What is a minus sign anyway?

John Baldwin

February 24, 2001

VERY PRELIMINARY DRAFT

How do we justify:

$$(40 + 12) - (20 + 5) = (40 - 20) + (12 - 5)?$$

I am writing some of these arguments with every step to illustrate a couple of points. The associative law can only work on two applications of the plus sign. We generalize it to say we can regroup any sequences of additions. It is quite plausible that this is a distinction one should not make for teachers – or at least the question should be at what level you want them to be aware of it. The second problem is that when minus signs are interspersed this gets more complicated. Since associativity fails for subtraction, some further rules are required. At the moment, mostly as mathematical whimsy, I am listing all the alternatives that are pointed out to me.

Method 1: Work in the integers with binary addition and multiplication and both a unary - (opposite) and a binary minus: $a - b$ abbreviates $a +^{-} b$. So this is a ring in the ordinary algebraic sense. Note that $(^{-}1)a =^{-} a$ since $a + (^{-}1)a = (1 +^{-} 1)a = 0$ by distributivity.

$$52 - 25$$

by definition of place value equals

$$(40 + 12) - (20 + 5) =$$

by definition of minus

$$(40 + 12) + (-1) \cdot (20 + 5) =$$

by distributivity

$$(40 + 12) + (^{-}20 +^{-} 5) =$$

by associativity

$$((40 + 12) +^{-} 20) + -5 =$$

by associativity

$$(40 + (12 +^{-} 20)) +^{-} 5 =$$

by commutativity

$$(40 + (^{-}20 + 12)) +^{-} 5 =$$

by associativity

$$((40 +^{-} 20) + 12)) +^{-} 5 =$$

by associativity

$$((40 +^{-} 20) + (12 +^{-} 5)$$

Method 1A: Working in the ring is unnecessary. John Wood pointed out that we can justify the step from

$$(40 + 12) - (20 + 5) =$$

to

$$(40 + 12) + (^{-}20 +^{-} 5)$$

entirely in the group of integers. Modulo multiple uses of associativity

$$^{-}x +^{-} y + y + x = 0$$

so $^{-}(y + x) =^{-} x +^{-} y$. By commutativity we have the result.

Method 2: Work in the natural numbers with binary addition and multiplication and partial subtraction: ($a - b$ is only defined if $a > b$.

In addition to associativity and commutativity of addition and multiplication and distributivity of multiplication over addition add that subtraction distributes over addition:

$$a - (b + c) = a - b - c.$$

The right hand side makes sense and the equation holds under the convention $a-b-c$ abbreviates $(a-b)-c$. This is (one of) a rule that Ma calls adding/taking away parentheses. Now the deduction appears much shorter.

$$52 - 25 = (40 + 12) - (20 + 5) =$$

by adding/taking away parentheses

$$= 40 + 12 - 20 - 5$$

by associativity and commutativity

$$= (40 - 20) + (20 - 5).$$

But the shortness is just compressing many steps using two short cuts (writing -5 for + -5) and using the general rule: associativity and commutativity allows you to perform a list of additions in any order.

I don't know if this a sufficient set of axioms for the equational theory here; it probably isn't.

Method 3: Finally one can work in the integers with binary multiplication, addition, and subtraction subject to the rules of method 3.

Method 4: Work in integers or natural numbers with binary functions addition, multiplication, subtraction and law:

$$(a - b) + b = a.$$

Now,

$$(40 + 12) - (20 + 5) = (40 - 20) + (12 - 5)$$

because,

$$((40 - 20) + (12 - 5)) + (20 + 5)$$

modulo multiple uses of associativity and commutativity gives

$$((40 - 20) + 20) + ((12 - 5) + 5)$$

which by $(a - b) + b = a$, twice gives $40 + 12$. You get the same on the left by the rule $(a - b) + b = a$ with $a = 40 + 12$ and $b = 20 + 5$.