ary function symbol. Following contemporary model theoretic practice, the truth of formula in a τ-structure M consists of a *valuation* map v_M from the constants in an expanded vocabulary τ_M that adds a name for each element of M, such that $M \models R(\overline{a})$ if and only if $v_M(\overline{a}) \in R^M$. This gives

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[6] Of course, [**?**] deals with many generalized quantifiers; but here all quantifiers are first order since we consider only $L_{\omega,\omega}$ and $L_{\omega_1,\omega}$ while modifying the inference rules for $L_{\omega,\omega}$. A key point of [**?**] is that in general the logics considered need not have rules of inference. Finding categorical such rules is one of the goals of our program.

[7] We consider only $L_{\omega,\omega}$ and $L_{\omega_1,\omega}$, For the second, one allows countable conjunctions and disjunctions.

[8] The terminology dates to Tarski in the early 1950's. [**?**, Chapter VII]

[9] Model theorists typically write M is a 'model' of T, and 'τ- structure' when not dealing with a particular theory. [**?**, 13] speaks of a model for a language as *any set* of valuations.

a truth value to each atomic formula and thus to each quantifier free τ_M-formula (using classical truth table rules for the propositional connectives). This collection of formulas is called the *quantifier-free diagram* (in contrast to the *complete diagram* which includes all first-order formulas in τ_M) of the structure. This definition is extended to define truth in M for arbitrary τ^M-formulas by induction on quantifiers as in [**?**, p.17]. In particular, for a sequence of constants $\overline{a}$ from τ_M and a formula of quantifier rank n, $M \models (\forall x)\varphi(x, \overline{a})$ if and only $M \models \varphi(c, \overline{a})$ for each $c \in Const(\tau_M)$. Thus, by induction, it holds for all formulas.

3. Inferential logic: An inferential logic contains a set $\mathbb{R}$ of rules of inference governing the logical operations (propositional connectives and quantifiers).

   (a) A *permissible* valuation v over a vocabulary τ is *admissible* if and only if it preserves the soundness of the relation of logical derivability defined by the set $\mathbb{R}$, i.e. for every rule $\mathbf{r} \in \mathbb{R}$, if v assigns truth to the premise of an instance of $\mathbf{r}$, it also assigns truth to the conclusion. In particular, it assigns truth to each logical theorem.

   We consider natural deduction introduction and elimination rules of inference for each propositional connective and each quantifier. These are mostly standard so we specify explicitly only those that are central to our argument. [10] We specify below the two sets of rules for first order logic that are relevant to our argument. We take the sub-derivations from assumptions, present in the rules of inference that discharge assumptions, to embody *a formally logical derivability relation*[11].

   All the logical calculi studied in this paper agree on the semantics for the *quantifier free* formulas as defined in Notation 2.4.2. An admissible valuation for a set of inference rules $\mathbb{R}$ will make M a model of a collection of sentences Σ formalized in the inferential logic defined by $\mathbb{R}$.

   (b) A set of logical inference rules is *categorical*[12] if and only if for all countable vocabularies τ and all admissible valuations for those rules the propositional

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[10] See [**?**]. Elegant presentations of systems of natural deduction for the propositional connectives and for the first-order quantifiers are given, for instance, in philosophical logic textbooks as [**?**, Ch. 4, Ch. 6] and [**?**, Sec. 20-3, 31-33 Ch. 6]. In contrast, the mathematical logic books generally rely on Hilbert style rules of inference and an unconscious 'Turing thesis' for proofs –all proof systems give the same validities.

[11] For cognoscenti: and not just to be *de facto truth preserving* (see Example 2.8 and [**?**, pp. 399-400]).

[12] The term 'categorical' is used in this paper in three related contexts. As above, as a property of an inference system, the usual philosophical use asserting that a theory (in some logic) has exactly one model, and the model theoretic notion of 'κ-categoricity' that a theory has exactly one model of cardinality κ.

connectives, universal, and existential quantifiers that these rules define have a unique interpretation[13].

4. Validity and proof {valproof}

   A τ-sentence is valid for a logic $\mathcal{L}_\tau$ if and only if it is true under every admissible τ-valuation.

   A sentence is proved in a logic $\mathcal{L}$ if and only if it can derived from the empty set of premises by using the rules of inference in $\mathcal{L}$.

   A sentence φ is derivable in $\mathcal{L}$ from a collection of sentences Σ if there is an $\mathcal{L}$-derivation from sentences contained in Σ leading to φ. We allow infinitary rules.

5. Theories and Axioms. Let Σ be a collection of τ sentences. The Σ is said to *axiomatize* the collection of all sentences that are derived from Σ. A deductively closed set of sentences is called a *theory*.

{sch}

Remark 2.5 (Scholia on Definition 2.4.2)**.** The definition of truth here corresponds to Robinson-Tarski-Geach semantics (see [**?**, Sec. 4]. It is easily expressed in terms of interpretations where an interpretation assigns to each constant symbol an element of a domain A and to each relation or function symbol, a relation or function on that domain. Working from the structure is natural in model theory where the aim is to formalize a class of mathematical structures and study definable relations on them.

[**?**] distinguish between the Tarski ([**?**, **?**] and Robinson ([**?**, **?**, **?**]) semantics depending on whether the domain of an assignment is variables or constants, but note that they yield the same truth values in each model for sentences [**?**, 1.7,1.8]. [**?**] label as 'hybrid approach' a Robinson approach in which one is careful to avoid identifying the names with the elements of the models that they name. We find this term a distinction without a difference. Thus we omit the word hybrid and write Robinsonian semantics (i.e. R-semantics, R-valuation, R-substitutional). Moreover, they suggest that there is an induction over languages and models. Rather, the truth of sentences on M and N is defined independently on each structure as we emphasize by the notation τ_M in our definition of truth in a structure. The induction is on quantifier rank in each structure. Under Definition 2.4.2, every permissible valuation v (structure M_v) determines a unique complete first theory on τ_M. Model theorists do not consider the issue of whether this extension is determined by inferential rules so the notion of *admissible* valuation/structure does not arise in model theory.

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[13] This unique interpretation is usually spelled out in most of the *philosophical* logic textbooks (see e.g., [**?**, Sec. 35], [**?**, Ch. 5], [**?**, Ch.6], [**?**, pp. 213-14]) in terms of the classical truth tables for the propositional connectives and in terms of 'objectual' (Tarski) or 'substitutional' (Carnap/Robinson) interpretations for the first-order quantifiers (see §3.2). In this paper we use the ideas and notations of the R-semantics (Definition 2.4.2) for expressing the truth-conditions of the connectives and quantifiers.

Indeed most contemporary model theory texts do not specify rules of inference and deal primarily with 'logical implication' ($\models$), preservation of truth.

For the benefit of model theorists, we provide considerable background on the notion of 'categoricity of inference rules'. This categoricity is a central step (Propositions 4.2 and 5.13) in the main arguments.

{varcon}

Definition 2.6 (Varieties of Constants)**.** *We consider several varieties of constants: τ-*constants, auxiliary constants, constant terms. *A τ-constant is a constant symbol in the vocabulary τ; the R-semantics introduces* auxiliary constants *in the vocabularies τ_M. Either case may also give rise to* constant terms*; these are definable unary functions of definable singletons, either a τ/τ_M - constant or the solution of a formula with no parameters that has a unique solution. E.g. the formula $\neg\exists y S(y) = x$ in the theory of an ω-sequence in the vocabulary $\langle S, =\rangle$. Depending on context 'constant' may refer to any of these.*

Example 2.7. *(Categoricity of $\&I$ and $\&E$-rules)* {conj}

The following set of rules of inference enforce that the logical connective '$\&$' behaves according to the 'classical' truth table definition. Consider the set of rules of inference that contains the following introduction and elimination rules for conjunction ("$\&$"):

$$\&I : \frac{\varphi \quad \psi}{\varphi\&\psi}; \qquad \&E' : \frac{\varphi\&\psi}{\varphi}; \qquad \&E'' : \frac{\varphi\&\psi}{\psi}.$$

In any admissible valuation for these rules, the conjunction "$\varphi\&\psi$" is true if and only if both "φ" and "ψ" are true. If "$\varphi\&\psi$" is true, then both "φ" and "ψ" are true (by $\&E$), and if both "φ" and "ψ" are true then "$\varphi\&\psi$" is true (by $\&I$). In the sense defined in Notation 2.4.3 , these rules are categorical and the truth function expressed by "$\&$" is fully formalized by the $\&I$ and $\&E$ rules.

Example 2.8. *(Non-categoricity of $\vee I$ and $\vee E$-rules)* {disj}

The following set of rules of inference does not enforce that the logical connective '$\vee$' behaves according to the 'classical' truth table definition. Consider the set of rules of inference that contains the following introduction and elimination rules for disjunction ("$\vee$"):

$$\vee I' : \frac{\varphi}{\varphi\vee\psi}; \qquad \vee I'' : \frac{\psi}{\varphi\vee\psi}; \qquad \vee E : \frac{\varphi\vee\psi \quad \begin{array}{c}[\varphi]\\ \vdots\\ \sigma\end{array} \quad \begin{array}{c}[\psi]\\ \vdots\\ \sigma\end{array}}{\sigma}.$$

The $\vee E$-rule licenses the derivation of the conclusion σ provided that it can be previously derived from the assumptions of both disjuncts φ and ψ (these two assumptions will be discharged by applying the $\vee E$-rule). The introduction rules for $\vee$ uniquely determine the first three rows from the 'classical' truth table for $\vee$. This means that in any admissible valuation for these rules, if one of the disjuncts "φ" or "ψ" is true, then the disjunction "$\varphi \vee \psi$" is true. However, the $\vee E$-rule does not uniquely determine the fourth row. This makes possible a non-standard valuation ($v^{\vdash}$) over the propositional language that makes a disjunction true, although its both disjuncts are false. Namely, fix any system of inference for propositional logic whose theorems are the set of tautologies and assign truth to each propositional theorem and falsehood to each propositional non-theorem. Thus, for a sentence A, $v^{\vdash}(A)$ and $v^{\vdash}(\sim A)$ are false (since they are non-theorems), while $v^{\vdash}(A \vee \sim A)$ is true (since it is a theorem). We take the deducibility relations from the premises of the $\vee E$-rule at face value, i.e. if there is a derivation of σ from both φ and ψ, then there is a derivation of σ from '$\varphi \vee \psi$'. Since '$\varphi \vee \psi$' is a theorem in $v^{\vdash}$, then σ will also be a theorem and thus true in $v^{\vdash}$ (see [**?**, pp. 399-400]). Thus, $v^{\vdash}$ preserves the soundness of this rule. In the sense defined in Notation 2.4.3, these rules are non-categorical and the truth function expressed by "$\vee$" is not fully formalized by the $\vee I$ and $\vee E$ rules.

Example 2.9. *(Non-categoricity of $\sim I$, $\sim E$ and double negation (DN)-rules)* {neg}

The following set of rules of inference does not enforce that the logical connective '$\sim$' behaves according to the 'classical' truth table definition. Consider the set of rules of inference that contains the following introduction and elimination rules for negation ("$\sim$"):

$$\sim I : \frac{\begin{matrix}[\varphi]\\ \vdots \\ \curlywedge\end{matrix}}{\sim \varphi}; \qquad \sim E : \frac{\varphi \qquad \sim \varphi}{\curlywedge}; \qquad DN : \frac{\sim \sim \varphi}{\varphi}.$$

The trivial valuation v^T that assigns truth to every sentence of the propositional calculus makes both sentences A and $\sim A$ true. Likewise, v^T assigns truth to the absurdity symbol[14] ($\curlywedge$). Thus, the first row of the classical truth table for negation is not determined by the rules. The valuation $v^{\vdash}$ from the above example makes both A and $\sim A$ false, and, thus, the second row of the classical truth table for negation is not determined by the rules[15].

It is well known that a full formalization of classical propositional logic depends both on the format of the rules and on how their expressive power is defined.[16] For the purposes

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[14] This refers to an arbitrary contradiction to avoid technicalities in finding a specific contradiction. Gentzen [**?**, p.70] takes this symbol to stand for what he calls 'the false proposition'.

[15] The relation between $v^{\vdash}$ and the (open-ended) rules for negation is discussed more extensively in [**?**].

[16] [**?**, p.20] presents a table that summarizes the results of each combination. In particular, [**?**], [**?**], and

of this paper, we assume that a categorical formalization of propositional logic is already given. This formalization can be obtained in various ways, for instance, by adopting Carnap's two disjunctive (i.e. multiple conclusion[17]) rules:

Definition 2.10. *(Categorical rules for $\sim$ and $\vee$)* {DisRules}

$$Refutation : \frac{\bigvee^{\&}}{\bigwedge^{\vee}}; \qquad \vee E_{MC} : \frac{\varphi \vee \psi}{\varphi \quad \psi}.$$

The expression '$\bigvee^{\&}$' stands for the conjunction of the set of all sentences in the specific propositional calculus and '$\bigwedge^{\vee}$' stands for the empty class of sentences taken in disjunction (which is always false, since it contains no (true) sentence). The valuation v^T is no longer admissible in the presence of the $Refutation$-rule, since it makes it unsound. The $\vee E_{MC}$-rule blocks the valuation $v^{\vdash}$ since it uniquely determines the forth row of the classical truth-table of disjunction (i.e. if the conclusion is false, then the premise itself is false). In addition, if disjunction has its classical truth-conditions, then negation will also have its classical truth-conditions ([**?**, p.80, Th.15-9] and, thus, all the other connectives will get their classical truth-conditions, since negation and disjunction form a functionally complete set of connectives.[18]

The problem of a full inferential formalization of first-order logic (FOL) is far more complex.[19] The natural deduction introduction ($\forall I$, $\exists I$) and elimination rules ($\forall E$, $\exists E$) for the first-order quantifiers are the following: {standrules}

Definition 2.11. *Let τ be a first-order vocabulary, $\varphi(t)$ be any sentence that contains the individual constant t, and x a variable in $\mathcal{L}$.*

$$\forall I : \frac{\begin{array}{c}\vdots\\ \varphi(t)\end{array}}{(\forall x)\varphi(x)}; \;\; \forall E : \frac{(\forall x)\varphi(x)}{\varphi(t)}.$$

[**?**] converge on the result that a multiple-conclusion formalization uniquely determines the classical truth-conditions of the propositional connectives while the 'axiomatic' (i.e. Hilbert style) proof systems fail for this task (they can determine the truth conditions of some simple connectives as 'and'-see [**?**, p.82]). In this paper we use what [**?**] calls 'deductive valuations/models', since we require an admissible valuation to preserve the soundness of the logical calculus, i.e. the soundness of each rule.

[17] The conclusion of a multiple conclusion rule is a set of formulas; any of them may be selected.

[18] Note that v^T and $v^{\vdash}$ are the only non-standard valuations admissible in a standard formalization of propositional logic. v^T disobeys the first row of the classical truth table for negation and $v^{\vdash}$ simultaneously disobeys the second row of the classical truth table for negation and the fourth row of the classical truth table for disjunction. The consequences for the other propositional connectives are presented by Carnap in a comprehensive table [**?**, p.82].

[19] See [**?**] and the references therein for details on this problem and on the inferentialist program. [**?**] and [**?**] analyze some recent proposals for obtaining a categorical formalization of propositional and first-order logic.

Restrictions on $\forall$I: t does not occur in any premise or assumption on which $\varphi(t)$ depends, and x does not occur in $\varphi(t)$. x replaces all and only occurrences of t in $\varphi(t)$.

$$\exists I : \frac{\varphi(t)}{(\exists x)\varphi(x)}; \quad \exists E : \frac{(\exists x)\varphi(x) \quad \begin{array}{c} [\varphi(t)] \\ \vdots \\ \psi \end{array}}{\psi}.$$

Restrictions on $\exists$E: t does not occur in i) $(\exists x)\varphi(x)$; ii) ψ, and iii) any premise or assumption on which the upper ψ depends (excepting the assumption $\varphi(t)$). The square brackets indicate that $\varphi(t)$ is an assumption that the $\exists$E-rule discharges. $\varphi(t)$ is obtained from $\varphi(x)$ by replacing all and only occurrences of x with t.

{identity}

Definition 2.12. *Let τ be a first-order vocabulary, $\varphi(t)$ be any sentence that contains the individual constant t. The introduction and elimination rules for $=$ are the following:*

$$= I : \frac{}{t = t}; \quad = E : \frac{t_1 = t_2 \quad \varphi(t_1)}{\varphi(t_2)}.$$

The $= I$-rule allows the derivation of the sentence $t = t$ from the empty set of premises, i.e. this sentence is a theorem. This means that on any line of a derivation we can introduce the sentence $t = t$.

Both Carnap [**?**, p.140, 148-50], [**?**, pp.231-32] and Garson [**?**, p.216, Th 14.3] showed that, due to the finitary character of the rules of inference, there are admissible valuations for the standard formalizations of first-order logic in which the first-order quantifiers have non-standard truth-conditions.

{non-categor

Example 2.13 (Non-categoricity of $\forall I$ and $\forall E$-rules)**.** (Carnap) Consider a vocabulary that has only two unary predicates F and G and a countable number of individual constants.

1. Let v be a standard (**R**-semantics) truth-theoretic valuation. In particular, it interprets $(\forall x)F(x)$ so that $v(F(c)) = T$ for each constant c; 'the values of $v(G(c))$ are irrelevant, for any constant c'.

2. Define a valuation v' that maps the individual constants in the vocabulary onto the objects from a denumerable domain and agrees with v on the quantifier-free formulas. Interpret $(\forall x)F(x)$ in v' as '*$v(F(c)) = T$ for any constant c and $v'(G(c)) = T$, for a particular c, namely b*'. In v' the universal quantifier has a richer content than that given by the usual truth conditions in v. We consider two options for how v interprets Gb, and thus how v' interprets $(\forall x)F(x)$, relative to the corresponding structure.

(a) If $v(G(b)) = v'(G(b))$ is true, then the $\forall E$ and $\forall I$ rules preserve their soundness under v' since Gb does not provide a counterexample.

(b) If $v(G(b))$ is false, then $\forall E$ is sound in v' since its premise, i.e. $v'((\forall x)F(x))$, is false. Likewise, if $v(G(b))$ is false, then the $\forall I$-rule preserves its soundness since both F and G are atomic predicates and there is no generalizable or free variable proof for Fc.

Since v' preserves the soundness of the $\forall I$ and $\forall E$-rules, v' is admissible, although it interprets the universal quantifier non-standardly[20].

Remark 2.14. Carnap's non-standard valuation v' is formulated in substitutional terms for a fixed countable vocabulary τ that involves no extension τ_M. This valuation offers an alternative to Notation 2.4.2 for assigning truth-conditions to a universally quantified sentence. The rules of inference for the universal quantifier preserve their soundness in all structures of the kind mentioned above when $(\forall x)F(x)$ in v' is interpreted as '$v(F(c)) = true$ *for any constant* c *and* $v'(G(c)) = true$*, for a particular* c*, namely* b'. The valuation v' is given by different semantic rules for the universal quantifier in the same structure.

It is well known by the Löwenheim-Skolem theorems that the formalizations of first-order theories are not powerful enough to determine the cardinality of their models. Thus, no first-order theory with an infinite model is categorical, i.e. not all its models are identical up to isomorphism. One may wonder, however, what happens if we strengthen the deductive system for FOL by adding more powerful rules of inference, such as the ω-rule, which is an infinitary rule of inference that allows the derivation of a universally quantified sentence from an infinite countable number of premises.

`{omega-logics`

Remark 2.15 (ω-logic, ω-rule, ω-model and standard model)**.** These terms appear with various definitions in various areas of logic. We attempt to differentiate. In all cases we let c range over a countable set C of constants.

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[20] Garson [**?**, p.237, Proof of Th. 14.3] also provided an example of non-standard valuation: Consider the set of instances of the sentence $(\forall x)F(x)$ in a certain vocabulary τ. The set $\{Fc_1, Fc_2, ...\} \cup \{\sim (\forall x)F(x)\}$ is syntactically consistent with the rules of inference for the universal quantifier. From a purely inferentialist perspective, we can define a valuation v'' that assigns truth to each member of this set. Thus, $v''(Fc_i)$ is true for each c_i, but $v''(\forall x)F(x)$ is false. v'' is non-standard since it provides $(\forall x)F(x)$ with nonstandard truth-conditions, but it is inferentially admissible since it preserves the soundness of the rules of inference for the universal quantifier. This valuation works for a deductive system that contains only the rules for the universal quantifier. However, from a standard model-theoretic perspective there is no corresponding underlying structure to this valuation since if $v''((\forall x)F(x))$ is false, then there is at least one object in the underlying domain that lacks the property F. Nevertheless, in Garson's valuation all objects are taken to instantiate the property F. Thus, since it has no corresponding structure, Garson's valuation is irrelevant from a standard model-theoretic perspective.

1. ω-logic and ω-rule. There are both model-theoretic and proof-theoretic approaches to ω-logic[21].
   (a) The more traditional proof-theoretic one adds the 'classical' ω-rule to the rules of inference of *one-sorted* first-order logic:

   $$\omega\text{-rule} : \frac{\{\varphi(c) : c \in C\}}{(\forall x)\varphi(x)}$$

   {classwrule}

   In the classical ω-rule, $\varphi(x)$ stands for any well-formed formula of the vocabulary τ in which the rule is formulated. Likewise, the variable 'x' is the only free variable in φ. The c_i are individual constants[22] from τ.

   This version of the rule was also used by Carnap [**?**, p.38, DC-2], [**?**, p.140, D30-3] and by Rosser [**?**, p.129], who calls it 'Carnap's Rule' -for historical remarks on different uses of the ω-rule see [**?**, pp. 101-5]. Precise formulations of variants of this rule will be given in Definitions 3.2, 4.1 and 5.12.

   (b) The model-theoretic version, stemming from ([**?**, **?**, **?**] as treated in [**?**, pp.28, 39], and [**?**, pp. 153-155]) is *two-sorted* with a designated predicate for 'the natural numbers'. In this case, an ω-model M is one where the predicate $N(M)$ consists only of a designated set of constants. Thus, the notion of ω-model is the same as in set theory [**?**, p 145], where $N(x)$ is replaced by $x \in \omega$ (So, not 2-sorted but with a distinguished predicate.).

   And the ω-rule is:

   $$\text{Generalized } \ \omega\text{-rule} : \frac{\{\varphi(c) : c \in C\}}{(\forall x)(N(x) \to \varphi(x))}.$$

   Chang and Keisler ([**?**, p.82-3]) expound both variants. We modify this rule to the $G - \omega$-rule in Definition 5.12. We use the single-sorted approach in Section 4 and the two-sorted in Section 5 but with a heavy emphasis on the rules of inference in both cases.

2. ω-model: There are several meaning of this term; the most important distinction here is whether the context is 1-sorted or 2-sorted.

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[21] While this notion developed from the study of arithmetic, this generalization has no reliance on arithmetic.

[22] Instead of using individual constants c, the ω-rule is sometimes formulated by using the so-called standard numerals. Since our interest in this paper goes beyond PA, we shall use the formulation of the ω-rule with individual constants c

(a) 1-sorted:

  (i) *ω-sequence ω-model* In the vocabulary $(0, S)$ the domain of the ω-model is the set of iterations of applications of a $1-1$ function to 0. See Definition 6.2.

  (ii) *arithmetic ω-model* The concept arose in studying arithmetic and the classical notion is: M is an ω-model of T if T interprets arithmetic in the vocabulary $(+, \times, 0, 1)$ of arithmetic and the universe of M is named by the numerals.

  (iii) *first order ω-model* This is generalized by replacing 'natural numbers' with 'the denotations of a specified countable set of constants' and T is any first order theory. This is the meaning of ω-model in Section 4.
  Set theorists work in a one-sorted world and an ω-model of ZFC is one which the only members of ω are the natural numbers.

(b) 2-sorted: [**?**] introduced the 2-sorted approach with a predicate N, a vocabulary containing countably many constants from N and an ω-model is one where these constants exhaust N. This is the meaning of ω-model in Section 5.

  This notation is especially important in reverse mathematics. The vocabulary contains two sorts, N, S (N for numbers and S for sets of numbers) and symbols for addition, multiplication, order, $0, 1$. An ω model is one in which the structure on the number sort is the standard structure for arithmetic, and the collection of sets is non-empty. [**?**, p 3] [**?**, Section 2.2]. That is, second order arithmetic with Henkin models rather than full second order quantification is studied.

3. Standard model: A *standard model* arises when there is an informal notion that has widely accepted formal counterpart. E.g., arithmetic, consider a vocabulary τ that includes the following non-logical terms $\{0, S, +, \times, <\}$ with the following abbreviations: $1 = S0, 2 = SS0, \ldots$. A standard model of arithmetic is an ω-model in which the domain is $N = \{0, 1, 2, 3, \ldots\}$, i.e. the model omits the set of formulas $\{x \neq 0, x \neq 1, x \neq 2, \ldots\}$.

   As [**?**, p. 28, 5B] point out, a confusion may arise when, influenced by Dedekind's axioms and the omnipotence of second order definability, one takes only $(\omega, 0, S)$ as the standard model of arithmetic.

It has been known as least since [**?**] that PA^{ω}, i.e. the consequences of the first order Peano axioms in first order logic with the ω-rule, proves TA (true arithmetic: all sentences that are true in the standard model of PA).[23] One might hope that since the ω-rule

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[23] See [**?**], [**?**], and [**?**] for later treatments.

expresses a more restrictive condition on its models, that PA^{ω} will uniquely determine the standard model of PA up to isomorphism. However, this dream is unfulfilled, since [**?**] TA, still has non- standard models[24].

We introduce in section 4 a novel version of the ω-rule giving what we call $I-\omega$-logic and show that its rules of inference for the universal quantifier are categorical (Proposition 4.2). Then we show that, formalized in the $I-\omega$-logic, PA^{ω} is categorical (Theorem 4.9). Before that, we describe in the next section two historical steps towards our $I-\omega$-logic (the first one is from provability to model-theoretic (objectual) semantics and the second one is from the latter to substitutional semantics).

3 Interludio: Tarski and Carnap on the ω-rule

{`Interludio`}

We discuss in this section Tarski's use of the ω-rule as an argument for a model-theoretic approach to logical consequence and we question his dismissal of the substitutional semantics. Then we discuss Carnap's application of the ω-rule for obtaining a categorical formalization of the substitutional truth-conditions for the universal quantifier in a fixed countable vocabulary.

3.1 Tarski on the ω-rule and Logical Consequence

{`Tarski`}

In order to show that the proof-theoretic concept of logical consequence does not coincide with the "common" concept of logical consequence, Tarski [**?**, p.410] provides an example of a sound inference that it is not provable on the basis of the finite rules of inference that define the usual concept of provability:

(ω) $A_0 : P(0)$
$A_1 : P(1)$
$A_2 : P(2)$
$\vdots$ (for all natural numbers)
A : For every natural number n, $P(n)$.

For Tarski, "intuitively" it seems certain that A follows from the totality of premises $A_0, A_1, A_2, \ldots$, and by this he refers to the fact that a counterexample to this inference must involve a situation in which the conclusion is false, which in turn would imply that there is at least one natural number n that is not P. However, for each such number there is a premise that asserts that n is P. Thus, the inference is sound, although not provable. If

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[24] For Shapiro's ([**?**, 40-1])'non-algebraic' theories, we use the term "standard model" in the usual sense, the model that provides the logical and non-logical expressions with their intended meanings. In the case of the real field, the standard model is the Dedekind reals.

we formalize this inference in first-order logic (FOL), however, by the methods that Tarski himself developed, we shall obtain an invalid (i.e. unsound) logical form: [**?**, p.415]

$$
\begin{array}{ll}
(\omega_{FOL}) & A_0 : P(a_0) \\
 & A_1 : P(a_1) \\
 & A_2 : P(a_2) \\
 & \vdots \text{ (for all natural numbers)} \\
 & A : (\forall x)(N(x) \to P(x)).
\end{array}
$$

A counterexample to this inference could be easily found if we let the non-logical constants $a_0, a_1, a_2, \ldots$ to denote $0, 1, 2, \ldots$; we take the extension of the predicate N to be $\{0, 1, 2, \ldots, \gamma\}$ and the extension of the predicate P to be $\{0, 1, 2, \ldots.\}$, where γ is an additional object in the domain. This counterexample is an objectual model of the theory of a pair of sets (N,P) with infinitely many distinct constants satisfying P and $P \subseteq N$. This objectual model is justified by Tarski's own model-theoretic account of the concept of logical consequence, for which a necessary condition is taken to be the following:

(F) If, in the sentences of the class K and in the sentence X, the constants —apart from, purely logical constants— are replaced by any other constants (like signs being everywhere replaced by like signs), and if we denote the class of sentences thus obtained from K by "K'" and the sentence obtained from X by "X'", then the sentence "X'" must be true provided only that all sentences of the class "K'" are true.

When taking the inference (ω_{FOL}) to be sound, Tarski seems to assume that both N and the numerals $0, 1, 2, \ldots$ are logical expressions and, thus, they will preserve their meanings in all admissible models of Peano Arithmetic (see [**?**, pp. 83-5]). However, this assumption is problematic from a logical point of view since it entails that the arithmetical terminology is also logical[25]. Making this assumption is equivalent with asserting that PA is categorical, i.e. all its models are isomorphic, but this should rather be the conclusion of the reasoning, rather than one of its premises. Moreover, although the condition (F) is taken by Tarski [**?**, pp. 415-16] to be a necessary condition, he acknowledges that it may also be taken to be sufficient, but "only if the designations of all possible objects occurred in the language in question. This assumption, however, is fictitious and can never be realized." Tarski does not explicitly mention why this assumption cannot be realized, but probably he considers that we cannot have more than a denumerable number of individuals[26], and criticizes Carnap [**?**] for making the concept of consequence dependent on the

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[25] For a very recent defence of a logicist position in connection to an inferentialist point of view, see [**?**].

[26] Note the use of variables rather than names in his definition of satisfaction (Remark 2.5.).

richness of the language investigated.

However, as it turns out in the case of the ω-rule, its intuitive soundness is also formally justified if we assume that the designations of all the objects from the domain occur in the language and appear in the premises of the ω-rule. As we shall see, this condition is embedded in a semantics that validates the ω-rule and is inferentially determined by it (Propositions 3.3, 4.2). This already provides us with a hint on how the categoricity of PA^{ω} can be inferentially obtained, with a certain reading of the ω-rule. From a logical inferentialist point of view, the richness of the language investigated will turn out to be essential in obtaining important properties, such as the categoricity of the first-order Peano Arithmetic.

3.2 Carnap's Use of the ω-rule in Substitutional Settings

{carinf}

Carnap [**?**, xv] initially suggested that any arbitrarily chosen rules of inference for a logical expression will determine what *meaning* is to be assigned to that expression. However, he later (in [**?**]) recognized that when a logical system is previously semantically defined, then a logical calculus has to be constructed that fully represents logical truth and logical consequence and, in addition, allows as admissible only those valuations that provide the logical expressions with their intended truth-conditions. These calculi are categorical or full formalizations of the system of logic that is semantically defined in advance.[27]

In addition to the multiple conclusion rules for propositional logic (Definition 2.10), Carnap [**?**, p.145] introduced a version of the ω-rule for providing a categorical formalization of the truth-conditions for the universal quantifier defined over a denumerable domain in which each object is named by an individual constant in the language. We reconstruct below Carnap's full formalization of the first-order quantifiers by formulating what we call the substitutional ω-logic (i.e. $S-\omega$-logic).

This Carnapian semantical approach to the first-order quantifiers (i.e. the substitutional interpretation) is an alternative to the Tarskian objectual interpretation of them and defines

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[27] [**?**, pp.3-7] calls this kind of approach *model-theoretic inferentialism*. Garson contrasts this approach with a *proof-theoretic inferentialism* that characterizes the meanings of the logical expressions in proof-theoretic terms (i.e. proofs, derivability conditions, etc.). Although a proof-theoretic inferentialist sees the talk about models and structures (at least) as a dispensable feature, the idea that the meaning of a logical expression is determined by the rules of inference that govern its use in a certain language (i.e. vocabulary τ) is compatible with a characterization of these meanings in model-theoretic terms. For instance, a semantic characterization of these meanings that completely displaces the standard objectual notion of *model* was developed in great detail by Leblanc [**?**]. However, since we want to approach the categoricity problem for a mathematical theory both from a model-theoretic and from a proof-theoretic point of view, we will use of the term *model* in the model theoretic sense, and of the relational and functional components of a valuation.

their truth conditions in terms of their instances in a certain vocabulary τ.[28]

If we define it for a vocabulary τ that contains a countable number of individual constants, a specific feature of the substitutional semantics is non-compactness, as every interpretation determines a countable model. Those interested in matching the substitutional semantics with the standard model-theoretic semantics for first-order logic have imposed different constraints in order to make this semantics compact, i.e., in order to invalidate the ω-rule (see for instance [**?**, p.183], [**?**, p.49], [**?**, p.215]. These constraints usually require mentioning the language in the definition of logical consequence, such that the rules remain sound in valuations over extensions of the initial language. However, since our goal is categoricity, and thus we are not interested in preserving compactness, while we continue to mention the language, we shall not deal with the status of these constraints here.

Recall from Notation 2.4.3 that *a valuation is admissible for a logic just if each inference preserves truth; that is, each inference rule is sound.* The substitutional truth-conditions of the universal quantifier (dating from Carnap), which, as noted just above, require that objects in each model are named in τ, are defined as follows:

Definition 3.1. *(Substitutional interpretation of the universal quantifier) For any permissible valuation v,* $v((\forall x)\varphi(x)) = true$ iff $v(\varphi(c)) = true$ for all c $\in \mathrm{Const}(\tau)$. {SubIntForal

This means that a universally quantified sentence is true in a valuation if and only if all its substitution instances obtained by substituting the bound variable ‘x’ with the individual

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[28] The distinction between objectual and substitutional interpretation of the quantifiers is mainly philosophical. This distinction is largely motivated by an ontological attitude, since some philosophers find the logical discourse about existing objects that have properties to be philosophically problematic and try to remain in a neutral language. For instance, objectually we say ‘$(\forall x)F(x)$’ is true iff all objects in the domain have the property F, while substitutionally: ‘$(\forall x)F(x)$’ is true iff all substitution instances ‘$F(a)$’ of this sentence are true.

The substitutional interpretation is dependent on the resources (i.e. individual constants) of the syntactic vocabulary we use. But if we have enough names and all objects are named, there is no logical difference between the two interpretations. However, there still remains a philosophical difference, namely: the substitutional approach takes as sufficient for interpreting the syntactic vocabulary a valuation that assigns truth-values to each sentence and show no interest in what makes ‘Fa’ true. It simply stipulates that $v(Fa)$ is true, without assuming that there is an entity called ‘model’ in which there is an object denoted by ‘a’ which has the property denoted by ‘F’, or belongs to the extension of ‘F’ (as the objectual interpretation requires -see [**?**, p.184], [**?**, pp. 39-51], [**?**, p.213]). The real difference between the objectual and substitutional interpretations is that the objectual one allows unnamed objects, while this idea is not intelligible from a substitutional point of view. We consider that the R-semantics we use cannot be classified as either of these traditional philosophical categories. On the one hand it begins with structures; on the other, it expands to vocabulary to substitute. In addition to this, the distinction between these two interpretations seems to be absent in mathematical practice. For instance, model-theorists duck the philosophical issue by: ‘we work in ZFC’ which makes the ‘objectual’ structures exist and the ‘we must define truth in them’ approach coherent.

constants from a vocabulary τ are true. However, this substitutional interpretation of the universal quantifier is not fully determined by the standard rules of inference in first-order logic due to the finitary character of these rules (see Example 2.13). Nevertheless, this substitutional meaning of the universal quantifier is fully determined by the relation of logical derivability if the deductive system of FOL is strengthened by introducing the following infinitary ω-rule.

Definition 3.2. *(Substitutional ω-rule) Let τ be* any *countable vocabulary. The collection of individual constants in τ is $\{c : c \in Const(\tau)\}$. Let φ stand for well-formed formulas in the vocabulary τ that have a unique free variable. The rules of inference for the universal quantifier in the substitutional inferential ω-logic are the following:* {Subrules}

$$S - \omega\textit{-rule} : \frac{\bigwedge\{\varphi(c) : c \in Const(\tau)\}}{(\forall x)\varphi(x)};$$

$$S - \forall E : \frac{(\forall x)\varphi(x)}{\varphi(c), \textit{ for each } c \in Const(\tau)}.$$

For having a complete picture, although we will not use the existential quantifier in this paper, the rules of inference that uniquely determine the truth conditions of this quantifier in the ω-logic are the following:

$$S - \exists I : \frac{\varphi(c), \textit{ for some } c \in Const(\tau)}{(\exists x)\varphi(x)};$$

$$S - \exists E : \frac{(\exists x)\varphi(x)}{\bigvee\{\varphi(c), c \in Const(\tau)\}}.$$

The conclusion of the S-$\exists$E-rule is a disjunctive set of instances, i.e. this rule is a multiple conclusion rule of inference that generalizes the disjunction elimination rule mentioned in section 1.[29]

We refer to the extension of first order logic by these rules as $S - \omega$-logic.

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[29] For a detailed analysis of multiple-conclusion logic see [**?**].

We have adopted categorical rules for the propositional connectives as in Definition 2.10; so to establish categoricity, we need only check the quantifier rules.

Proposition 3.3. *(Categoricity of S-ω-rule and $S-\forall E$-rule) The $S-\omega$-rule and the $S-\forall E$-rule uniquely determine the substitutional interpretation of the universal quantifier as given in Definition 3.1.* `{infsubmeanin`

Proof. ($\Longrightarrow$) Assume that for any admissible valuation v, $v((\forall x)\varphi(x)) = true$. It follows from the soundness of the $S-\forall E$-rule in the $S-\omega$-logic that $v(\varphi(c)) = true$, for all $c \in Const(\tau)$.

($\Longleftarrow$) Assume that for any admissible valuation v, $v(\varphi(c)) = true$, for all $c \in Const(\tau)$. Then, from the soundness of the $S-\omega$-rule in $S-\omega$-logic, we obtain that $v((\forall x)\varphi(x)) = true$.[30]

□

Definition 3.2 above is a syntactic definition of the rules of inference for the quantifiers in the S-ω-logic. Definition 3.1 is a semantic definition that states the truth-conditions of the universal quantifier in this logic, while Proposition 3.3 shows that the rules of inference for the universal quantifier uniquely determine the truth-conditions stated in Definition 3.1 such that no alternative non-standard interpretation of this quantifier is admissible. In particular, the non-standard valuations from Example 2.13 formulated by Carnap and Garson are no longer admissible since they violate the soundness of the $S-\omega$-rule. Consider Carnap's valuation v' that interprets $(\forall x)F(x)$ as '$v(F(c)) = T$ for any constant c and $v'(G(c)) = T$, for a particular c, namely b', while v'(Gb) is false in the corresponding structure. The premises of the $S-\omega$-rule are true in v' (since $v'(F(c))$ is true in the corresponding structure for any constant c), while the conclusion is false (since v'(Gb) is false). Thus, the structure corresponding to v' makes the rule unsound and, consequently, v' is not admissible. Consider Garson's valuation v'' that assigns truth to all countable instances of a universally quantified sentence, but falsehood to the sentence itself. v'' makes the premises of the $S-\omega$-rule true, while its conclusion false. Thus v'' is not admissible. These considerations also apply to the $I-\omega$-rule that we introduce in the next section, since Carnap's and Garson's non-standard valuations do not deal with extensions of the vocabulary τ.

The $S-\omega$-logic illustrates the match between the substitutional semantics for the universal quantifier in a fixed countable vocabulary and the substitutional rules of inference

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[30] This proof shows that the $S-\omega$-rule and elimination rule for the universal quantifier in the substitutional omega logic uniquely determine its semantic meaning as it is defined by the substitutional semantics. In Carnap's terms [**?**, p.4], all semantic properties of the universal quantifier are "fully formalized" by these rules, such that a user of these rules cannot misinterpret the meaning of the sign '$\forall$'.

(Proposition 3.3), but it is inadequate to obtain the categoricity results for first-order theories. We obtain these results by formulating two more powerful rules in the next two sections.

4 The Inferential ω-rule and the Categoricity of PA^{ω}

{subquant}

We have fixed on the R-semantics in Notation 2.4.2 and Remark 2.5 as a definition for valuations/interpretations/structures. We now describe two systems of inference rules: inferential (I) (here) and G-inferential (G) (§5). The admissible valuations for these rules assign truth-conditions for the quantifiers governed by these rules and we prove that the rules uniquely determine these truth-conditions (Propositions 4.2, 5.13).

The inferentialist idea that the truth-conditions of a logical expression is uniquely determined by the class of admissible valuations (see Notation 2.4.3) can be easily evaluated in the case of the propositional connectives, where a valuation is simply a function that assigns truth values to each well formed formula from the vocabulary τ. Extending this idea to the quantifiers is particularly difficult when we consider valuations in expansions by constants of the original vocabulary.

The formal rules of inference for the universal quantifier and its substitutional interpretation in the $S-\omega$-logic were formulated in the previous section in a very general manner, i.e. for an arbitrary vocabulary τ, but without specifying the relation between a valuation v defined over a particular vocabulary τ and valuations defined over extensions of this vocabulary. Carnapian substitutional interpretation takes it for granted that we have a fixed countable set C_{τ} of constants given by the constants c and each valuation v must map these constants onto a domain (and implicitly the relations and functions in the vocabulary). The countable number of c and the 'onto' condition is assumed to enforce the idea that all models are countable.

In short, these rules are simply not sufficient for model theory. In order to deal with valuations over extended vocabularies and for its usefulness in studying non-standard models, we shall use in our approach a certain version of the substitutional semantics, namely, the Robinsonian semantics (Definition 2.4.2).

The Inferential ω-rule we now introduce is considerably stronger than the usual versions. While the instantiations that must be satisfied in the hypothesis of the ω-rule are the same as usual, the formula φ now may have parameters $\overline{d}$ from $Const(\tau(C)) - Const(\tau)$. We require $Const(\tau)$ to be countably infinite.

Definition 4.1 below is a syntactic definition of the rules of inference for the quantifiers in the inferential ω-logic.

Proposition 4.2 shows that the rules of inference for the universal quantifier uniquely determine the truth-conditions for $\forall$, making the rules in Definition 4.1 categorical.

As in Section 3.2, we restrict ourselves to vocabularies with countably many constants. Unlike Carnap, we do not require all constants from C to be used in every valuation. We only require the constants from $Const(\tau)$ to be used in every valuation. Thus, our rules are actually rule schema, indexed by specifying a set of constants.

We state the rule for vocabularies with infinitely many constants. We can study theories T in vocabularies with only finitely many constants such that all models of T have the definable closure of the $\emptyset$ infinite. E.g. $Th(\omega, S, 0)$, an ω-sequence.

Definition 4.1. *(Inferential ω-rule) Fix a vocabulary τ and use the notation of R-semantics as in Notation 2.4.2 and Remark 2.5. For any countable set of constants $C \supseteq Const(\tau)$ and any well-formed formula $\varphi(x, \overline{d})$ in the vocabulary τ expanded by the constants C (denoted $\tau(C)$):* `{ourinfrule}`

$$I - \omega\text{-}rule : \frac{\bigwedge\{\varphi(c, \overline{d}) : c \in Const(\tau)\}}{(\forall x)\varphi(x, \overline{d})};$$

$$I - \forall E : \frac{(\forall x)\varphi(x, \overline{d})}{\varphi(c, \overline{d}),\ \textit{for each } c \in C}.$$

The rules for the existential quantifier in the notation of the R-valuations are the following:

$$I - \exists I : \frac{\varphi(c, \overline{d}),\ \textit{for some } c \in C}{(\exists x)\varphi(x, \overline{d})};$$

$$I - \exists E : \frac{(\exists x)\varphi(x, \overline{d})}{\bigvee\{\varphi(c, \overline{d}) : c \in C\}}.$$

Proposition 4.2. *(Categoricity of I-ω-rule and $\forall E$-rule) The I-ω-rule and the I-$\forall E$-rule for the universal quantifier in the inferential ω-logic uniquely determine:* `{infRmeaning`

For any countable set of constants $C \supseteq Const(\tau)$, any admissible valuation v_C, $v_C((\forall x)\varphi(x, \overline{d})) = true$ iff $v_C(\varphi(c)) = true$ for each constant $c \in Const(\tau)$.

Proof. ($\Longrightarrow$) For any admissible valuation v_C, if $v_C((\forall x)\varphi(x,\overline{d})) = true$, then $v_C(\varphi(c,\overline{d}))$ is true, for each c from C (by the I-∀E-rule) and, in particular, for each τ-constant c.

($\Longleftarrow$) For any admissible valuation v_C, if $\varphi(x,\overline{d})$ is a formula in the vocabulary $\tau(C)$ and $v_C(\varphi(c)) = true$ for each τ-constant c, then $v_C((\forall x)\varphi(x,\overline{d})) = true$ (by the I-ω-rule).

□

Notation 4.3. *For any first order theory, We denote the set of its consequences under an ω-rule as T^ω. There is a certain ambiguity in the T^ω notation as its meaning depends on which variant of ω-logic is being considered. In this section we use T^ω as a first order theory closed under the $I-\omega$-rule and in the next one as closed under the $G-\omega$-rule.*

{stdmod}

Remark 4.4 (Scholia on Remark 2.15.3 (Standard Models))**.** [**?**] distinguishes between ‘algebraic’ and ‘non-algebraic’ structures. An ‘algebraic’ structure such as a group is one of a class of structures for which there is no prototype. In contrast, we have a widely accepted notion of what the isomorphism type of a ‘non-algebraic’ object is. E.g, there is a construction in set theory that is widely agreed to be, e.g. the standard model of Peano arithmetic. These are the structures that prompted the notion of categoricity in early twentieth century. One of our goals here is to provide mathematical rather than historical conditions on a structure which yields categoricity.

Example 4.5. *The theory of a countably infinite set A theory T_0 whose models are all infinite sets it is easily axiomatized in $L_{\omega_1,\omega}(\tau)$, where τ has only the equality symbol. Take the conjunction of the sentences φ_n asserting there are n distinct objects, e.g., φ_2 is $(\exists x, y)x \neq y$. This theory has arbitrarily large models but is categorical in $\aleph_0$.* {set}

We now show that with their reading in terms of R-valuations the rules of inference for the universal quantifier in the inferential ω-logic make PA^ω categorical.

A fundamental result for the ω-logic is the ω-completeness theorem (see [**?**, p.82]), where ω-model is in the sense of Remark 2.15.2.a.ii):

{omegamodexis

Proposition 4.6. *A theory T in the vocabulary τ is consistent in ω-logic if and only if T has an ω-model.*

It is important to mention that the ω-completeness theorem simply tells us that the presence of the ω–rule guarantees the existence of an ω-model, but it does not tell us that *all* models of PA^ω are ω-models (see [**?**, Th.2], [**?**, Th.3]). The argument that we put forward for the categoricity of PA^ω is formulated from an inferential point of view, namely, we take the soundness of the $I-\omega$–rule together with the assumptions of nameability and countability of the set of constants that we work with as primitive, and this allows us to read off the truth-conditions of the universal quantifier in the $I - \omega$-logic (these assumptions will be discussed in section 7). With these truth-conditions we provide a positive answer to the question of the categoricity of PA^ω. Roughly, the argument goes as follows:

P1. The truth-conditions of the universal quantifier are uniquely determined by the $I - \omega$-rule and $I - \forall E$-rule in the inferential ω-logic (Proposition 4.2).

P2. With these truth-conditions of the universal quantifier we obtain a categorical characterization of the natural numbers (Theorem 4.9).

C. Thus, with the $I - \omega$-rule and $I - \forall E$-rule in the $I - \omega$-logic we obtain a categorical characterization of the natural numbers.

The first premise of this argument has already been established above in this section (Proposition 4.2). We show now that the second premise also holds. For clarity, we remind the reader that we work with countable vocabularies τ and we assume that all objects from the domain that someone can 'refer to' are named by a countable number of individual constants. We prove below that PA^ω is categorical. The proof we give below (in Theorem 4.9) shows that a non-standard model for PA does not satisfy the rules of inference for the universal quantifier in the inferential ω-logic.

`{prime}`

Definition 4.7. *There are two notions of 'prime model'; both will be used.*

1. *A τ-structure M is an* algebraically or Robinson prime *model of T if it can be embedded in every model of T,*
2. *A τ-structure M is* Vaught prime *(usually, just prime) if it can be elementarily embedded in every model of T.*

Definition 4.8 (Robinson's **Q**)**.** *By* **Q***, we mean the finitely axiomatized first-order theory, considerably weaker than Peano arithmetic (PA), whose axioms contain only one existential quantifier. See [**?**], [**?**, Ch. 10-11] or the splendid wikipedia article on Robinson Arithmetic. Like PA, it is incomplete and incompletable in the sense of Gödel's incompleteness theorems, and essentially undecidable. The vocabulary $\tau_{\mathbf{Q}}$ for* **Q** *has a single constant symbol* 0 *and function symbols S (unary) and $+, \times$ (binary). By $\mathbf{Q}^\omega$ we mean* **Q** *in the inferential ω-logic.*

`{wrulebworks`

Theorem 4.9. With the inferential ω-rule (Definition 4.1), the standard model of Robinson's Q has no proper extension satisfying Q^ω, thus Q^ω is categorical.

Proof. Note that in any model of **Q**, the $\tau_{\mathbf{Q}}$-substructure generated from 0 is isomorphic to $\mathbf{N} = \langle \mathbf{N}, \mathbf{0}, +, \times \rangle$. That is, N is an algebraically prime model of **Q**. Thus, an arbitrary countable non-standard model M of **Q** extends (an isomorphic copy of) N. Fix $Const(\tau_N)$ as a countable set of constants containing 0 and a valuation v_N that maps $Const(\tau_N)$ onto N. Let v_M be an extension of v_N which enumerates the remaining elements of the τ_M-structure M by constants $e \in Const(\tau_M)$ where $Const(\tau_M) = Const(\tau_N) \cup D$ and D is a countable set disjoint from $Const(\tau_N)$. Fix a particular $d \in D$. Consider the formula $\varphi(x, d) : x \neq d$.

Thus, if v_M is admissible, then it satisfies the following instance of the inferential ω-rule.

$$I - \omega\text{-rule} : \frac{\{\varphi(c, d) : c \in Const(\tau_N)\}}{(\forall x)\varphi(x, d)};$$

Again, if v_M is admissible, it also satisfies the following instance of the inferential $\forall$E-rule:

$$I - \forall E : \frac{(\forall x)\varphi(x, d)}{\varphi(e, d),\ \text{for each } e \in Const(\tau_M)}.$$

By this instance of the I-$\forall E$, we have $\varphi(d, d)$. But this is a contradiction since $d = d$. Thus v_M violates the rules for the universal quantifier in ω-logic and is not admissible.

Note that v_N is clearly admissible since the hypothesis of the ω-rule contains all the instances that can be derived from $\forall E$. □

Of course, this also implies PA^ω is categorical, since it is a consistent extension of $\mathbf{Q}^\omega$.

{expmain}

Remark 4.10. The proof of Theorem 4.9 assumes, for reduction to contradiction, that both v_N and v_M are admissible valuations over $\mathbf{Q}^\omega$. That v_N is an admissible valuation is justified by the ω-completeness theorem. The I-ω-rule and the I-$\forall$E-rule are legitimately applicable in v_M since the soundness of these rules was defined for arbitrary countable vocabularies τ and the formula $\varphi(x) : x \neq d$ is a syntactically well-formed formula in the vocabulary τ_M of the valuation v_M. However, since a contradiction is derived, then v_M is not an admissible valuation.

To give general conditions on a theory T for categoricity with the $I - \omega$-rule requires abstracting two elements of the proof of Theorem 4.9. We made explicit that the theory **Q** has an algebraically prime model. It was also crucial that every element of that model was named. With this observation it is straightforward to deduce:

{apcat}

Corollary 4.11. *If a first order theory T in a vocabulary τ has an algebraically prime model with every element named by a τ-constant, then T plus the $I - \omega$-rule is categorical.*

In particular this applies to the theory of $(\omega, 0, S)$ with axioms that S is $1 - 1$ and every point is in the range except 0.

We can show all models of a theory T are countable without the algebraically prime hypothesis but with no restrictions on the number of countable models. The argument for Theorem 4.9 immediately shows.

{allmax}

Theorem 4.12. *Fix a vocabulary τ with $\aleph_0$ constants and suppose that for any model of T, the substructure consisting of the elements named by those constants is a model of T. Then, no model of T^ω has a proper extension.*

The following examples show both hypotheses of Corollary 4.11 are essential. Items 1) and 2) show why we require there be infinitely many constants (or a least infinitely many elements definable over the empty set) in τ. Item 3) show the algebraically prime model is essential.

{neccond}

Example 4.13.

1. The pure theory of an infinite set (i.e. $\tau = \{=\}$ with axioms asserting there are at least n elements for every $n < \omega$) shows the necessity of the second requirement. It is categorical in all cardinalities, but the absence of τ-constants make the ω-rule powerless. In contrast, Theorem 4.9 applies in two related cases: i) if the vocabulary $\{=\} \cup C$ with C a countably infinite set of constants with axioms $c \neq d$ for distinct constants in C or ii) the theory of (ω, S) in a finite vocabulary even without a constant in $\tau = \{S\}$ (Cf. Definition 2.6).
2. Consider the theory of the rational order (dense linear order without endpoints) in the vocabulary $\{<\}$. Like the theory of an infinite set, Theorem 4.9 does not apply. And like Example 2, there are continuum many countable completions each of which is categorical.
3. The vocabulary has two unary predicates F, G and countably many constants. The (incomplete) theory T asserts only that the constants are distinct. Clearly T satisfies Theorem 4.12. But there are continuum many countable models since any distribution of the constants among the four disjoint sets given by the boolean combinations of F and G yield a distinct model of T.

{catrat}

Remark 4.14. *Since Example 4.13 is a canonical example of categoricity (in $\aleph_0$), it presents in a graphic way the problems of reference we discuss in Sections 6 and 7. In this case there is a natural choice of naming the constants as $c_{n,m}$ for integers n and m. Namely, we name the rational number $\frac{n}{m}$ with n and m relatively prime by $c_{n,m}$ for n, m integers.*

5 Categorical structures in $L_{\omega_1\omega}$ with an Inferential $G{-}\omega$-rule

{infinitary}

In this section we translate $L_{\omega_1,\omega}$-sentences to theories in first-order logic and add an inferential $G - \omega$-rule. This allows us to translate infinary sentences to finite sentences at the cost of an infinitary rule of inference.

As in the previous sections, unlike a general abstract logic, we require a syntax of the usual sort[31]. Our theories will be formulated in countable vocabularies. Using the $\mathbf{R}$-semantics, arbitrarily many constants are available but our central arguments will only involve countable expansions of the vocabulary.

$L_{\omega_1,\omega}$ extends first order logic by allowing countable conjunctions and disjunctions of countable sets of formulas Φ with the restriction that only finitely many free variables can occur in Φ. This guarantees that quantifying by a finite string of quantifiers is permissible. The notion of valuation in Definition 2.4 is extended by $M \models \bigwedge \Phi$ if and only each $\varphi \in \Phi$ is true in M and dually for disjunction.

To avoid confusion, we do not discuss the normal infinitary proof rules (for conjunction) in $L_{\omega_1,\omega}$ and the resulting Karp completeness theorem establishing the equivalence of provability and validity for $L_{\omega_1,\omega}$, [**?**, Chapter 4]. However, we do rely on the theorems from Fact 5.3 about $L_{\omega_1\omega}$, which are found in such sources as [**?**, **?**, **?**], to translate $L_{\omega_1\omega}$ classes to those definable in $L_{\omega,\omega}$ with our $G - \omega$-rule. To facilitate their understanding, we now describe the model theoretic notion of *type*. This notion has nothing to do with Russell's type theory or any of its descendants.

{fot}

Definition 5.1 (First order types). *A type $p(\overline{v}/A)$ is a description of the relation between a point b (finite sequence $\overline{b}$) in a model N and a set A with $A \subseteq M \prec N$. Suppose M is a τ-structure. Let τ_A be the vocabulary obtained by adding constants for each element of A. Formally, a type p over A is a consistent set of formulas*

$$p(\overline{v}) = \{\varphi(\overline{b}, \overline{a}) : N \models \varphi(\overline{b}, \overline{a})\}$$

where $\varphi(\overline{v})$ is a τ_A-formula and $\overline{v}$ is a $k = \lg(\overline{b})$ sequence of variables.

Often such a type p is shown consistent by observing that each finite subset of p is satisfied in M.

Example 5.2. *Examples of Types*

{extyp}

1. *Let M be a countable set of points named by $\{a_i : i \in \omega\}$. We include in the theory of M all sentences $a_i \neq a_j$ if $i \neq j$.*

---

[31] The two sorted structures here are even more special because they do not admit relativization.

a) *Let* p_1 *be the type* $\{x = a_{17}\} \cup \{x \neq a_i : i \neq 17\}$.

b) *Let* $p_2 = \{x \neq a_i : i \in \omega\}$.

By the compactness theorem, each p_i *is consistent. A* realization *of* p_i *must occur in an elementary extension of* M*; say,* $M' = M \cup \{b\}$.

Example a): p_1 *is a* principal *or* atomic *type because a single formula implies the others.*

Example b): p_2 *is* nonprincipal *because it requires infinitely many formulas.*

2. *(Ehrehfeucht's example) [**?**] Let* $M = (\mathbf{Q}, <)$ *be the ordered set of rational numbers. Let* $A = \{a_i : i \in \omega\}$ *name an increasing sequence.*

   Let $p = \{v > a_i : i < \omega\}$. *The* L_A *theory has three models. In one,* p *is omitted, in one it is the limit point of the* a_i *and in the third above the limit point.*

3. *Let* $\mathbf{Q}$ *be the set of rational numbers contained in the set* $\mathbf{R}$ *of real numbers in a vocabulary with* $<$*-symbol in its normal interpretation. Then* $\sqrt{2}$ *realizes the type over* $\mathbb{Q}$,

   $$\{x < q : q^2 > 2\} \cup \{x > q : q^2 < 2\}.$$

   Note that multiplication was used in the external definition (choosing the q's), but does not appear in the formulas in the type.

We rely on the following background facts; we try to explain them below and give a detailed proof of a variant of item 4.

{mathback}

Fact 5.3. 1. *Scott's theorem: For any countable structure* A *for a vocabulary* τ*, there is an* $L_{\omega_1,\omega}(\tau)$ *sentence* φ_A*,* the Scott sentence of A*, such that all countable models of* φ_A *are isomorphic to* A. φ_A *is* complete *in that for any sentence* ψ *of* $L_{\omega_1,\omega}$, $\varphi_A \vdash \psi$ *or* $\varphi_A \vdash \neg\psi$*. This holds by the extended completeness theorem (see [**?**, Sec. 2] or [**?**, §6.1]), the downward Löwenheim-Skolem theorem, and the uniqueness of the countable model.*

2. *Chang [**?**, p 48]: For any* $L_{\omega_1,\omega}$ τ*-sentence* φ*, there is a vocabulary* $\tau^\varphi \supset \tau$*, a first order* τ^φ*-theory* T^φ*, and a countable collection of types* Γ *such that the models of* φ *are exactly (in particular, no two non-isomorphic atomic models of* T^φ *have isomorphic* τ*-reducts) the* τ*-reducts of the models of* T^φ *that omit each type in* Γ*. Moreover, if* φ *is complete, we can consider the reducts of the atomic models of* T^φ*. See [**?**, §1.2] for the first and Theorem 6.1.12 and Chapter 18 in [**?**] for the atomic case.*

3. *Applying the classical paper of [**?**] to T^φ, φ has an uncountable model if and only if the unique (up to isomorphism) countable model has a proper submodel isomorphic to itself.*

4. *Morley: [**?**] Work in a two-sorted vocabulary σ with a predicate N, a countable set C of constants each satisfying N, and a predicate V with a τ^φ-structure from (2) such that $V \restriction \tau^\varphi \models T^\varphi$. Each non-principal type (Example 5.2.1.b) p over the empty set can be coded*[32] *in a theory $T^{\hat{\varphi}}$ so that p is omitted in a model M if and only if the type $\{x \neq c, N(x) : c \in C\}$ is omitted. Thus the reducts to τ of models of T^φ (from Chang) that arise as the $V(B)$ for $G - \omega$-models (as defined in Definition 5.11) B of $T^{\hat{\varphi}}$ are exactly reducts of atomic models of T^φ and models of φ. There is no additional rule of inference in [**?**]. (See Notation 5.7.5 below for $T^{\hat{\varphi}}$).*

{catdef}

Definition 5.4. *We say a* complete *sentence ψ of $L_{\omega_1,\omega}$ is* categorical *if it has a unique model (up to isomorphism).*

Since $L_{\omega_1,\omega}$ satisfies the downward Löwenheim-Skolem theorem, necessarily, the unique model is countable. However, *mentioning that ψ is a* sentence *is essential.*

Example 5.5. *The $L_{\omega_1,\omega}$ theory of the structure $(\mathbb{R}, 0, 1, +, \times, <)$ is categorical; the only model has cardinality $2^{\aleph_0}$. The uncountable set of axioms assert that for each cut in the rationals (individual rationals are named as $\frac{n}{m}$, although there is no predicate for the set of rationals) there is a unique point in each cut.*

We recall some basic model theoretic notions from [**?**].

{atomicdef}

Definition 5.6. *A structure M is an* atomic *model of a first order theory T if for every finite sequence $\overline{m}$ from M, there is a formula $\varphi(\overline{y})$ depending only on $p = \operatorname{tp}(\overline{m}/\emptyset)$ such that $(\forall \overline{y})\varphi(\overline{y}) \rightarrow \psi(\overline{y})$ for each $\psi(\overline{y}) \in p$.*

The following notation refers to the classes of models that arise in the (Fact 5.3.2) reduction of complete sentences of $L_{\omega_1,\omega}$ to atomic models.

{classnot}

Notation 5.7 (five classes of models)**.** *The class of τ-structures that satisfy*

1. $\mathbf{K}_\varphi$ *is the class of models of the* complete *$L_{\omega_1,\omega}$-sentence φ.*

2. $\mathbf{K}_{T^\varphi}$ *those that satisfy the τ^φ-theory T^φ.*

3. $\mathbf{K}^{at}_{T^\varphi}$ *is those that are atomic models of the τ^φ-theory T^φ.*

---

[32] A variant of this coding is described in Definition 5.11

4. *$\mathbf{K}^{\tau}_{T^{\varphi}}$ is those that are reducts to τ of models in $\mathbf{K}^{at}_{T^{\varphi}}$.*

5. *$\mathbf{K}^{\tau}_{T^{\hat{\varphi}}}$ is the class of σ-structures in Definition 5.11. $T^{\hat{\varphi}}$ is the first order theory of $\mathbf{K}^{\tau}_{T^{\hat{\varphi}}}$.*

The Chang theorem asserts that $\mathbf{K}_{\varphi} = \mathbf{K}^{\tau}_{T^{\varphi}}$. Note that while $\mathbf{K}_{T^{\varphi}}$ has arbitrarily large models, $\mathbf{K}^{at}_{T^{\varphi}}$ may not; although it does if it has models up to the cardinal[33] $\beth_{\omega_1}$.

{ordex}

Example 5.8 (Distinguishing the classes)**.** Start with the structure of the integers with order $M = (\mathbb{Z}, <, 0)$. There are two non-principal types over the empty set for the first order theory of M. $p_{+\infty(x)}$ ($p_{-\infty(x)}$) says x is greater (less) than 0 and infinitely far away. Let φ be the $L_{\omega_1,\omega}$ sentence characterizing this structure. I.e. the axioms for discrete linear orders that omit $p_{\pm\infty}$. T^{φ} is the theory of discrete linear order, with additional symbols $P_{+}(x)$ ($P_{-}(x)$) with axioms saying $P_{+}(x)$ ($P_{-}(x)$) implies $p_{+\infty(x)}$ ($p_{-\infty(x)}$) respectively. T^{φ} has arbitrarily large models. But, the unique atomic model of T^{φ} omits both types. Thus, as asserted in general in Fact 5.3.2, $\mathbf{K}_{\varphi} = \mathbf{K}^{\tau}_{T^{\varphi}}$.

{catexs}

Example 5.9 (Examples of categorical sentences in $L_{\omega_1,\omega}$)**.**

1. *Peano arithmetic: In particular φ includes $(\forall x) \bigvee_{n<\omega} x = S^{n}(0)$ and the quantifier-free diagram of $(N, +, \times, 0, 1, <)$.*

2. *The theory of a single bijective function S that has exactly one cycle of length n for each n and no ω-sequence.*

3. *The examples of Marcus and Knight [**?**, Ex 18.9] of complete $L_{\omega_1,\omega}$-sentences that are categorical (univalent) but the unique (necessarily countable) model N is not homogeneous. In particular, there is no isomorphic substructure of N.*

Any sentence of $L_{\omega_1,\omega}$ whose first order consequences satisfy the hypotheses of Theorem 4.12 has only countable models by that theorem. But these hypotheses are too restrictive. In order to remedy this situation, we turn to the notion of a (generalized) ω-model in Remark 2.15.1 and to a new inferential rule modifying Definition 4.1. This variation will allow examples with no constants in the base language τ (even $\mathrm{dcl}(\emptyset) = \emptyset$). Key points of Definition 5.12 are that the instances of the formula in the G-ω-rule are required to come from the index set N and not from the structure being investigated V, but the relation R uses parameters from V.

Recall that since the 1950's a class is of τ-structures is called *pseudoelementary* if it is the collection of τ reducts of a first order theory in an explansion τ' of τ.

---

[33] This cardinal is defined by induction. $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, for limit δ, $\beth_\delta = \bigcup_{\alpha<\delta} \beth_\alpha$. Morley proved [**?**] there are sentences of $L_{\omega_1,\omega}$ that have models only up to κ for any $\kappa < \beth_{\omega_1}$. But any larger and there are arbitrarily large models.

`{sedef2}`

Definition 5.10. *Let* $\mathbf{K}$ *and* $\mathbf{K}'$ *be (pseudo)-elementary classes of models in the same vocabulary* τ*, but* determined *by theories* T*,* T' *in possibly different logics. We say the classes* $\mathbf{K}$ *and* $\mathbf{K}'$ *are* structurally equivalent*, if they have the same class of models.*

We say 'determined' because in the application $\mathbf{K}'$ is the class of τ-reducts of T'.

τ-reducts are atomic models of T^{φ}, not of T. In particular, if T is first order τ-reduct will be an arbitrary model of T. ???

`{Morcode}`

Definition 5.11. Fix a Scott sentence φ in a vocabulary τ with a countable model A. We work in a two-sorted[34] vocabulary σ (σ depends on τ) with disjoint sorts (N, V) where N contains the set of images of the constants $Const(N) =_{\mathrm{df}} \langle c_{n,i} : i, n < \omega \rangle$ and V consists of a τ^{φ}-structure satisfying the theory T^{φ} constructed from φ by Fact 5.3.2 (Chang). Each τ^{φ}-constant becomes a σ-constant satisfying V.

Adapting [**?**][35], we construct a theory $T^{\hat{\varphi}}$ such that a $G-\omega$-model of $T^{\hat{\varphi}}$ omits each non-principal type over $\emptyset$. In particular, if $B \models T^{\hat{\varphi}}$, the restriction of B to $V(B)$ satisfies T^{φ}. Extend the vocabulary σ of $T^{\hat{\varphi}}$ to include, for each n, an $(n+1)$-ary-relation R^n on $N \times V^n$ in 2-sorted G-ω-logic.

The theory $T^{\hat{\varphi}}$ is given by T^{φ} relativized to V along with axioms saying the $c_{n,i}$ are distinct elements of N; the following axioms on the R^n ensure that the elements of N code all finite τ^{φ}-types over the empty set of the elements of V.

$$(1)\ (\forall v_0)[R^n(c_{n,i}, \overline{v}_0) \leftrightarrow \varphi_i(\overline{v}_0)].$$

where $\varphi_i(\overline{v})$ generates the ith principal n-type (in τ^{φ}) over $\emptyset$ for a given injective enumeration of those types[36].

$$(2)\ (\forall \overline{v}_0)[V(\overline{v}_0) \rightarrow \exists v_1[N(v_1) \wedge R^n(v_1, \overline{v}_0)]$$

We call a model B of $T^{\hat{\varphi}}$ with $Const(N) = \{c_{n,i} : i, n < \omega\}$ denoting the elements of $N(B)$ a $G-\omega$-model.

As we now show, axioms (1) and (2) in Definition 5.11 guarantee that in a model satisfying the $G-\omega$-rule (Definition 5.12), each non-principal τ^{φ}-type over is omitted; thus $V(B)$ is an atomic model of T^{φ}. Observe that each element of N codes a type over the empty set because of axiom (2).

---

[34] Constants and variable will be restricted to specific sorts.

[35] Morley coded a single non-principal type; we code countably many types of arbitrary finite length so we use $n+1$-relations for all n, rather a single binary R. The constants $c_{n,i}$ satisfying $N(x)$ code all T^{φ} principal types over $\emptyset$.

[36] $R^n(c_{n,i}, \overline{d})$ with $\overline{d} \in V$ means: $\overline{d}$ realizes the $n-\tau^{\varphi}$-type over the empty set indexed by $c_{n,i}$.

Definition 5.12 below is a syntactic definition of the rules of inference for the quantifiers in the $G-\omega$-logic. Proposition 5.13 shows that the rules for the universal quantifier uniquely determine the truth-conditions, making the rules in Definition 5.12 categorical (see Notation 2.4.3.b.). As in Section 4, we do not require all constants to be used in every valuation. Thus, our rules are actually rule schema, indexed by specifying a set of constants.

`{ourinfrule2`

Definition 5.12 (Inferential $G-\omega$-rule). *σ is a 2-sorted vocabulary as in Definition 5.11; For any countable set of constants $B \supseteq Const(\sigma)$ and for any $\sigma(B)$-formula $\lambda(x,\overline{v})$ and, in particular, instances $\psi(x,\overline{d})$ of a formula $\psi(x,\overline{v})$ with $\lg(\overline{v})=n$:*

$$G{-}\omega\text{-}rule : \frac{\bigwedge\{\psi(c_{n,i},\overline{d}) : c_{n,i} \in Const(N), n, i < \omega\} \ \ \& \ \ (\forall x)(\forall v_i)(\psi(x,\overline{d}) \to (N(x)\& \bigwedge_{i<n} V(v_i)))}{(\forall x)(N(x) \to \psi(x,\overline{d}))};$$

$$G - \forall E : \frac{(\forall x)\lambda(x,\overline{b})}{\lambda(e,\overline{b}),\ \textit{for each } e \in B}.$$

Note that there are constants in B that are not in N, but they are not instances of the hypotheses of the $G-\omega$-rule. The G-$\exists I$- and G-$\exists E$-rules are parallel to those in Definition 4.1.

`{G-infRmeani`

Proposition 5.13. *(Categoricity of G-ω-rule and G-$\forall E$-rule) The G-ω-rule and the G-$\forall E$-rule for the universal quantifier in the inferential G-ω-logic uniquely determine:*

For any admissible valuation v_B, $v_B((\forall x)(N(x) \to \psi(x,\overline{d}))) = true$ iff $v_B(\psi(c_{n,i})) = true$ for each N-constant $c_{n,i}$, provided that $v_B((\forall x)(\forall v_i)(\psi(x,\overline{d}) \to (N(x)\& \bigwedge_{i<n} V(v_i))) = true$.

Proof. ($\Longrightarrow$) If $v_B((\forall x)(N(x) \to \psi(x,\overline{d}))) = true$, then $v_B(N(c_{n,i}) \to \psi(c_{n,i},\overline{d})) = true$, for each $c_{n,i}$ from $Const(N)$ (by the G-∀E-rule). Thus, $v_B(\psi(c_{n,i},\overline{d})) = true$ for each constant $c_{n,i} \in Const(N)$.

($\Longleftarrow$) If $\psi(x,\overline{d})$ is a σ-formula and $v_B(\psi(c_{n,i},\overline{d})) = true$ for each $c_{n,i} \in Const(N)$ and $v_B((\forall x)(\forall v_i)(\psi(x,\overline{d}) \to (N(x)\& \bigwedge_{i<n} V(v_i))) = true$, then $v_B((\forall x)(N(x) \to \psi(x,\overline{d}))) = true$ (by the G-ω-rule).

□

The argument requires that V realizes only countably many τ^{φ}-types over the empty set; this follows from the Chang's theorem 5.3.2 and Definition 5.11 since each realized τ^{φ} type is principal.

{omegaruleatc

Theorem 5.14. *Fix a complete $L_{\omega_1,\omega}$-sentence φ. ~~categorical~~ complete. With Definition 5.12, if a model of $T^{\hat{\varphi}}$ satisfies the $G-\omega$-rule then its τ^{φ}-reduct is an atomic model of T^{φ}. I.e., each $p \in S(\emptyset)$ realized in V is principal as a τ^{φ}-type. Thus, by Fact 5.3.2, the τ-reducts of models of $T^{\hat{\varphi}} + G - \omega - rule$ are exactly the models of φ. That is, φ and $T^{\hat{\varphi}} + G - \omega$-rule are structurally equivalent.*

Proof. First we note that by the omitting types theorem the countable atomic model A of T^{φ} has an admissible valuation. Now, for an arbitrary $B \models T^{\hat{\varphi}}$, working in τ_B, let v_B be an admissible valuation with range B assigning $c_{n,i}$ for some $n, i < \omega$ to each member of $N(A)$. For sake of contradiction, let B be a model of T^{φ} extending A that is not atomic; thus for some n there is an n-tuple $\overline{d} \in V(B)$ realizing a non-principal τ^{φ}-type p over $\emptyset$. Applying Löwenheim Skolem, choose a countable elementary submodel B' of B that contains $\overline{d}$. By the last line of Definition 5.11 and since the elements of $N(A)$ code all and only the principal types of T^{φ}, it is coded by an element e of $N(B') - N(A)$, i.e. $B' \models R^n(e, \overline{d})$. Since p is non-principal, $B \models \neg R^n(c_{n,i}, \overline{d}) \wedge N(c_{n,i})$ for each standard $c_{n,i}$. Applying the $G-\omega$-rule with the formula $\psi(x, \overline{d})$ as $\neg R^n(v_0, \overline{d})$, $B' \models (\forall v_0)(N(v_0) \rightarrow \neg R^n(v_0, \overline{d}))$. Hence, such a $\overline{d}$ cannot exist and so $\overline{d}$ realizes a principal type in $V(B')$ so in $V(B)$. From the contradiction we conclude $V(B)$ is atomic.

Chang's theorem shows $\mathbf{K}_{\varphi} \subseteq \mathbf{K}^{\tau}_{T^{\varphi}}$ (Notation 5.7); the converse is immediate in $G-\omega$-logic since every model of the $G-\omega$-rule is atomic. So φ and $T^{\hat{\varphi}} + G - \omega$-rule are structurally equivalent. □

Note that this shows that if T is a first order theory categorical in $\aleph_0$, then the associated T^{φ} will satisfy the G-ω-rule; there are no non-principal types to omit. This applies even to the much studied totally categorical theories (there exists a unique up to isomorphism model in each cardinality). But none of these theories are categorical in the sense of his paper. We describe the well-known characterization of those structures that are categorical (in $L_{\omega_1,\omega}$).

{gendef}

Definition 5.15. *A structure M is said to be generative if it is isomorphic to a proper substructure (equivalently extension) of itself.*

Immediately from the structural equivalence (Definition 5.10, Theorem 5.14):

{coupdegras}

Corollary 5.16. *φ is categorical if and only if $T^{\hat{\varphi}}$ is categorical in $G-\omega$-logic if and only if the countable model of φ is non-generative.*

Proof. Only the second equivalence needs argument. By Theorem 5.14 both M and M_1 are atomic (and without loss countable). So $M_1 \approx M$. But then following [**?**] we can build an uncountable chain $\langle M_\alpha : \alpha < \omega_1$ of pairwise isomorphic elementary extensions $\langle M_\alpha : \alpha < \omega_1\rangle$. So M_{ω_1} is uncountable and violates categoricity. ☐

Any complete sentence of $L_{\omega_1,\omega}$ is $\aleph_0$-categorical. But it will be categorical only if the countable model is not generative. In particular, the standard model of arithmetic is atomic since for any finite sequence $\overline{a} \in N$, $\bigwedge_{i<n} v_i = a_i \rightarrow \mathrm{qtp}_\emptyset(a_0, \ldots a_{n-1})$; the model is in the definable closure of the empty set.

The structure $(\mathbb{Z}, +, 0)$, the free abelian group on one generator, is certainly canonical and so standard in the sense of Remark 4.4. Since there is only one constant term, it cannot be proved categorical by the methods of Section 4. But is easily axiomatized in $L_{\omega_1,\omega}$ as a torsion free Abelian group generated by a single element x satisfying $\bigwedge_{w_1,w_2}(w_1(x) = w_2(x) \rightarrow (\forall y)(w_1(y) = w_2(y))$ where w_1, w_2 range over all words with a single argument.

Example 5.9.2 shows that unlike Peano Arithmetic, there are simple examples of $L_{\omega_1,\omega}$ categorical structures that are not in the definable closure of a finite set.

6 Structures, Models, and the Modelists Doxological Challenge

{Dochall}

We introduced in Definition 2.1 a notion of structure that is identical *mutatis mutandis* to the one defined by [**?**, Definition 1.2]. We re-emphasize that this notion of structure is methodologically distinct from the notion of model of a given mathematical theory formalized in a certain system of logic. The notion of structure is well-defined with respect to a specific signature, but it is not dependent on any formal system of logic. As [**?**, p. 8] suggest, writing ‘as usual in set theory’, this notion of structure is given in (naive?) set theory. Our aim in proving categoricity above for *theories* was to choose (or given) a vocabulary τ and a τ-structure, M, find a logic $\mathcal{L}$ and an $\mathcal{L}(\tau)$-theory T whose unique model up to isomorphism is M. We aimed to find substantially weaker choices of $\mathcal{L}$ that can give categorical of appropriate countable structures.

Although they are situated in different frameworks (Cf. §6.2), the notions of *structure* and *model of a theory* are not always clearly and coherently kept distinct, and this generates philosophical misunderstandings. For instance, even though they define the notion of τ-structure both informally just before ([**?**, Definition 1.2]) and then semiformally in set theory exactly as in a model theory text, [**?**] formulate a problem (*The Modelist’s Doxological Challenge*) that obscures the difference between the informal notion of structure and the formal notion of model, and likewise between the different frameworks to which these two notions belong. We introduce the context in which they formulate this challenge and provide an answer to the challenge. For the clarification, we specify 'frameworks' in §6.2 to distinguish between pre-formalized mathematics, the identification and uniqueness of structures – on the one side and formalized mathematical theories in a certain logic, uniqueness of models up to isomorphism and categoricity of formal theories – on the other side.

6.1 The Modelists and the Doxological Challenge

{moddc}

[**?**, p. 143-4] introduced the notion of modelism, as an attitude towards model-theory, with this claim as to how 'structure' should be explicated:

> *The modelists manifesto.* Mathematicians frequently engage in structure-talk, and model theory provides precise tools for explicating the notion of *structure* they invoke. Specifically, mathematical structure should be understood in terms of *isomorphism*, in the model theorist's sense.

This commandment brings joy to the ears of a model-theorist, especially since both the notions of structure and isomorphism ([**?**, Definition 2.2]) are defined in naive set theory exactly as in a model-theory texts. However, it is then rather surprising to read:

> *The Modelist's Doxological Challenge.* How can we pick out particular isomorphism types? [**?**, p 145]

The challenge is raised for both kinds of modelists that [**?**] consider, i.e. objects modelists and concepts modelists. Each of these modelists accepts that mathematicians deal with structures[37], but the first claims that the talk about structure is about proper (abstract)

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[37] [**?**] identifies five forms of structuralism: methodological, set-theoretic, ante rem, modal, and logical. This taxonomy is more fine-grained and partly orthogonal to [**?**]'s classification. Methodological structuralism, according to [**?**, p.371], extends beyond algebra to arithmetic, geometry, and other domains, and it emphasizes a formal, axiomatic method of presentation of mathematics rather than metaphysical commitments. We can think of it as background for the distinction between set-theoretical and logical structuralism. Reck's set-theoretic structuralism mirrors what we have called the model-theoretic approaches, but he adds certain metaphysical and semantic theses. The metaphysical thesis asserts that mathematics concerns only sets, while the semantic thesis identifies structures such as the natural or real numbers with sets, understood as any member of the isomorphism type. Model theorists, however, remain metaphysically neutral. More significant is the distinction between set-theoretic and logical structuralism. Dedekind's account exemplifies the latter: he takes N as any representative of the isomorphism type and then 'creates' the natural numbers, in a *logical* manner. The *constitutive properties* of these numbers consist only in those imposed by successor. While other members of the isomorphism type (e.g. the von Neumann ordinals) have other constitutive properties evident in the set theoretic construction of the structure.

objects, while the latter maintain that this talk is about mathematical concepts. [**?**] argue that each modelist fails to explain i) how we refer to *bona fide* objects or ii) how we acquire and use concepts that are fine-grained as the isomorphism types, respectively.

We take the doxological challenge to be only partly meaningful, since we consider it to be based on the confusion between internal and external questions that [**?**] warned us about long ago. Briefly, once we set up a certain theoretical framework, the question either is internal to that framework, and thus gets a precise answer within the framework, or it is external to it, and it is thus meaningless, i.e. it lacks cognitive content. We expand this discussion of i) in Section § 6.2. [**?**], based on evidence from Cognitive Science, advance convincing arguments towards answering ii) which we expand in Section § 6.3.

Writing $M \approx N$ to indicate that there is a bijection between M and N that preserves (in both directions) the atomic $\mathcal{L}$-formulas, the standard (model theorists) answer to the so called *doxological challenge* is the following:

{answer}

Procedure 6.1 (Identifying a isomorphism type)**.** *a) Any particular structure is defined/described by its construction [**?**, p 9]. E.g., the finite Von Neuman and Zermelo ordinals are distinct but isomorphic structures (See Definition 6.2 for such a construction.). b) An isomorphism type of a structure M is the collection*[38] *of all N with $N \approx M$.*

We argue here that the resources necessary to prove that a structure exists suffice to pick out its isomorphism type. We do not find answer b) of Procedure 6.1 in [**?**]. They [**?**, p. 151-2] first distinguish a moderate objects modelist from an immoderate modelist on the criterion that the first denies a special faculty of 'mathematical intuition'. Without the aid of such intuition, they suggest that a moderate modelist must reject 'reference by acquaintance' and 'must make do with reference by description', i.e. they must answer the doxological challenge 'by laying down *some*[39] formal theory'. Button and Walsh take this second path and interpret the formal theory as a categorical theory and conclude that it cannot be first order by Löwenheim-Skolem considerations. No argument is given for the shift in meaning from *structures* to *models of a formal theory in a specific logic*, but in the context of page 152, it can be seen as an additional condition that must be met by a moderate-modelist.

There seems to be no other option in [**?**] aside from invoking intuition or laying down a formal theory. However, Procedure 6.1 provides such an option: a construction of the structure in set theory without making meta-theoretical investigations (ZFC, though much weaker ones suffice), not a theory in the same vocabulary as the structure whose models

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[38] Technically, a proper class.

[39] Our emphasis.

include the structure (and only ones that share its isomorphism type). As we already mentioned, categoricity is a different notion altogether. It is a property, not of a structure or an isomorphism type, but of a theory T in a specific logic: such that there is only one isomorphism type of models of T. It is true that one could pick out an isomorphism type by finding a logic, a theory, and a structure $M \models T$ such that T is categorical. But we are visiting Robin Hood's barn and the distracting trip around is more complex than the straightforward answer provided by Procedure 6.1 in naive set theory. However, the modelist has a more difficult task that we clarify with the discussion of frameworks in Section 6.2.

One may say that Procedure 6.1 is a mathematician's answer, one who overlooked a key word, the '*modelist's* doxological challenge', i.e. the challenge is for those who want to *explicate* mathematical structures by model theoretic tools. This recognition also clarifies the word 'doxological', which refers to the fact that the explication is not intended as an *ontological* hypothesis ([**?**, p. 38]). This means that the challenge is not an epistemological question ([**?**, pp. 145, 149]), but one that requires an answer for how we form mathematical beliefs[40]. Can a mathematician be part of the modelists' community? As [**?**, pp. 143, 164] acknowledge, modelism is 'an attitude towards model-theory' and 'no one ever *called* themselves a 'moderate modelist''. Thus, a practicing mathematician who has in his moments of reflection on his work this sort of philosophical attitude towards model-theory falls willy-nilly under this modelist label. Consequently, the rejection of intuition does not imply the need of answering the challenge by means of a formal theory in a certain logic. A particular isomorphism type can be *picked out* through its construction in set theory.

One might object to Procedure 6.1: 'Set theory has raised its ugly head.' Answer: It already has[41]. The formal definition of a structure requires a weak set theory (neither the axioms of replacement nor choice). Here is a canonical example.

{omdef}

Definition 6.2. *[The structure $N = (\omega, S)$] Let τ contain a single unary function S. Let S_N denote the function $\{\}$ mapping x to $x \cup \{x\}$* [42].

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[40] [**?**, p. 145] emphasize that model theory is not of much help in answering epistemological questions concerning mathematical knowledge, but it is rather curious that someone would expect an answer from a modelist concerning the way we form mathematical beliefs. The answer of the modelist must certainly go beyond the resources of model theory.

[41] This is acknowledged by Shapiro, as quoted at [**?**, p 181] in describing the 'holistic' attitude to second-order logic.

[42] This is the Zermelo ordinal version, for which careful use of induction eliminates the possibility of finite cycles; see, e.g. [**?**, p. 51]. The Von Neuman ordinals take S_N as the function $\{\}$ mapping x to $\{x\}$. For this, one needs the axiom of regularity: $x \neq \emptyset$ implies $\exists y \in x(x \cap y = \emptyset)$ to eliminate cycles. Being more careful, on could invoke regularity again as in [**?**, III.§5] to observe that if there were two structures satisfying the definition, the function f fixing the $\emptyset$ and with $f(x) = x$ is the identity between them. ZF

It should be clear by now that the original doxological challenge is a) distinct from any notion of categoricity and b) it is easily met by a (set-theoretic/logical ([**?**])) modelist. In the next section, after we clarify the difficulties that confront the modelist in referring to a structure by specifying the different frameworks involved, we show she can give a stronger answer to the Doxological Challenge via categoricity. While categoricity is ‘out of reach for the first order theories we most care about’ [**?**, 145] (those with finitary inference rules), moving to second order logic is an unnecessary step since using the weaker $I - \omega$ and $G - \omega$-logics from Sections 4 and 5 one regains the goal of characterizing canonical structures by categoricity.

6.2 The Doxological Challenge and Carnap’s Linguistic Frameworks

{carling}

Carnap’s ([**?**, §2]) account of *linguistic frameworks* makes a fundamental distinction between two kinds of existence questions that one can formulate. To speak meaningfully about a new class of entities —numbers, properties, or propositions— one must first introduce a corresponding linguistic framework, that is, a system of expressions and rules governing the use of these expressions. Within such a framework, *internal questions* concern the existence or identity of particular entities as determined by the framework’s rules. These are legitimate theoretical questions, answerable by logical or empirical methods depending on the nature of the framework. By contrast, *external questions*—those asking whether the entire system of entities is ‘real’ or ‘exists’—are, for Carnap, not genuine theoretical questions but pragmatic choices about adopting a mode of speech or a theoretical framework as such. They lack cognitive content and are therefore pseudo-questions.

Carnap illustrates this distinction with examples of frameworks for: things, natural numbers, propositions, thing properties, the integers and rationals, the real numbers, and the spatio-temporal coordinate system for physics. An internal statement such as ‘these two pieces of paper have at least one color in common’ is an empirical claim within the framework of thing properties. By contrast, the metaphysical question whether *properties themselves* are real exemplifies an external question and, in Carnap’s view, it has no factual or cognitive significance. Similarly, within the physical framework, empirical questions about spatial or temporal relations are internal, while questions about the ‘reality’ of space and time themselves are external pseudo-problems. In the framework for natural numbers one may likewise ask an internal question as ‘Is there a prime number greater than a

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(Zermelo-Fraenkel) extends Zermelo set theory by adding regularity and replacement. ω is the set containing the $\emptyset$, for each $x \in \omega$, $\{x\} \in \omega$ and satisfying induction:

$$(\forall y \subseteq \omega)\Big((\emptyset \in y \land \forall z \in y(S(z) \in y)) \rightarrow \forall u \in \omega(u \in y)\Big).$$

The axiom of choice is required to show Dedekind finite is equivalent to ‘embeddable in ω’. Zermelo set theory has many equivalents; see [**?**, **?**] or the summary in [**?**, §3.2].

hundred?' and this will be precisely answered by analysis based on the rules for the arithmetical expressions. However, if someone asks 'Are there numbers?', as a question concerning the *ontological status of numbers as independent entities that have a reality of their own*, then this question simply lacks cognitive content for the natural numbers framework ([**?**, §2]). Carnap's discussion of this last example illustrates very well the peculiarities of the external questions:

> "Therefore nobody who meant the question "Are there numbers?" in the internal sense would either assert or even seriously consider a negative answer. This makes it plausible to assume that those philosophers who treat the question of the existence of numbers as a serious philosophical problem and offer lengthy arguments on either side, do not have in mind the internal question. And indeed, if we were to ask them: "Do you mean the question as to whether the framework of numbers, if we were to accept it, would be found to be empty or not?" they would probably reply: "Not at all; we mean a question prior to the acceptance of the new framework." They might try to explain what they mean by saying that it is a question of the ontological status of numbers; the question whether or not numbers have a certain metaphysical characteristic called reality (but a kind of ideal reality, different from the material reality of the thing world) or subsistence or status of "independent entities." Unfortunately, these philosophers have so far not given a formulation of their question in terms of the common scientific language. Therefore our judgment must be that they have not succeeded in giving to the external question and to the possible answers any cognitive content. Unless and until they supply a clear cognitive interpretation, we are justified in our suspicion that their question is a pseudo-question, that is, one disguised in the form of a theoretical question while in fact it is a non-theoretical; in the present case it is the practical problem whether or not to incorporate into the language the new linguistic forms which constitute the framework of numbers.'

We consider the doxological challenge raised by Button and Walsh in relation to four *nested frameworks*, i.e. contexts that contain expressions for different categories of objects and rules that govern their use. Take the framework of 'natural number arithmetic. Analogously to Carnap's treatment of the rationals, internal questions here are those expressible in the first-order theory of $(N, 0, +, \times)$. Certainly, questions asked/answered in the metatheory concerning the decidability, completeness, or categoricity of various theories such as Q, PA, or PA^2 that are satisfied by the structure N are, from the arithmetical perspective, external; yet they are internal to any set theory used as a metatheory to formally construct $(N, 0, 1, +, \times)$. The assertion that the continuum hypothesis is true, for example, can be formulated for PA^2 but not for weaker arithmetical systems.

Thus, our basic objection to the Button-Walsh doxological challenge 'How can we pick out particular isomorphism types?' is exactly that given by Carnap for questions of the 'reality of properties or numbers'. It is devoid of *cognitive content*, if frameworks are ignored. We reconstruct the study of mathematics as taking place in three different frameworks and address the Challenge in relation to them in the fourth framework, the one in which the philosophical reflection takes place.

[**?**, p. 152] make a distinction between 'reference by acquaintance' and 'reference by description'. The first one is supposed to be obtained with the aid of mathematical intuition, but since the moderate modelist denies a special faculty of mathematical intuition, she aims to 'pick out' reference by description.

We find the use of the word 'reference' problematic in the context of mathematics and we consider that its use generates misunderstandings since people tend to use this expression as standing for a sort of 'object'. For us, 'mathematical objects' are concepts of different levels of complexity (depending on the framework to which they belong) that are constructed and grasped by the human mind. In each of the three frameworks below people who work in those frameworks are acquainted with and can describe the mathematical concepts from that framework.

1. *Structure framework.* This is the informal (natural language) framework in which children -and likewise practicing mathematicians- consider intuitive or naive mathematical concepts, such as numerals, functions or operations that children work with. Children grasp these concepts in the process of learning (and so are acquainted with them) and can describe them by using the means of natural language. In particular, there is a naive notion of the natural numbers endowed functions $(+, \times)$ and relations $(<)$. We refer to concepts at this level as *naive*.

2. *Theoretical framework of naive set theory.* This framework is employed in ordinary mathematical practice by mathematicians when they construct and compare various structures; in particular when they identify an isomorphism type.[43] Mathematicians

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[43] The number theorists Barry Mazur [**?**, p.222] describes very nicely the work of the practicing mathematician in this framework: 'To define the mathematical objects we intend to study, we often—perhaps always—first make it understood, more often implicitly than explicitly, how we intend these objects to be presented to us, thereby delineating a kind of *super-object*; that is, a species of mathematical objects garnished with a repertoire of *modes of presentation*. Only once this is done do we try to erase the scaffolding of the presentation, to say when two of these super-objects—possibly presented to us in wildly different ways— are to be considered *equal*. In this oblique way, the objects that we truly want enter the scene only defined as *equivalence classes of explicitly presented objects*. That is, as specifically presented objects with

define in naive set theory [44] both individual structures (an ω-sequence) and classes of structures in naive set theory (the class of semigroups, structures $(A, +)$ such for all $a, b, c \in A$, $(a + b) + c = a + (b + c)$). Note there is no formal language present. The first corresponds to what [**?**] calls a non-algebraic theory (i.e. univocal theory) and the second to an algebraic theory. We avoid 'theory' in this context, reserving it for formal theories. We suggest that mathematicians have *acquaintance by definition/construction* to structures defined in this manner. Following [**?**, p. 9] we refer to the concepts used in this framework as *informal*. Some mathematicians might prefer 'formalizable' as they are consciously working loosely in ZFC.

3. *Metatheoretical framework of formalized theories in various logics*. This framework contains formal mathematical concepts defined by the mathematical logician. She is acquainted with them by defining a formal axiomatization of the informal concepts from Framework 2 and can describe some of those concepts by specifying a formal logic, a vocabulary and a categorical set of axioms. More specifically, those who work in this framework are directly acquainted with the *formal concept* and can refer by a *formal description* [45] to the concept from Framework 2 that is categorically formalized.

4. *Philosophical framework.* This is the framework in which one can reflect on any philosophical features arising in the first three. Sections §6 and §7 of this paper take place in this philosophical framework. For instance, in this framework we can also state the following principle about the previous three frameworks:

   Meta-framework principle: If a question is properly asked and answered in framework n, then it remains so in higher frameworks.

The Doxological Challenge gets a clear answer once we set up a precise and adequate framework, and is meaningless otherwise. Consider Frameworks 1 and 2. As children learn the informal nested framework of things, properties, and natural numbers and understand that a structure contains functions and relations, then they can talk about that particular structure, i.e. they use the 'structure framework'. In this framework children refer to natural numbers by simply using the language and they can answer simple (internal) questions. It is meaningless to ask them whether these numbers 'really' exist. The mathematician can answer the same internal questions, but he can also describe what his

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the specific presentation ignored, in the spirit of "ham and eggs, but hold the ham."' In the paper he suggests category theory rather set theory as the most suitable organization or scaffold (See [**?**] for 'ignoring').

[44] We use the term 'naive set theory' in the sense of [**?**], or the mathematician who is unfamiliar with any details of axiomatic set theory.

[45] As [**?**, p 152] require of a moderate or conceptual modelist.

arithmetical language refers to once he has adopted the theoretical framework of naive set theory. In the latter framework of set theory the mathematician can simply refer to a particular structure due to the fact that he actually constructs it. This is the reason that the Doxological Challenge gets a clear and precise answer by Procedure 6.1 in Framework 2.

However the modelist has a more difficult task, which requires Framework 3 (and Framework 4 for the discussion). If someone wants to formally represent a certain structure, she has to know (i.e. have a concept of) what she wants to formally represent. The model theory approach provides a common ground for all signatures and many (all?) logics and even for the formalism-free approaches to model theory. The categoricity approach requires a different approach and possibly different logic to describing the isomorphism type for each structure.

The move to Framework 3 may both sharpen and extend the mathematical conception of a rougher notion. For example, Euclid, with axioms that are informal in our sense imagines a line with some irrational numbers described by specific algorithms; Dedekind's 2nd order logic approach imagines a line with irrational numbers described by arbitrary 'cuts'. Both are working in rudimentary versions of Framework 3. While Dedekind fits well[46] in Framework 2, by stressing definitions and axioms Euclid presages Framework 3. Framework 3 has allowed model theorists to make deep contributions to analysis, number theory, differential equation, combinatorics etc. Sections §4 and §5 of this paper are conducted in this framework.

The categoricity, decidability, completeness, et al. questions are internal questions in Framework 3, where formalized theories in various logics are analyzed, but these questions are meaningless (external) questions outside this metatheoretical framework (for instance in Frameworks 1 and 2). In particular, the question about uniqueness of *models* up to isomorphism and categoricity of formal theories are internal questions in metatheoretical frameworks of formalized theories in a certain logic, but external to Framework 2. The questions about the identification and uniqueness of *isomorphism types* are internal questions in both Frameworks 2 and 3. The difference is that in Framework 2 the mathematician identifies the isomorphism type by constructing/defining it, while in Framework 3 the mathematical logician identifies it by choosing an adequate logic to obtain a categorical theory that it satisfies.[47] This step from Framework 2 to 3 introduces formalized mathematics and is the answer by the moderate modelist to the doxological challenge. It provides *reference by (formal) description* in the sense of [**?**]. We discuss in Section 7 for which logics the construction of the logic invokes the concepts aimed to be categorically

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[46] Certainly, if we adopt Quine's view of second order logic.

[47] We note in passing that when someone provides derivations in a formal theory in a given logic, he does not need the whole resources of Framework 3. This is a more narrow framework that comprises the axioms of the theory and the deductive apparatus of its underlying logic.

formalized.

In discussing the doxoloxical challenge, our emphasis is on the distinction between the first two frameworks: the Challenge is external to the structure framework and easily answered in the framework of naive set theory. Categoricity of theories is simply irrelevant in these two frameworks since it presupposes a theory formalized in a certain system of logic.

In Framework 3, one can consider the question of categoricity of theories in a particular logic. The question is external to the logic itself, where one works in the object language of that logic; it is addressed and answered in the metatheoretical framework of set theory. Likewise, in this framework one can compare the resources needed and the results provable or at least stateable in the logic. But questions motivating such comparisons arise in Framework 4.

6.3 Cognitive Modelism

{cogmodelism}

Our own philosophical view is closer to what [**?**] call *concepts modelism* since we see classical mathematics as a complex process of constructing and developing a certain kind of concepts. The claim in this sentence is not intended to be an ontological one, namely, when we say 'structure' in this paper we actually have in mind something that has a conceptual nature, not an object that inhabits Plato's realm. We refer to concepts defined and grasped by human mind, whose linguistic expression[48] is a constitutive component. We do not make the further step in postulating a realm of objects that correspond to these concepts and we see this as a problematic step that objects modelists make in accordance with their *metaphysical credo*. The fact that certain mathematical structures defined by mathematicians are suggested by configurations found in the physical world or that some mathematical concepts are sometimes found to be partly instantiated in the physical world does not undermine the conceptual nature of the mathematical structures. Mathematical concepts are likewise important for objects modelists since, for instance, the assertion that there actually are ω-sequences presupposes that we possess the concept of ω-sequence.

[**?**] use terms such as 'natural number structure' to designate a particular isomorphism type. However, this is meaningful just if there is a concept of natural number structure. [**?**, p. 150] discuss how humans acquire, express, share such concepts as redness and write 'In the case of redness, though, we can point to both red things and also to organs which are good at detecting red things. In the case of mathematical concepts, such as ω-*sequence*, it is doubtful that there is anything similar to point to.' We adopt the name, *cognitive*

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[48] For practical purposes, we shall not defend here a systematic philosophical conception on concepts. For a general discussion on concepts see [**?**] and their §4 for the relation between language and concepts.

modelism given by [**?**], for their description of 'what to point to' while refining the notion using the frameworks established in Section 6.2.

The quote above from [**?**] oversimplifies the development of the concept 'red'. A long time ago people identified certain properties of objects as colors and others as shapes. Much later[49], a particular group of people decided a certain family of colors that were not too different should be called 'red'. Of course there are disagreements over whether a particular scarf is red or scarlet. Eventually, an international agency decides on a certain family of frequencies. Mathematical concepts evolve in the same way.

[**?**] give an account of how children acquire the concept of an ω-sequence. Of course, the specific organ(s) must include the brain. But rather than merely asking, 'What color is the ball?', the study of concept acquisition requires much more complicated analysis. [**?**] partially base their account on the work of such scientists as [**?**] analysis of children's development of number concepts.[50] The book, [**?**], argues extensively with wide ranging references to cognitive science: 'the features of the brain that allow us to do mathematics are the very same features that enable us to use language – to speak to others and understand what they say'.

While the early steps of this development is known through observations of babies, the process continues through school. But even more, it continues through history. Although Euclid was well-aware of ratio and proportion, he admitted neither the unit nor rationals as numbers [**?**, §2.1,§3.1]. Today, we maintain the second exclusion in our notion of 'natural' number, but not the first. In a non-empirical direction, [**?**] analyzes how the mind moves from sensory perception to abstract concepts.

There are other ways of grasping a structure than having a picture in mind of its quantifier free diagram (Notation 2.4.2), which works for ω-sequence. Grasping the structure of natural number arithmetic by 'visualizing the diagram' is problematic. One only knows the addition and multiplication by rules, there are sentences of arbitrarily high quantifier complexity, and easily stated but difficult to picture let alone prove such as Fermat and twin primes.

The practice of mathematics illustrates very well the fact that mathematical concepts rarely have a rigorously defined content from the very start. The process of *concept formation* is dynamic and a categorical formal concept is an advanced stage in the process of making a mathematical concept precise. It is thus natural to invoke some preformalized/informal mathematical concepts in laying down a (categorical) formal theory in a certain logic. Even in formulating the semantics of first-order logic we use notions as 'interpretation', which relies on a previous grasp of the notion of 'function'.

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[49] Likely in the last 2000 years; an internet glance at 'red in Indo-European' showed a commonality - the first letter was usually 'r'.

[50] In addition to the bibliography in [**?**] consult such authors as [**?**, **?**].

There are, however, in special cases other approaches to categoricity in framework 2 beyond procedure 6.1. Suppose one has a class of algebras $\mathbf{K}$ that is closed under the operations of homomorphism, subalgebra, and direct product. We say an algebra $A \in \mathbf{K}$ is free on κ generators if there is a homomorphism from A onto B for every $< \kappa$ generated algebra B. There is a unique free algebra on κ for each κ [**?**, Chapter 4].

Thus, returning to arithmetic, on our view there is a concept of natural number structure[51]. This concept is partly grasped by those who work in Framework 1 and fully grasped by those who work in Framework 2. The goal of categoricity is achieved by choosing a vocabulary, a theory and a logic in that vocabulary, specified by its rules of inference and axioms, such that all models of those rules are isomorphic. We have provided two such logics and appropriate rules of inference and axioms. Certainly, those who work in Framework 3 also grasp a concept of natural number structure and simply aim to formally capture it by means of a categorical theory.

[**?**] argue that the standard natural numbers are to be found in the concepts that cognitive agents acquire in a learning processes. In particular, they reconstruct the cognitive process as modeled by a function between numerals and suitable collections corresponding to them that is generalized by induction to the entire number sequence. This rational reconstruction of the learning process concludes that cognitive agents do grasp the concept of an omega sequence. We are sympathetic with this approach and we take the evidence provided by Cognitive Science as a very reasonable justification for the idea that in the learning process children become acquainted with *the intuitive* notion of an ω-sequence. This idea in turn justifies the intelligibility of the use of the ω-rule in our approach since grasping its premises will be no mystery even for children, who will quickly understand its premises as asserting that each object in the sequence they learned has a certain property.[52]

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[51] Actually, there are several. In a footnote to [**?**, §2], but not in the book, Button and Walsh are clear to distinguish ω-sequence $\langle 0, S\rangle$ from arithmetic $\langle 0, +, \times\rangle$. That is crucial, as it is only with some sort of induction axiom that one can define the arithmetic operations on an ω-sequence. Both concepts appear in the first framework and are grasped by most adults, though perhaps not differentiated. {fnamb}

[52] [**?**, Ch. 10] also proposed an account based on the classical generalized ω-rule for obtaining the categoricity of PA. The categoricity of PA is obtained by taking the ω-rule as an *open-ended* rule schema (i.e. a rule that remains sound in extensions of the initial language/vocabulary). A non-standard model is blocked by considering an instance of the rule with the one-place predicate '$SM(x)$' that means 'x is a standard number of model M'. The conclusion of this instance of the rule (i.e. $\forall(x)(N(x) \rightarrow SM(x))$) will, naturally, be false in a non-standard model M and, consequently, the model will not be admissible. From a mathematical point of view, we don't find intelligible the use of this predicate in an instance of the rule as applied in arithmetic. It may be attractive as a philosophical argument (in Framework 4), but not as means for categorically formalizing PA in Framework 3. In particular, since categoricity is a property of theories formalized in a certain logic, it must be achieved by mathematical means in Framework 3, i.e. *there is no philosophical substitute for mathematics* (see also [**?**, §7] for discussion). On the philosophical side, Warren's argument for the possibility of following the ω-rule is interesting and we are sympathetic to

This may also explain Tarski's natural attitude (see Section 3 above) of taking the classical ω-rule as *intuitively* valid.

Unlike the rules of inference, the concepts required for axiomatization depend on the topic and lay out in the vocabulary the fundamental notions being axiomatized. This explanation is lost in second order logic as quantifying over concepts allows the formal definition of virtually anything conceivable.

As elaborated in [**?**], the vocabulary is chosen based on the informal concept held by the formalizer. Two informal concepts are often conflated as 'natural numbers', but properly distinguished in [**?**, §2]: ω-sequence and arithmetic. The conflation occurs because of the immense power of second order definition [**?**, §5.B] or even first order induction schema. We consider first order axioms for each (Theorem 4.9 and Corollary 4.11) with specified vocabulary for ω-sequence as an $(S, 0)$ structure and for arithmetic $(S, 0, <, +, \times)$. With our (non-circular) $I - \omega$-rules we establish categoricity of the theories (Proposition 4.2, 5.13).

However, the simplest first order axiomatization of an ω-sequence is to require that S is 1-1 and onto every element except 0 ([**?**, Definition 1.9: Q1-Q3]) and require that there are no finite cycles. This gives an first order axiomatizable $\aleph_1$ (and so uncountably)-categorical structure whose prime model is (ω, S). The structure is even strongly minimal (every definable subset is finite or cofinite). Such definable sets are the building blocks of theories categorical in uncountable cardinalities. This is the 'trivial' case of the Zilber trichotomy (below). That is, far from being a wild theory such as arithmetic, the unique structure given by the $I - \omega$-rule and these axioms is an iconically simple decidable theory. On the other hand, adding the induction schema (in particular, eliminating finite cycles) of [**?**, Definition 1.9] gives full first order Peano Arithmetic.

{ZilHru}

Remark 6.3 (Three examples of mathematical concept formation)**.** The concept of countability[53] has a long history. Aristotle distinguished between 'potential' and 'completed infinity' and rejected the latter. The notion of various 'orders of infinity' appears at least by the 12th century writing of Grosseteste [**?**, p 134]. But the modern fully worked out concept of comparing infinities arrives only with [**?**].[54]

[**?**], argues that a structure/isomorphism type can be known 'more directly than as the structure of all models' of a certain categorical theory. He lays out how a cognitive

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it (see [**?**] for a recent critical discussion). We agree with [**?**] that 'it is clearly not possible for the users of a language systematically to follow a non-effective rule in practice', and that these rules 'have a place, and perhaps an important one, in theoretical syntax.', i.e. in Framework 3.

[53] That is, able to be counted. So mathematically, it means finite or of cardinality $\aleph_0$.

[54] The relation between natural numbers and the operation of 'counting' has been recently discussed by [**?**], who argues that the meanings of the numerals are essentially tied to counting. We agree with this idea as an explanation of what happens in Framework 1, when children learn numerals, but we see no conflict between this idea and the approaches conducted in Frameworks 2 and 3.

agent can visualize finite structures, abstract to ‘templates’ and then organize sequences of templates to obtain knowledge of a structure, as that of natural numbers. This knowledge has an experiential constituent and it is thus different from ‘knowledge by a description of the form *the structure of such-&-such axioms’*[**?**, p 57], which requires knowing that the formally axiomatized theory is categorical.

Here is a modern example of developing the concept of a specific structure. In proving in the 1980’s that there is no finitely axiomatizable first order theory that is categorical in both $\aleph_0$ and $\aleph_1$, Zilber introduced a classification of strongly minimal theories. Strongly minimal theories are the simplest type of theory that is categorical in all uncountable powers. A strongly minimal theory admits a combinatorial geometry induced by the notion of algebraic closure ($a \in \mathrm{acl}(B)$ if for some $\varphi(x, \overline{b})$ with $\overline{b} \in B$, $\varphi(a, \overline{b})$ and $\varphi(x, \overline{b})$ has only finitely many solutions). Roughly speaking, the geometry is discrete if $a \in \mathrm{acl}(B)$ implies $a \in \mathrm{acl}(b)$ for a single element of B and locally modular if the closure relation behaves like a vector space. Otherwise the closure is non-locally modular. Zilber conjectured that every non-locally modular geometry of a strongly minimal set behaves like a field.

In the early 1990’s Hrushovski refuted this conjecture by defining a class of finite structures with strong substructure relation[55]. The direct limit under $\leq$ of the members of the class is a is a countable structure. It is strongly minimal, not $\aleph_0$-categoical, not locally modular and not field-like. Model theorists grasp this structure. Hrushovski used a similar technique to refute the conjecture that every stable $\aleph_0$ categorical theory is superstable. These methods have been used in hundreds of papers since. These papers adapt the construction to find theories in many classes of the classification hierarchy. There is no need for being able to visualize the structure, although some extremely geometric examples were constructed with a partial visualization in mind. The moral of the story is that an explicit description of the structure is not essential.

It is by no means clear that a Hrushovki theory has an algebraically prime model that would make it categorical in $I - \omega$-logic. However, the Scott sentence of the saturated countable model in a vocabulary with constants naming a basis for the geometry, is non-generative and so it is categorical in $G - \omega$-logic.[56]

7 Against the Circularity Argument

{AgCA}

In this section we address what we call the ‘circularity argument’ of Button and Walsh. We work here in Framework 4, since we are discussing philosophical issues about the Frame-

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[55] The Fraïssé limit (a 1950’s construction) moves from a uniformly locally finite class of finite structures to a countable $\aleph_0$-categorical structure.

[56] The second order theory of an infinite set of constants is not categorical. The first paragraph of [**?**] begins with a short argument to this effect by Shelah.

work 3 analysis of formal theories. They [**?**, p.164] argue that the appeal to a categoricity theorem for explaining how we grasp 'by formal description' certain mathematical concepts is useless since for proving it one must semantically set forth a logic that 'will invoke precisely the kind of mathematical concepts that were at issue in the first place, and which they were hoping to secure by appeal to a categoricity theorem'.

This circularity argument is also described by [**?**, p.151] as a reiteration of the Doxological Challenge at the level of selecting a logic because the logics stronger than first order logic 'must be characterized using mathematical concepts'. They don't specify what concepts are involved. Clearly finite, countable, and the inductive definition of formulas and satisfaction are included as they are used to formulate the syntax of first order logic.

We observe in this section that our inferential approach (§ 4 and § 5) requires only the notions of finite and countably infinite and so does not presuppose the notion of an ω-sequence or arithmetical notions. For this we consider the precise definition of the notions used in formulating our inferential rules.

{carddef}

Definition 7.1. *We say two sets have the same* cardinality *if they can be put in* 1-1 *correspondence. Following Cantor and Dedekind*[57]*, we say a set D is* (Dedekind finite) *iff every one-one map of D into itself is onto. We take this as our notion of finite*[58]*. Further, we say a set is* infinite *if it is not finite. Finally a set X is countable if it is infinite and every proper subset is either finite or has the same cardinality as X.*

{cardstr}

Definition 7.1 is firmly[59] in Framework 2. Thus the notions of (equi)-cardinality and countable are available to construct logics in Framework 3. We described in Notation 2.4 (1)-(3) the three components of a system of logic (syntax, semantics and rules of inference). We work with inferential logics so item (3) is central, the theorems on categoricity of the inference rules make (2) a by-product.

Remark 7.2. *Our rules are based on cardinality of at most countable sets not arithmetic* {cardrules}
Recall the sketch of the history of the concept 'countable' in Remark 6.3. Note that our ω-rules, (4.1, 5.12), all have the form: if a formula holds for all constants in a countable set C,

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[57] Cantor defines the cardinality of an aggregate by 'abstracting from the nature of its elements and the order in which they are given' [**?**, p 86]. Two pages later, he defines the cardinality of two aggregates to be equal if they are in 1-1 correspondence and that of their sum as that of their disjoint union. Finally he says a set is finite if is not of the same cardinality as any proper subset.

[58] It is well-known that the equivalence of Dedekind finite with the standard definition in e.g. [**?**], $|A| < \omega$, where $|A|$ is defined as the least ordinality of a set in 1-1 correspondence with A requires the axiom of choice. For the moment, we disregard this search for unique reference. It is necessary for the study of larger cardinalities but not for the justification of our rules or the construction of an ω-sequence.

[59] By 'firmly' we mean they depend only on the absolute (not affected by set theoretic forcing) notions of 1-1 and cardinality.

then the universal closure of the formula also holds. Thus, our rules of inference[60] depend only on the notion of finite and countable cardinality. Neither the rules of inference, nor the truth conditions for infinite conjunction or disjunction have any dependence on the crucial concepts of well-order and induction aimed to be secured by the categoricity theorem for PA^ω. Moreover, this approach extends to $L_{\omega_1,\omega}$ via Theorem 5.14.

We now contrast our logics with other weak logics discussed in quantifier ([**?**, p 163]). [**?**, §7.10] do not mention ω-logic, because it has an infinitary rule of inference. But they argue that each of seven logics weaker than second order are 'just more theory'[61]. They consider that all these weaker logics must be semantically specified and so are dismissed for invoking in their semantics precisely the concepts aimed to be 'secured' by a categoricity theorem.

In contrast to the previous seven, our logics are inferentially specified. Moreover, our logics are no more semantically defined than first order; perhaps less, our ω-rules yield categoricity of the quantifiers and the $\mathbf{R}$-semantics are thus justified syntactically.

[**?**, pp. 163-4] argue against $L_{\omega_1,\omega}$ that 'grasping an infinite disjunction is just like grasping the natural number sequence'. We reject this equivalence. Note that an infinite disjunction is derivable from any of its disjuncts and, on the semantic side, the truth value of an infinite disjunction depends on the set of formulas, not their order. In Theorem 5.14 we reduced $L_{\omega_1,\omega}$ to $G-\omega$-logic. As we argued earlier each of our $I-\omega$ and $G-\omega$-rules depend only on a notion of a countably infinite infinite set of constants, not any relations that hold among those constants. And, on the notion that there is a point that is not in a given set.

Koellner argues that the ω-logic is strongly connected with the standard model of arithmetic and, thus, it is difficult to maintain that the results in the ω-logic depend only on the syntactic form. Working with the historical notion of ω-logic (see Definition 2.15.2.a.i.) that concerns only theories that interpret arithmetic, Koellner writes:

> A "proof" in ω-logic is an infinitary structure. The reason that Σ^0_1-complete theories T are complete for the statements of arithmetic in ω-logic is simply because these infinitary proofs can trace out the complete diagram of the standard model of arithmetic. On the model-theoretic side, the completeness

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[60] Definition 5.11 uses a numbering of the constants that smells of arithmetic (although only a pairing function). But this is used in the proof of Theorem 5.14, not in the formulation of the rule.

[61] [**?**], especially §2.3 on 'just more theory' provides an argument very similar to ours. By distinguishing the construction of the structure in Framework 2 from the categoricity in Framework 3, our argument is more general. Our argument for the concepts of cardinality and infinity being formalizable in Framework 2 justifies the semantic characterization of Speitel's logic $L(Q_0)$ (as well as the Härtig equicardinality quantifier) and authorizes the logics, not just the categoricity of arithmetic.

> theorem for ω-logic shows that $T \vdash_\omega \varphi$ if and only if φ is true in all ω-models of T, where an ω-model is simply a model that is correct in its computation of the natural numbers. In either case, whether one views the matter "proof-theoretically" or model-theoretically, the entanglement of the logic with the standard model of arithmetic is clear. Given this, it is hard to maintain that results in ω-logic depend on syntactic form alone. [**?**, p.19]

This argument simply does not apply to our inferential logics; Kollner is using the notion of ω-model in Definition 2.15.2.a.i. Our inferential ω-rules, Definition 2.15.2.a.ii and Definition 2.15.2.b. apply to any first order theory – not just to those interpreting first order arithmetic. Although there is a strong historical connection between the ω-rule and the standard model of arithmetic, our ω-rules use only the notions in Definition 7.1.

In building up our inferential ω-logics we have added to the syntactical instruments constructing first-order logic by the restriction (§ 4) to a countable number of individual constants -an idea that is common in the discussion of logical inferentialism since [**?**]. Philosophically speaking, we may say that we have assumed 'nameability' and 'countability'.

We noted above (Remarks 6.3 and 7.1) that the notion of 'countable' does not rely on arithmetic. Nameability is a constitutive feature of a natural language, in the sense that ordinary speakers use many expressions with the intention of naming something. In addition, it is part of the usual linguistic practice to introduce a new name in the language when there is some evidence that a new object has entered into the universe of discourse. Although this assumption may be seen as a model-theoretic or referential one, we take it to be compatible with an inferentialist perspective, since introducing a name for something does not immediately provide the name with a meaning. The meaning of that name will be determined by its inferential connections with the other items of the language and, more precisely, by the rules that govern its use in linguistic and inferential practices. In addition to this, nameability is certainly not an arithmetical concept.

Thus, the formalization of first-order theories in the $I - \omega$-logic and $G - \omega$-logic is, indeed, 'more theory', but it is not 'just more theory', since the categoricity problem gets a clear solution by theoretical tools that do not presuppose notions they aim to justify.

Does the choice of logic give more information than is needed or encode irrelevant questions?[62] This is a sin of second order logic. Many less expressive logics suffice to prove categoricity of arithmetic. While, second order offers the categoricity of the reals as a bonus. And the continuum hypothesis has been decided. But, which way will depend on our choice of the ambient set theory as a metatheoretical framework.

We compare our approach with Dedekind's second order axiomatization of arithmetic.

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[62] See [**?**, §11.2] for 'immodesty of Dedekind's axiomatization of geometry.

1. We obtain categoricity in a first order logical framework, i.e. with quantification over individual variables.
2. We use rules of inference and we univocally read off the truth-conditions of the quantifiers from their rules. Second order logic (SOL) and most of the weaker logics use semantic notions. SOL needs the standard semantics for categoricity, while its rules of inference are complete for the Henkin semantics.
3. This approach is much more parsimonious than the others.
4. On the semantical side, the ontological commitments (that is, our quantifiers only give points as in first order logic, while second order arithmetics allows the definability of arbitrarily complex relations [**?**, §5.B]) of the logic we use are the same as in first order logic. So, ontologically, we are as conservative as possible.

While the classical ω-rule provides the negation completeness of PA ([**?**],[**?**]) but not its categoricity, our approach shows that with an inferential version of the ω-rule, interpreted in terms of R-semantics, we obtain the negation completeness of PA and its categoricity. In SOL we obtain only the second meta-theoretical property. With these criteria, a costs-benefits analysis favors our approach.

8 Conclusion.

We studied in this paper two main notions of categoricity: the categoricity of a logical calculus and the categoricity of a first-order theory formalized in it. Each logical calculus was considered as an inferential logical system with natural deduction rules. By results of Carnap and Garson, the standard axiomatic and natural deduction calculi for classical propositional and first-order logic are known to be non-categorical, i.e. they allow non-standard admissible valuations that provide most of the logical connectives and quantifiers with non-standard truth-conditions. In addition, no first-order theory with an infinite model is categorical.

Our solution for obtaining a categorical formalization of the first-order quantifiers was to introduce a new form of a well-known infinitary ω-rule of inference, the $I-\omega$-rule. With the $I-\omega$-rule we obtained a categorical formalization of the first-order universal quantifier and, at the same time, we proved that all first-order theories that have an algebraically prime model in which all elements are named by constants are categorical when formalized in the inferential $I-\omega$-logic. Likewise, we proved in the two-sorted generalized ω-logic that each complete $L_{\omega_1,\omega}$ sentence defines the same class of structures as a corresponding first-order theory with the $G-\omega$-rule. Moreover, we proved (Corollary 5.16) that a sentence φ of $L_{\omega_1,\omega}$ (and so the associated (Theorem 5.14) theory in $G-\omega$-logic) is categorical just if the countable model of φ has no proper isomorphic extension modeling φ.

Finally, we argued in the last two sections that these mathematical results answer the doxological challenge and the circularity argument raised by Button and Walsh [**?**]. In particular, we argued that the doxological challenge is generated by conflating the structure and naive set theoretic frameworks described in Section 6.2. Likewise, we argued that the circularity argument does not apply to the proofs of the categoricity theorems in our ω-logics since our ω-rules do not presuppose arithmetical concepts that were in need of determinacy by a categoricity result.

References