The Entanglement of Model Theory and Set Theory
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[1] Thanks to J. Kennedy and A. Villeveces

Goal

In Second Philosophy Maddy writes,

> *The Second Philosopher sees fit to adjudicate the methodological questions of mathematics – what makes for a good definition, an acceptable axiom, a dependable proof technique?– by assessing the effectiveness of the method at issue as means towards the goal of the particular stretch of mathematics involved.*

We discuss the choice of definitions of model theoretic concepts that:

1. reduce the set theoretic overhead
2. while providing tools to solve mathematical problems
3. and an organizing scheme for mathematics.

In this talk, we emphasize 1).

Entanglement

Such authors as Parsons, Kennedy, and Väänänen have spoken of the entanglement of logic and set theory.

Theses

There is a deep entanglement between (first-order) model theory and cardinality.

There is No such entanglement between (first-order) model theory and cardinal arithmetic.

There is however such an entanglement between infinitary model theory and cardinal arithmetic and therefore with extensions of ZFC.

Equality as Congruence

Any text in logic posits that:
Equality '=' is an equivalence relation:

Further it satisfies the axioms schemes which define what universal algebraists call a congruence.

The indiscernibility of identicals

For any x and y, if x is identical to y, then x and y have all the same first order properties.
For any formula ϕ: $\forall \mathbf{x} \forall \mathbf{y} [\mathbf{x} = \mathbf{y} \rightarrow \phi(\mathbf{x}) \leftrightarrow \phi(\mathbf{y})]$

Equality as Identity

The original 'sin'

The inductive definition of truth in a structure demands that the equality symbol be interpreted as identity:

$$M \models a = b \text{ iff } a^M = b^M$$

The entanglement of model theory with cardinality is now ordained!

This is easy to see for finite cardinalities.

$$\phi_n : (\exists x_1 \dots x_n) \bigwedge_{1 \le i < j \le n} x_i \neq x_j \wedge (\forall y) \bigvee_{1 \le i \le n} y = x_i$$

is true exactly for structures of cardinality n.

Cardinality

Three examples of the entanglement with cardinality.

1. Downward Löwenheim Skolem –not so much
2. Upward Löwenheim Skolem
   Yes! Look at the proof.
3. Only finite structures are categorical.

Entanglement with Cardinal arithmetic and extensions of ZFC

In 1970, model theory and axiomatic set theory seemed intrinsically linked. Shelah wrote

> *"... in 69 Morley and Keisler told me that model theory of first order logic is essentially done and the future is the development of model theory of infinitary logics (particularly fragments of $L_{\omega_1,\omega}$). By the eighties it was clearly not the case and attention was withdrawn from infinitary logic (and generalized quantifiers, etc.) back to first order logic."*

Shelah: Set theory and model theory

Shelah again:

> *During the 1960s, two cardinal theorems were popular among model theorists. . . . Later the subject becomes less popular; Jensen complained when I start to deal with gap n 2-cardinal theorems, they were the epitome of model theory and as I finished, it stopped to be of interest to model theorists. I sympathize, though model theorists has reasonable excuses: one is that they want ZFC-provable theorems or at least semi-ZFC ones the second is that it has not been clear if there were any more.*

Two Questions

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II. Why by the mid-70's had those apparent links evaporated for first order logic?

I. Apparent dependence on set theory

Löwenheim Skolem for 2 cardinals

Vaught: Can we vary the cardinality of a definable subset as we can vary the cardinality of the model?

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Two Cardinal Models

1. A two cardinal model is a structure M with a definable subset D with $\aleph_0 \leq |D| < |M|$.
2. We say a first order theory T in a vocabulary with a unary predicate P *admits* (κ, λ) if there is a model M of T with $|M| = \kappa$ and $|P^M| = \lambda$. And we write $(\kappa, \lambda) \rightarrow (\kappa', \lambda')$ if *every theory* that admits (κ, λ) also admits (κ', λ').

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Hypotheses included:

1. replacement: Erdos-Rado theorem below $\beth_{\omega_1}$.
2. GCH
3. $V = L$
4. Jensen's notion of a morass
5. Erdös cardinals,
6. Foreman [1982] showing the equivalence between such a two-cardinal theorem and 2-huge cardinals AND ON

1-5 Classical work in 60's and early 70's; continuing importance in set theory.

The links dissolve

Why did it stop?

Theorem: revised problem solved in ZFC.

Suppose

1. [Shelah, Lachlan $\approx$ 1972] *T* is stable
2. or [Bays 1998] *T* is *o*-minimal

then $\forall(\kappa > \lambda, \kappa' \geq \lambda')$
if *T* admits (κ, λ) then *T* also admits (κ', λ').

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Reversing the question

set theorist:
For which cardinals does $P(\kappa, \lambda, T)$ hold for all theories ?

model theorist:
For which theories does $P(\kappa, \lambda, T)$ hold for all cardinals ?

Really, Why did it stop?

Definition

[The Stability Hierarchy:] Fix a countable complete first order theory T.

1. T is stable in χ if $A \subset M \models T$ and $|A| = \chi$ then $|S(A)| = |A|$.
2. T is
   1. ω-stable[2] if T is stable in all χ;
   2. superstable if T is stable in all $\chi \geq 2^{\aleph_0}$; That is, for every A with $A \subset M \models T$, and $|A| \geq 2^{\aleph_0}$, $|S(A)| = |A|$
   3. stable if T is stable in all χ with $\chi^{\aleph_0} = \chi$;
   4. unstable if none of the above happen.

[2] This 'definition' hides a deep theorem of Morley that T is ω-stable if and only if it stable in every infinite cardinal.

So what?

Sacks Dicta

“... the central notions of model theory are absolute and absoluteness, unlike cardinality, is a logical concept. That is why model theory does not founder on that rock of undecidability, the generalized continuum hypothesis, and why the Łos conjecture is decidable.”

Gerald Sacks, 1972

More precisely

While the stability spectrum function is another function about cardinality,
The notions defining the hierarchy are all absolute.

1. ω-stability (Morley rank defined: Π^1_1)
2. superstability (D-rank defined: Π^1_1)
3. stability (no formula has the order property: arithmetic)

The success of the hierarchy

Theorem

[Stability spectrum theorem] Every complete first order theory falls into one of the 4 classes just defined.

Actually, studying a few more, simplicity and NIP (without the independence property) has extended the range to a much wide range of mathematically important topics.

The success of the hierarchy

A crucial consequence of stability is the ability to define family of dimensions and classify structures.

The stability classification of T gives detailed information about the fine structure of definable sets in each model of T. This information is encoded by stability ranks that are in many cases (e.g. algebraic geometry) the same as those arising in other content areas.

A sophisticated theory for studying the interactions of these various dimensions has had applications in many fields.

Mathematically relevant areas of mathematics can be axiomatized by complete first order theories of various stability classes.

Model theory entangles with Algebra

Theorem (Hrushovski 1989) Let T be a stable theory. Let $\tilde{p} \not\perp \tilde{q}$ be stationary, regular types and let n be maximal such that $\tilde{p}^n \perp^a \tilde{q}^\omega$. Then there exist p almost bidominant to $\tilde{p}$ and q dominated by $\tilde{q}$ such that:

- $n = 1$ q is the generic type of a type definable group that has the regular action on the realizations for p.
- $n = 2$ q is the generic type of a type definable algebraically closed field that acts on the realizations for p as an affine line.
- $n = 3$ q is the generic type of a type definable algebraically closed field that acts on the realizations for p as a projective line.
- $n \geq 4$ is impossible.

The Entanglement with group and field theory: Importance

The hypotheses are purely model theoretic.

There is no assumption that a group or ring is even interpretable in the theory.

The conclusion gives precise kinds of group and field actions that are *definable* in the given structures.

There are important consequences in model theory, diophantine geometry, differential fields, . . .

Entanglement of Infinitary Logic and Axiomatic Set Theory

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   - C Consistency may imply truth.

ABSTRACT ELEMENTARY CLASSES

A class of L-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an *abstract elementary class: AEC* if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism plus:

1. If $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ and $A \subseteq B$ then $A \prec_{\boldsymbol{K}} B$;

Examples

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2. Closure under direct limits of $\prec_{\mathbf{K}}$-chains;
3. Downward Löwenheim-Skolem.

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Shelah infinitary categoricity theorem

No Assumption of upwards Löwenheim-Skolem

Theorem [Shelah]

1. (For $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$) A complete $L_{\omega_1,\omega}$-sentence which has few models in $\aleph_n$ for each $n < \omega$ is excellent.
2. (ZFC) An excellent class has models in every cardinality.
3. (ZFC) Suppose that ϕ is an excellent $L_{\omega_1,\omega}$-sentence. If ϕ is categorical in one uncountable cardinal κ then it is categorical in all uncountable cardinals.

Boney/Vasey eventual categoricity theorems

Theorem (Boney)

If κ is a strongly compact cardinal and $LS(\boldsymbol{K}) < \kappa$ then if $\boldsymbol{K}$ is categorical in some $\lambda^+ > \kappa$ then $\boldsymbol{K}$ is categorical in all $\mu \geq \lambda^+$.

Theorem (Vasey)

Assuming, κ is a strongly compact cardinal and $LS(\boldsymbol{K}) < \kappa$, VWGCH, and the result of a long preprint of Shelah, if $\boldsymbol{K}$ is categorical in some $\lambda > \kappa$ then $\boldsymbol{K}$ is categorical in all $\mu \geq \lambda^+$.

The Dependence on cardinality

First order (Morley)

$\aleph_0$ is exceptional:

1. Categoricity is $\aleph_1$ implies categoricity in all uncountable cardinals.

Infinitary: Shelah, Boney/Vasey

Some small cardinals may exceptional:

1. (VWGCH) Categoricity is all cardinals below $\aleph_\omega$ implies categoricity in all uncountable cardinals.
2. Categoricity beyond a strongly compact implies categoricity in all uncountable cardinals.

Which cardinals are exceptional?

Any $\aleph_n$. (Hart-Shelah; B-Kolesnikov)

The Paradigm Shift

Fundamental Distinctions

Logics

1. second order logic
2. infinitary logic (aec)
3. first order logic

The choice of logics presents a trade-off between greater ability to control the structure of models (via e.g. compactness) and lesser expressive power.

The Paradigm Shift

Model theory in the 1960's concentrated on the properties of logics.

This resulted in many problems being tied closely to axiomatic set theory.

The switch to classifying a theory T according to whether there were good recipes for decomposing models of T into simpler pieces resulted in

1. a divorce from axiomatic set theory
2. a fruitful interaction with many other areas of mathematics.

The study of infinitary logic offers more expressive power to study mathematics at a possible cost of set theoretic independence.

Related Work

Completeness and Categoricity (in power): Formalization without Foundationalism
The Bulletin of Symbolic Logic 2014

Formalization, Primitive Concepts and Purity
Review of Symbolic Logic vol 6, 2013

Axiomatizing Changing Conceptions of the geometric continuum I and II
First order justification of $C = 2\pi r$
submitted

`http://homepages.math.uic.edu/~jbaldwin/model11.html`