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Abstract

In this paper we present three aspects of the autonomy of geometry.

(1) An argument for the geometric as opposed to the 'geometric algebraic' interpretation of Euclid's Books I and II; (2) Hilbert's successful project to axiomatize Euclid's geometry in a first order geometric language, notably eliminating the dependence on the Archimedean axiom; (3) the independent conception of multiplication from a geometric as opposed to an arithmetic viewpoint.

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- The Autonomy of Geometry
- Abstract –at beginning

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The Autonomy of Geometry

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By the autonomy of geometry we mean that geometry involves distinct concepts from and does not depend on either arithmetic or algebra. We address this autonomy from three perspectives: historical, foundational, and conceptual. After describing these perspectives in Section 1, we address the historical issues in Section 2, the foundational in Section 3, the conceptual in Section 4 and then summarise the argument in Section 5.

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1 Introduction

In Baldwin (2017a), we argued that an easily intertranslatable set of mathematical results could arise from different conceptions (in that paper of the continuum) at different times. And, while recognition of this rough equivalence is important; it is essential to understand the different conceptions. In Section 2 we object[1] to Van der Waerden's description of Book II of Euclid as he ignores a similar change over time from geometric to algebraic conceptions and notations.

> We studied the wording of the theorems and tried to reconstruct the original ideas of the author. We found it evident that these theorems did not arise out of geometrical problems. We were not able to find any interesting geometrical problem that would give rise to theorems[2] like II 1-4. On the other hand, we found that the explanation of these theorems as arising from algebra worked well. Therefore we adopted the latter explanation. (Van der Waerden, 1976, 203-204)

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[1]Our attention was drawn to this topic by the incisive Katz Katz (2020) critique of Unguru's position in the Van der Waerden-Unguru controversy. With the extremes of an historian's perspective well dealt with, we focus here on the extremes on the other side. We hold that mathematicians, historians, and linguists all make essential contributions to the history of mathematics.

[2]This is more circumspect than the statement in his book. 'When one opens Book II of the Elements, one finds a sequence of propositions that are nothing other than geometric formulations of algebraic rules.' (Van der Waerden, 1954, 118).

In supporting this objection we follow the lead of Arpad Szabo (Szabo, 1978, Appendix 4) and Piotr Błaszczyk (Błaszczyk, 2019, Section 3) who argue for the geometrical motivation of Book I and II. In particular Szabo argues that just as Book I is aimed at a proof of the Pythagorean theorem, Book II is aimed at constructing from an arbitrary polygon a square that has the same area. This construction is accomplished, as is Book I, without any use of proportion but just by the rearrangement of polygons. The theory of proportion for magnitudes appears only in Book V with geometric consequences in Book VI on similarity of triangles and thereafter. For this discussion, we should clarify the meaning of the word 'algebra'. Van der Waerden writes,

> When I speak of Babylonian or Greek or Arab algebra, I mean algebra in the sense of Al- Khwarizmï, or in the sense of Cardano's "Ars magna", or in the sense of our school algebra. Algebra, then, is:
>
> *the art of handling algebraic expressions like* $(a+b)^2$ *and of solving equations like* $x^2 + ax = b$.
>
> (Van der Waerden, 1976, 199)

The difficulty with applying such a definition to Euclid is that 'handling algebraic expressions' is precisely what does not occur in Euclid. Points, lines, polygons, and circles are handled.

Studying the changes in algebra through time has resulted in some more precise terminology. Christianidis (2000), borrowing the term from Oaks, lays out a careful distinction between *premodern* and *modern* algebra[3] in abstract terms rather than by identification with specific mathematical cultures. Among his characterizations of premodern is (Christianidis, 2000, page 39), 'Despite the fact that the problems solved by algebra may have been stated in terms of arithmetic, mensuration, commerce etc., the unknowns were always numerical measures of quantifiable objects. Accordingly, it is not surprising that premodern algebra was considered by its practitioners as part of arithmetic.' But Euclid deals in congruence and 'equal area', not numerical measurement.

Christianidis further writes (Christianidis, 2000, page 39) 'The notion [premodern] itself was created to describe primarily the medieval Arabic, Latin, and Italian algebra. By adopting this notion (mainly from the works of Jeffrey Oaks), my aim in the present study is to show that the notion of 'premodern algebra' provides a suitable contextual framework for conceptualizing Diophantus' *Arithmetica* as well.' He speaks of Diophantus, not Euclid.

Second (Section 3), we discuss Hilbert's reformulation of Euclid in 1899. In contrast to the *arithmetization project* of Dedekind and Weierstrass which attempted to secure the foundations of analysis in arithmetic, Hilbert (1971) clarifies Euclid's axiomatization of geometry as an autonomous subject. In particular, he removes Euclid's dependence on the axiom of Archimedes and gives

[3] This terminology would naturally not be used by Van der Waerden. His book *Moderne Algebra*, Van der Waerden (1949) began what contemporary mathematicians call the much more abstract *modern algebra*.

an entirely first order axiomatization. Of course this last sentence is anachronistic; the careful distinction between first and second order logic was twenty years in the future. But Hilbert made the distinction clear in his 1917/1918 lecture notes (Hilbert (2013)) by formalizing his geometry in the restricted predicate calculus. He accomplishes independence from Archimedes by interpreting a field into each model of his first order[4] axioms. Thus he can assign an element of the field to each line segment (length) or polygon (area). Then he defines proportionality by using division in the modern sense. This allows a straight-forward proof of Euclid's VI.1 and VI.2, which apply the notion of proportion to calculate the area of a triangle and show corresponding sides of similar triangles are proportional. Thus, he establishes the theory of similarity without Euclid's recourse to the axiom of Archimedes.

In Section 4 we discuss the conceptual role of geometry in students' understanding of multiplication. At least in United States schools, students are taught in the elementary school that multiplication is repeated addition. This has several negative effects. 'Then, how can you divide a bigger number into a smaller?' And when the student passes from arithmetic of the natural numbers to dense order, suddenly every magnitude is divisible by any natural number. The difficulty is that there are three distinct intuitions of multiplication: repeated addition for the natural numbers, area and scaling/similarity in geometry.

2 The geometric motivation for Euclid II

Euclid's Book I is generally understood to deal with geometry: congruence, area, and eventually the Pythagorean theorem. In contrast, the propositions of Book II have often been interpreted as based in algebra. Indeed, Van der Waerden (Van der Waerden, 1954, 118) opens a section of *Science Awakening* titled 'Geometric Algebra' with the statement 'When one opens Book II of Euclid, one finds a sequence of propositions which are nothing but geometric formulations of algebraic rules.' Of course, that is how the first four propositions[5] might appear to us, but algebraic notation was far in the future during Euclid's time. These are rules (e.g., II.1 asserts in modern language: scalar multiplication distributes over vector addition) which assert that area is conserved by disjoint union. Algebraic notation is far more general than geometric argument. Indeed, the essence of algebra is that often, in the course of an algebraic derivation, *one must lose* the explicit meaning of each statement. See a simple example from high school algebra in Baldwin et al. (2010). In contrast, the geometric argument refers to a sequence of actual diagrams.

On reflection, there is a natural geometric motivation for the main themes of Book II: *Determine a precise method for determining which of two disjoint rectilinear figures[6] (polygons) has the greater area.* Błaszczyk (2019) and Szabo

[4]Technically his 2nd order continuity axiom must be replaced by circle-circle intersection.

[5]Perhaps, Van der Waerden is falsely accused of viewing the entire Book II as 'geometric algebra'; in his book he stops his presentation of equations with Propositions II.9 and 10.

[6]Euclid Definition I.19: Rectilinear figures (See A in Figure 2.) are those which are con-

(1969) agree that the goal is show any rectilinear figure has the same area as a square. We emphasize that Euclid's argument implies that the rectilinear figures are totally ordered by what we call area. A key step took place already in Proposition I.45: any rectilinear figure has the same area as a parallelogram and indeed as a rectangle. This result is proved by an implicit induction, described on pages 9-10 of Błaszczyk (2019). So we need only to determine which of two rectangles has the greater area. But the rectangles are determined by two parameters, width and length[7] The search for a canonical single paramater representation leads to the following, strange to modern ears, definition and proposition.

Definition 2.1 *Any rectangular parallelogram is said to be* contained *by the two straight lines containing the right angle.*

This definition allows us to study the areas of arbitrary rectangles by indexing each rectangle by its semi-perimeter and a cut point in that line. This indexing restricts from parallelograms to rectangles, which clarifies the focus on rectangles in Book II. And if we can replace an arbitrary rectangle by a square, since side length determines area, we have solved the comparison problem. The first lemma in studying this representation asserts:

Proposition 2.2 (Proposition 2) *If a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole.*

Here, 'cut at random' and 'contained by' give us a pair of rectangles with the same height but arbitrary length summing to that height and describes their areas. In modern terms it yields the formulas ab, $(b-a)b$ for the areas. Thus, Proposition II.2 certainly implies that if a square is split into two non-overlapping rectangles the sum of the areas of the rectangles is the area of the square. But while Van der Waerden (Van der Waerden, 1954, 118) reads[8] this as $a(b+c) = ab + ac$ (where $a = b + c$); we regard it as emphasizing for rectangles, that when one joins two geometric figures along a common side the area of the resulting figure is the sum of the two. This is the basic premise for developing a theory of areas by decomposition of polygons. Crucially, this development has no dependence on a theory of proportions but is able to handle irrational lengths.

Szabo[9] (Szabo, 1978, Appendix 3, Example 3) lays out some of the important geometric techniques involved in the proof. In particular the 'carpenter's

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tained by straight lines.

[7]In fact, one of these can be specified (Euclid, 1956, 346-347).

[8]Joyce says that if $x = y + z$ then $x^2 = xy + xz$, (Guide to Book II in online version of Euclid (1956)).

[9]We do not go into the extensive philological analysis of Szabo nor his speculations on the origins of the Greek theories of proportion (Szabo, 1978, Chapter 1). When speaking of Greek thought we mean in the era discussed here. That is, in the early 3rd century B.C.E. Van der Waerden describes the Golden Age as the fifth century BC in the first paragraph of his Chapter V, but then discuses Book II of Euclid in detail.

square' or *gnomon* appears whenever a smaller square is subtracted from a larger. Namely, it is the remainder of the square and can be decomposed into another smaller square and two congruent rectangles[10]. In figure 1, $CBFGHL$ is a gnomon.

Much of Book II considers the relation of the areas of various rectangles, squares, and gnomons, depending where one cuts a line. While gnomons[11] have a clear role in decomposing parallelograms, the algebraic representation for the area of gnomon, is not a tool in polynomial algebra. That is, while such equations as $(a+b)(a-b) = a^2 - b^2$ or the formula for product of binomials are tools in algebra which have nice geometric explanation, the area of a gnomon has an algebraic expression, $2ab + a^2$, which does not recur in algebra (e.g., as a method of factorization).

Propositions II.5 provides the first step by replacing the area of a rectangle by that of a gnomon. We take the cut point D in AB to be between the midpoint C and B.

Theorem 2.3 (Proposition II.5) *Let C be the midpoint of AB and D a point between C and B. The area of the rectangle with length AD and height DB plus the area of the square on CD equals the area of the square on CB.*

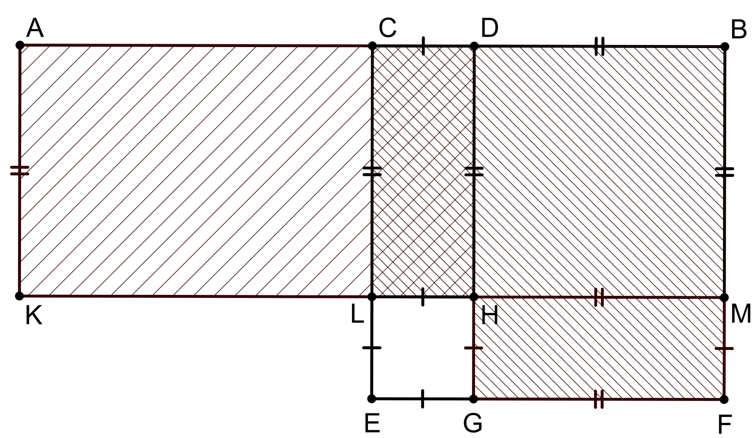

Figure 1: II.5

That is, we show the rectangle shaded // in Figure 1 to have the same area as the gnomon shaded \\. Figure 1 is constructed in the proof by making $DB \cong BM$, $CD \cong MF$ and those lines which appear parallel to be parallel.

[10] In fact, Euclid uses gnomon in even more generality; a gnomon is an L-shaped figure made by removing a parallelogram from a larger similar parallelogram.

[11] Incidentially, (Berggren, 1984, 298) reports that, in contrast to degree measurement, this concept was transferred from the Babylonians at an early date.

Leaving aside the details of the construction, the result is clear: The rectangle $ADHK$ is composed of $ACLK$ and $CDHL$. Moreover, $ACLK \cong BFGD$. Now the square on CB ($CBFE$) is composed of $BFGD$ plus $CDHL$ plus $LHGE$ (which has the same area as the square on CD).

Note the decomposition of $ADHK$ uses Proposition II.2 implicitly. As pointed out in Heath (Euclid (1956)), if we take a as the length of AD and b as the length of DB, Proposition II.5 yields the algebraic equation.

$$(\frac{a+b}{2})^2 = ab + (\frac{a-b}{2})^2 \tag{1}$$

But since the Greeks did not assign numbers as lengths in their geometry (but only discussed the 'equality' of magnitudes of the same dimension), it is hard to see algebra as the Greek understanding, although it is convenient for modern readers skilled in algebra.

(Szabo, 1978, Appendix 4) draws an interesting comparison of the proofs of Proposition II.5 and of the existence of a mean proportional in Proposition VI.13. He argues at (Szabo, 1978, 46-48) (citing Heiberg (1904) and Heath (1949) that i) VI.13 depends on proportion (Thales/ VI.2), which for Euclid depends on Archimedes, and ii) the proof in book II came later than the proof of VI.13 and was designed to avoid proportion.

Euclid's proof of II.14 essentially repeats the proof of II.5 before continuing to the conclusion. One could summarize that II.5 constructs a gnomon from a rectangle (gnomon $CBFGHL$ from rectangle $ADHK$ in Figure 1) and the new part of II.14 constructs a square from gnomon (square $JXYZ$ from gnomon $UVPJSR$ in Figure 2).

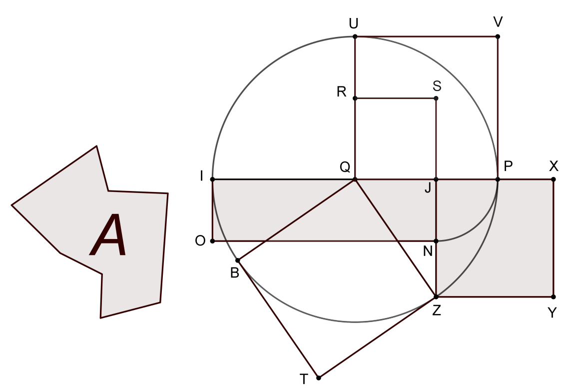

Figure 2: Proposition II.14

Together, they reduce the comparison of areas of polygons to that of squares. This summary aims to clarify the mathematical connection rather than historically analyze Euclid's argument. In more detail,

Theorem 2.4 (Proposition II.14) *To construct a square equal to a given rectilinear figure.*

Proof. As discussed near the beginning of this section, by Proposition I.45 we may assume the rectilinear figure A is a rectangle $IJNO$.

In Figure 2 we modify Figure 1 to apply Proposition II.5. The old $ADHK$ becomes $IJNO$; but the square on QP is above IX while the square on CB was below AB. So $LHGE$ of Figure 1 corresponds to $QJSR$ in Figure 2. And, the gnomon $CBFGHL$ corresponds to the gnomon $UVPJSR$. We show the latter gnomon has the same area as the square $JZYX$. By Proposition II.5, the rectangle $IJNO$ plus the square on QJ ($QJSR$) has the same area as the square on QP ($QPVU$). But $QP \cong QZ$, so we can consider the square on QZ ($QZTB$)). By the Pythagorean theorem (I.47) the square on QZ is the sum of the squares on QJ and on JZ. So, taking away the square on QJ from the square on QP yields a square with the same area as the gnomon $UVPJSR$ and so as $IJNO$. Namely, the square on JZ ($JZYX$). $\blacksquare$

Euclid's geometrical motivation is evidenced by a further generalization of II.14, which still doesn't need proportion:

Theorem 2.5 (Euclid: III.35) *If in a circle two straight lines cut one another, then the rectangle contained by the segments of the one equals (has same area) the rectangle contained by the segments of the other.*

This further consequence of II.5 creates a new result about chords of circles. And when the chords are perpendicular and one is a diameter, III.35 yields a new proof of II.14. We can see the proof from the same Figure 2, if we take the rectangle contained by the segments to again be $IJNO$ with one chord the diameter IP and the other chord ZJ extended to meet the circle.

Propositions II.12 and 13 can be interpreted as the law of cosines. Although Euclid precedes the definition of sine and cosine by generations, these are surely more geometric than algebraic results.

Thus, Book II culminates in Proposition II.14. It relies on only a few of the earlier results. It crowns a series of results which transform rectangles (usually given by cutting with prescribed conditions) into squares or gnomons with intricate relations of the area of the result to the hypotheses. By showing every polygon determines a square of the same area, it provides a method for comparing the areas of arbitrary polygons with a single parameter, the side of the resulting square.

3 Hilbert's Geometry

We claimed[12] in Baldwin (2017a,b), that in the discussion of area Hilbert and Euclid are treating in some rough sense the same topic but in very different ways.

[12]These matters are explained in much more detail in Hartshorne (2000); Baldwin (2018), in Baldwin and Mueller (2012), which are notes specifically for secondary teachers, and of course in Hilbert (1962).

A crucial distinction is in the use of the word number. In Greek mathematics, numbers are what we now think of as $2, 3, 4\ldots$. Even the unit is distinct; a number is 'a multitude of units'. Magnitudes are a different species. Thus Euclid addresses the theory of proportionality in both Books V (magnitudes) and VII (numbers). A fundamental difference is that magnitudes can be divided into an arbitrary finite number of equal pieces. Clearly this does not hold for members of the system of natural numbers. Over several thousand years western mathematics arrived at a notion of real numbers which contain the natural numbers as a subsystem. But the full development occurs only in the late 19th century. Hilbert chooses to express his measurement of lengths and area by an arithmetic on segments[13]. But, unlike Euclid, who says the area of a triangle is proportional to the base and the height, he uses the formula $\frac{1}{2}bh$ for the area of a triangle to be evaluated in the segment arithmetic (by counting the number (perhaps fractional or indeed irrational) of unit squares that cover the triangle).

A key point of Hilbert's foundations involved a distinction that was fully formulated only 20 years after his geometry. The distinction is between first and second order logic. First order sentences only quantify over individuals. Thus, Hilbert would rephrase Euclid's first proposition, I.1, as: for any two points A, B there is a third point C, not on AB, so that AB, BC and CA are congruent (ABC is an equilateral triangle.).

Hilbert proves *'almost all' of the actual theorems of Euclidean geometry from a set of first order axioms.* The first order axioms are reformulations of Euclid plus two additional axioms that are described below. After explaining the non-first order axioms, we will turn to the first order axioms and explicate 'almost all'.

The first non-first order axiom, Hilbert's axiom of continuity (Dedekind completeness), is equivalent to the second order statement: for every pair of arbitrary non-empty subsets X, Y of the rationals $\boldsymbol{Q}$ such that every element of X is strictly less than every element of Y and $X \cup Y = \boldsymbol{Q}$, there is a point a in the reals such that $a \geqslant x$ for every $x \in X$ and $a \leqslant y$ for every $y \in Y$. This axiom is used in the Grundlagen only to prove that Hilbert's geometry is isomorphic to the geometry over the reals studied in high school geometry.

The second non-first order axiom is the *axiom of Archimedes*[14]: if the smaller one of two given segments is marked off a sufficient number of times, it will always produce a segment larger than the larger one of the original two segments. This may seem quite similar to our translation Euclid's first proposition, I.1. The difference is the 'sufficient number of times'. The full statement results by saying 'marked off 2 times' or 'marked off 3 times', etc. It might also seem that this is a quantification over segments (which are subsets). But, by mark off 2

[13] Descartes is a transitional figure; he considers multiplication as an operation on segments but defines it from Euclid's (Archimedean) proof of the existence of a 4th proportional. Hilbert had considered using the points of the geometry as the elements of the field but elected segments in the *Grundlagen*.

[14] In geometry the completeness axiom implies Archimedes. But completeness can be stated in terms of order alone while Archimedes in geometry requires a notion of congruence and is usually formulated in the presence of a group.

times, we mean, given A_0, A_1, there exist A_2, A_3 so that $A_0A_1 \cong A_1A_2$ and $A_0A_1 \cong A_2A_3$, where $\cong$ denotes congruence. This is not first order because it involves an infinite conjunction. Euclid embeds this axiom in his Definition V.4. aIt might be paraphrased as, in geometry no two points are infinitely far apart.

There is also a very basic issue with respect to Proposition I.4 of Euclid: if two corresponding sides and the included angle are equal then the triangles are congruent. Euclid proves this by the method of superposition, which is unclear at best. This can be remedied in a number of ways. Hilbert chose to take the proposition SAS (Euclid I.4) as an axiom[15].

We work below in an axiom system we call HP5 (Hartshorne (2000)). Here HP denotes Hilbert's incidence, betweenness and congruence axioms (Hilbert Plane) and HP5 also includes the 5th (parallel) postulate. We write Euclidean geometry (EG) if we also add the circle-circle intersection property (implicit in the construction of an equilateral triangle in Euclid I.1).

The 'almost all' is summarized as follows[16]. We restrict our attention to plane geometry and so omit Books V, VII and X (proportionality and number theory and Books XII (except XII.2) and XIII (solid geometry). Thus, below we select from Books I-IV, VI, XII.1, 2 and consider certain geometrical aspects of V and X. Now the results provable are

Euclid I, polygonal geometry Book I (except I.1 I.22), Book II.1-II.13, Book III (except III.1 and III.17), Book VI

Euclid II, circle geometry I.1, I.22, II.14, III.1, III.17 and Book IV.

Archimedes, arc length and π XII.2, Book IV, (area of a circle proportional to the square of the diameter), approximation of π, circumference of circle proportional to radius[17] Euclid, Archimedes' axiom.

The conclusion is:

Theorem 3.1 *1. The sentences of Euclid I are provable in HP5.*

2. The sentences of Euclid II are provable in EG.

3. The sentences of Archimedes, arc length and π: Euclid XII.2, area of circle are provable in Hilbert I-IV plus Archimedes and also in the first order theories EG_π.

The actual proofs of 1) and 2) appear in Section 12 and Sections 20-23 in Hartshorne (2000). And 3) is proved in (Baldwin, 2018, 10.1,10.2), where EG_π is defined as a first order theory with a name for π.

The key to Hilbert's elimination of the axiom of Achimedes is to define a field from first order geometric principles. Euclid had the basic idea (`https://mathcs.clarku.edu/~djoyce/java/elements/bookVI/propVI12.html`): Given

[15] Baldwin and Mueller (2012) takes SSS because it makes copying angles particularly easy.

[16] (Baldwin, 2018, 218-220) or a variant in Baldwin (2017a).

[17] Of course,this result is not in Euclid but see Baldwin (2017b).

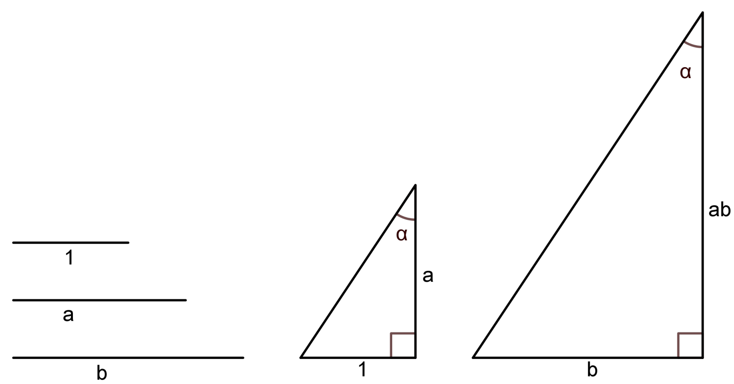

Figure 3: Definition of multiplication

line segments a, b, c; construct the 4th proportional, a segment d so that $a : b = c : d$. Euclid proceeded by thinking entirely geometrically; his proportions are between magnitudes, while we now think of a, b, c as being the *lengths* of the segments. And Euclid does this construction only after he has developed an abstract Archimedean theory of proportional magnitudes and the theory of area. Descartes takes the same course but calls the 4th proportional of $1 : b = c : d$ the product of b and c.

In contrast, Hilbert begins the study of proportion by defining the notion of multiplication of segments geometrically. He works from HP plus what we now call the theorem of Desargues. Following Hartshorne we adopt the more restrictive parallel postulate rather than Desargues and call the result of adding it to HP, HP5. By a *segment class* we mean a set (equivalence class) of all congruent segments. Neither proof assumes the Archimedean axiom.

Definition 3.2 *[Multiplication] Fix a unit segment class* 1. *Consider two segment classes a and b. To define their product, define a right triangle*[18] *with legs of length* 1 *and a. Denote the angle between the hypotenuse and the side of length a by α.*

Now construct another right triangle with base of length b with the angle between the hypotneuse and the side of length b congruent to α. The segment class of the vertical leg of the triangle is ab.

The crucial theorem is that this multiplication is associative, commutative (this uses the parallel postulate; without that axiom Hilbert gets only a division ring), has inverses, and distributes over segment addition. Thus there is a definable field.

[18]Since we assumed the parallel postulate (which implies Desargues) the right triangle is just for simplicity; we really just need to make the two triangles similar.

We now compare the study of area in Hilbert, Euclid, and Hartshorne. We follow Hartshorne's definition of figure which includes polygons but is more generous (E.g. the union of two separated polygons is a figure.).

Definition 3.3 *(Hartshorne, 2000, 196) A* figure *is a subset of the plane that can be represented as a finite union of non-overlapping triangles. A point D is in the interior of a figure P if there is a triangle ABC entirely contained in P such that D is in the interior of ABC. We say two figures are* non-overlapping *if no point is in the interior of both.*

As in Błaszczyk (2019) and Euclid (1956), we take two figures to have equal area by the following definition[19].

Definition 3.4 (Area of polygons)

1. *Two figures are* scissors-congruent *if you can cut one up (on straight lines) into a finite number of triangles which can be rearranged to make the second.*
2. *Two figures P, Q have* equal area *if there are figures $P'_1 \dots P'_n$, $Q'_1 \dots Q'_n$ such that none of the figures overlap, each pair P'_i and Q'_i are scissors congruent and $P \cup P'_1 \dots \cup P'_n$ is scissors congruent with $Q \cup Q'_1 \dots \cup Q'_n$.*

Note that the natural definition of equal area from Euclid's practice, deletes the word 'scissors' from Definition 3.4.2.

Theorem 3.5 (Euclid: I.35) *Parallelograms which are on the same base and in the same parallels equal one another.*

The argument, using the following diagram, is explained in modern language in (Błaszczyk, 2019, Section 3).Proposition 22.3 of Hartshorne (2000) asserts that his formalization of the equivalence relation incorporates Euclid's use of CN2 and CN3 (addition/subtraction of equals from equals are equal) in Books I and II, specifically in I.35.

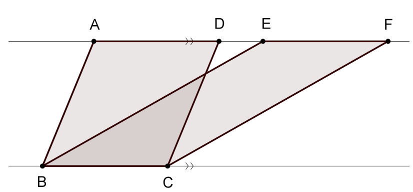

Figure 4: I.35 Finite Parallelogram

Hilbert shows that while Euclid's proof of I.35 is perfectly correct, the figures cannot be proved to be scissors congruent without using Archimedes' axiom (i.e.

[19]Hilbert and Hartshorne call the first of these notions *equal area* and the second *equal content.*

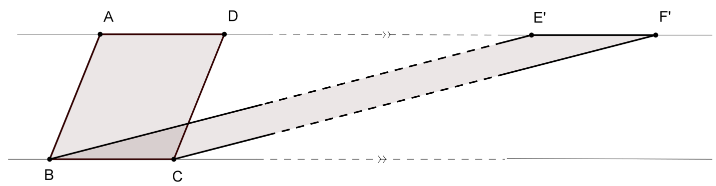

Figure 5: 1.35 Infinite Parallelogram

without denying points at infinite distance). To see this, suppose that E and F are translated to E' and F' infinitely far to the right of D (Figure 3). Then line BE will have to be covered by infinitely many segments, each shorter than the diagonal BD, the longest possible segment in $ABCD$, since these segments are on the finitely many triangles covering $ABCD$. But that is impossible.

In fact, the two notions of area are the same under the Archimedean axiom. Welsh (2016) points out that Wallace (1807) proved the Wallace–Bolyai–Gerwien theorem that polygons with the same area are scissors-congruent. All these authors assume Archimedes' axiom. It might appear, that her exposition depends on Archimedes because (Welsh, 2016, Lemma 2.4) appeals to the theorem of Thales (Figure 4): the lengths of the sides of a triangle created by cutting a triangle by a line parallel to the base are proportional to the sides of the original triangle. But while Euclid uses his theory of area and a theory of proportion depending on the axiom of Archimedes to establish VI.2, Hilbert proves VI.2 using his notions of proportion and area without this appeal. The problem lies rather in the paragraph and diagram after (Welsh, 2016, Lemma 2.5) where two stacks of finitely many rectangles are created and the sums of the bases of the rectangle of each stack are assumed commensurable.

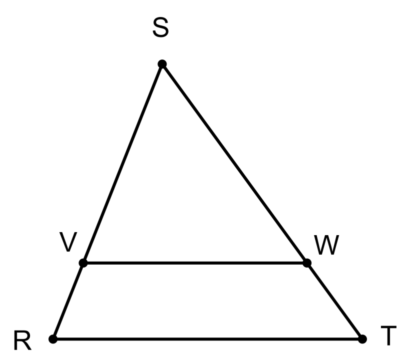

Figure 6: Euclid VI.2: $SV : SR = SW : ST$

Hilbert constructs a specific ordered field with an element greater than infinitely many copies of a unit segment (as in Figure 5.) But the algebra is unnecessary. Working in the language of geometry with a unit segment AB we have for each n a point F_n such that AF_n contains non-overlapping copies of n copies of AB. So write formal sentences $AF > AB+AB$, $AF > AB+AB+AB$, etc. Since we have axiomatized geometry in first order logic, by the compactness theorem, there is a model of geometry and an F which makes all these sentences true at once. The advantage of this argument is that it replaces a very particular argument for violating Archimedes that depends on the geometry being sufficiently strong to define a field and on technical algebra with a general argument depending only on a Euclidean description of the Archimedean axiom and basic logic.

We now discuss an additional complexity that arises in Hilbert's treatment. He has defined multiplication before introducing the notion of area. Thus, he can define the area of a rectangle by the formula $A = \frac{bh}{2}$ and then compute the area of a polygon by triangulating it and taking the sum of the areas of the triangles. But there is a fine point in defining the formula. Does its value depend on which altitude is chosen? Euclid doesn't have this problem because he says (Proposition VI.1) only proportional while Hilbert specifies the proportionality constant. (Hartshorne, 1977, 197-205) argues that a few propositions at the end of Book I and, in particular, the uniform choice of the proportionality constant depend on de Zolt's axiom. We hope to analyze this argument in a future paper. For now, we only describe the invariance of the area formula.

Axiom 3.6 (Z: De Zolt) *If Q is a figure contained in another figure P, and if $P - Q$ has non-empty interior then P and Q do not have equal area.*

Definition 3.7 (measure of areas) *(Hartshorne, 2000, p.205) A* measure of area function *is a map α from the set of all figures into an ordered field such that*

1. *For any triangle T, $\alpha(T) > 0$.*
2. *Congruent triangles have the same area.*
3. *If two figures do not overlap then $\alpha(P \cup Q) = \alpha(P) + \alpha(Q)$.*

Among the easy conclusions from the existence of a measure of area function are the following.

Theorem 3.8

1. *(Hartshorne, 2000, Proposition 23.1) If a plane satisfies HP and has a measure of area function then it satisfies de Zolt's axiom.*
2. *(Hartshorne, 2000, Theorem 23.2) If a plane satisfies HP5 then it has a unique measure of area function defined by $\alpha(T) = \frac{1}{2}bh$ (based on a unit area of a square with side 1).*

The next lemma, using de Zolt's axiom, is crucial for showing the Hilbert's measure of area function computes the area of rectangles 'correctly'. The parenthetical letters in Figure 7 show where the original $WXYZ$ is copied to.

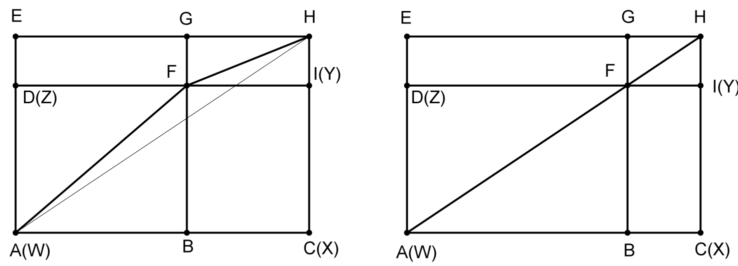

Figure 7: Lemma 3.9: Using de Zolt's axiom

Lemma 3.9 *If two rectangles $ABGE$ and $WXYZ$ have equal area there is a rectangle $ACID$, congruent to $WXYZ$ and satisfying the following diagram. Further the diagonals AF and FH are collinear.*

Proof. Suppose AB is less than WX and AE is more than YZ. Then make a copy of $WXYZ$ as $ACID$. This will give the left hand diagram in Figure 7. Let F be the intersection of BG and DI. Construct H as the intersection of EG extended and IC extended. Now we prove F lies on AH and so the right hand diagram is actually correct.

Suppose F does not lie on AH. Subtract $ABFD$ from $ABGE$ and $ACID$; $DFGE$ and BCIF have the same area. The diagonals AF and FH divide each of the rectangles $ABFD$ and $FIHG$ into a pair of congruent triangles. So $AFD \cup DFGE \cup FHG$ has the same area as $ABF \cup BCIF \cup FIH$, both being half of rectangle $ACHE$ (Note that the union of the six figures is all of $ACHE$.). Here, $AEHF$ is properly contained in AHE and $ACHF$ properly contains ACH. This contradicts (Z), which holds by Theorem 3.8; hence F lies on AH. ■

Claim 3.10 *If $ABGE$ and $ACID$ are as in the right diagram (i.e., have the same area), then in segment multiplication $(AB)(BG) = (AC)(CI)$.*

We omit the fairly straightforward proof of Claim 3.10 (Baldwin and Mueller, 2012, 6.19) and apply the result to show the area formula is correct.

Theorem 3.11 *Any of the three choices of base for a triangle give the same value for the product of the base and the height.*

Proof. Consider the triangle ABC in Figure 8. The rectangles in the small figures 1, 3, and 5 are easily seen to be scissors congruent, while the triangle ABC is half of each them. Claim 3.10 shows each product of height and base for the triangle is the same. That is, $(AB)(CD) = (AC)(BJ) = (BC)(AM)$. But these are the three choices of base/altitude pair for the triangle ABC. ■

In writing the congruence postulates Hilbert assumed that triangles are not oriented. In Hilbert (1971), Hilbert deals also with oriented triangles. And he

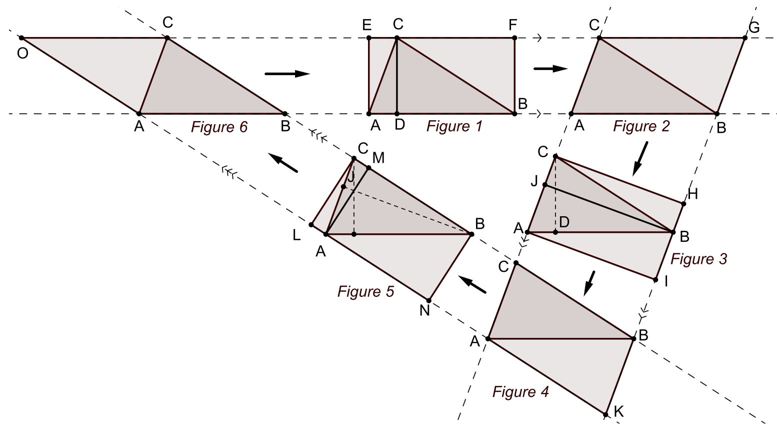

Figure 8: Area does not depend on choice of base and height

shows in Appendix II, that assuming SAS only for similarly oriented triangles is strictly weaker. Indeed, the axiom of Archimedes is required to obtain the usual SAS if the axiom is stated only for oriented triangles.

As discussed earlier in Section 2, Euclid develops in Books I and II a theory of area that does not depend on proportion. Thus, there is no reliance on the Archimedes Axiom. Hilbert demonstrated that slight changes, such as 'equal area' to 'scissors congruence' can produce a theory where the Axiom of Archimedes is essential but which yields no better results. In Book V, Euclid developed a theory of proportion that relied on the Archimedes Axiom to deal with incommensurable magnitudes. And thus, so does his development of similarity in Book VI. By defining field multiplication, and thus defining proportion geometrically, Hilbert eliminated this dependence.

4 Multiplication is not repeated addition

Elementary school students are ordinarily introduced to multiplication of natural numbers by examples such as $2 + 2 + 2 = 6$ and $3 \times 2 = 6$. That notion of multiplication is correct for its context: multiplication of natural numbers. But, that notion of multiplication does not admit an inverse. While, multiplication of real numbers does. Repeated addition motivates multiplication of a real number *by* a natural number; but it does not motivate multiplication of a natural number *by* an arbitrary real. As, multiplication of two arbitrary line segments naturally has a multiplicative inverse. To find the inverse of a, ask what is the length b of the base of a rectangle with height a and area 1? Or compute by Definition 3.2 as illustrated in Figure 9

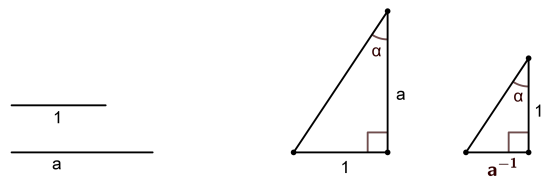

Figure 9: Computing the multiplicative inverse

Example 4.1 (Area) The area of a rectangle with rational sides should be introduced in elementary school by dividing the rectangle into congruent smaller squares (so that both sides of the square are multiple of the side of the small square) and counting them.

This comment is motivated by the experience of each author with students who, when presented in pre-calculus or calculus with min-max problems, would react, ‘I know A is either ℓw or $2\ell + 2w$ but which is it?’.

Example 4.2 (Similarity in the real world (slightly contrived)) A triangular clothes hanger and its reflection in a mirror.

Figure 10: Similarity Demonstrated

Example 4.3 (Similarity Problem) The second intuition is similarity. How can you measure the height of a street light? In the picture the man is d meters tall; his shadow is c meters long and the shadow of light pole is a meters long. The height b of the tower is $\frac{ad}{c}$ meters. See figure 11. Such problems are fundamentally geometrical. Crucially, computing areas motivates the concept of multiplication.

Figure 11: Shadow problem

In this short section, we have emphasized that geometry provides two distinct conceptual understandings of multiplication that do not arise in arithmetic. It is important that these conceptions be inculcated into elementary school students.

5 The autonomy of geometry

Without careful linguistic documentation but with a consideration of the general line of Greek thought we argue the autonomy of geometry in Greek mathematics and an even stronger autonomy discovered by Hilbert.

Van der Waerden asserted that Greeks learned algebra from the Babylonians[20] and Egyptians and translated that algebra into geometric form. And, that the Greeks introduced the idea of proof. But he asks what causes 'the push to geometrization' and answers 'the discovery of irrational numbers'. The separate developments of proportion for magnitude in Book V and proportion for numbers in Book VII illustrate the great Greek distinction between number

[20] We address neither whether Babylonian mathematics can properly be called algebra nor the extent of its influence on Greek thought.

and magnitude. The Greek mathematicians didn't have a common vocabulary for both while the Babylonians did. Van der Waerden makes this clear.

> That they [Greeks] did not consider $\sqrt{2}$ as a number was not a result of ignorance, but of a strict adherence to the definition of number. Arithmos meant quantity, therefore whole number. Their logical rigor did not even allow them to admit fractions; they replaced them by ratios of integers.
>
> For the Babylonians, every segment and every area simply represented a number. They had no scruples in adding the area of a rectangle to its base. When they could not determine a square root exactly, they calmly accepted an approximation. But the Greeks were concerned with exact knowledge, with 'the diagonal itself' as Plato expresses it, not with an acceptable approximation.
>
> (Van der Waerden, 1954, page 125)

This is precisely why we reject the notion of a Euclidean 'geometric algebra'. Algebra requires a uniform range of interpretation for the variables. The (early 3rd century B.C.E.) Greek mathematicians' refusal to allow areas and lengths to be directly comparable as numerical quantities disallows the adjective 'algebraic'. While many of the results in Book II can be expressed as quadratic equations (and cubic equations in Book XI), there is a reason that there are no fourth degree equations; for them, there is no fourth dimension. This is not an objection to dimensional analysis; it is a remark that the geometry determines the degrees of the algebraic equations we can force on the data. In contrast to 'the Greeks' in the Van der Waerden quote, this objection does not apply to Viéte. To cite a convenient source, 'Viéte saw the symbols in equations could represent either numbers, or geometric quantities, and that this was a powerful tool for analyzing and solving geometric problems.' (Stedall, 2008, 738). And by the time of Viéte 4th degree equations were actively studied.

On the other hand, Euclid develops a systematic theory of area in Books I and II, in Book V a theory of proportions, and derives from them in Book VI a theory of similarity. Here, there is a link with number theory, the axiom of Archimedes. But, Hilbert demonstrates that this link is unnecessary. By taking the construction of the fourth proportional as a basic definition of multiplication of segments he grounds the arithmetic of the rational, the real, and, indeed, any Euclidean field in geometry.

Tarski and Gödel make the autonomy even clearer. Gödel (1931) proves arithmetic is undecidable and has no finitary foundation (Zach (2019)). Tarski (1931, 1959) proves that geometry[21] has a constructive foundation and is decidable.

[21] Specifically, both EG and $\mathcal{E}^2$, the first order theory $\mathcal{E}^2$ whose models are the geometries over real closed fields, are finitarily consistent (i.e provably in Primitive recursive arithmetic Tait (1981)) and the second is decidable. See (Baldwin, 2018, Chapter 10) for summary. Note however that by Ziegler (1982) every finitely axiomatized extension of HP5 is undecidable; see Makowsky (2018) for an exposition.

Thus, we have argued the autonomy of geometry in three modes. Historical: Books I and II of Euclid are motivated by the problem of developing a theory of areas that provides a mean to compare the areas of arbitrary polygons. But it does not rely on a theory of proportion, nor is it a 'translation of Babylonian algebra'. Foundational: Euclid's geometry is constructively consistent and does not rely on arithmetic for its justification. Conceptual: geometry provides distinct intuitions for the notion of multiplication.

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