Axiomatizing changing conceptions of the geometric continuuum II: Archimedes – Descartes – Hilbert – Tarski

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Abstract

In Part I of this paper we argued that the first-order systems HP5 and EG are modest complete descriptive axiomatization of most of Euclidean geometry. In this paper we discuss two further modest complete descriptive axiomatizations: Tarksi's for Cartesian geometry and new systems for adding π. In contrast we find Hilbert's full second order system immodest for geometrical purposes but appropriate as a foundation for mathematical analysis.

This two part paper analyzes the axiomatic foundation of the 'geometric continuum', the line embdedded in the plane. For this, we built on Detlefsen's notion of *complete descriptive axiomatization* and defined in Part I a *modest complete descriptive axiomatization of a data set* Σ (essentially, of facts in the sense of Hilbert) to be a collection of sentences that imply all the sentences in Σ and 'not too many more'. Of course, this set of facts will be open-ended, since over time more results will be proved. But if this set of axioms contradicts the conceptions of the original era, we deem the axiomatization immodest.

Part I [Baldwin 2017a] dealt primarily with Hilbert's first order axioms for polygonal geometry and argued the first-order systems HP5 and EG (defined below) are 'modest' complete descriptive axiomatization of most of Euclidean geometry. Part II concerns areas of geometry, e.g. circles, where stronger assumptions are needed. Hilbert postulated his 'continuity axioms' – the Archimedean and completeness axioms in extensions of first order logic; we pursue weaker axioms.

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In Section 1, we reprise our organization of various 'data sets' for geometry and describe the axiom systems. We contend: 1) that Tarski's first-order axiom set $\mathcal{E}^2$ is a modest complete descriptive axiomatization of Cartesian geometry (Section 2); 2) that the theories $EG_{\pi,C,A}$ and $\mathcal{E}^2_{\pi,C,A}$ are modest complete descriptive axiomatizations of extensions of these geometries designed to describe area and circumference of the circle (Section 3); and 3) that, in contrast, Hilbert's full second-order system in the *Grundlagen* is an immodest axiomatization of any of these geometries but a modest descriptive axiomatization the late 19th century conception of the real plane (Section 4). We elaborate and place this study in a more general context in our book [Baldwin 2017b].

1 Terminology and Notations

Part I provided the following quasi-historical description. Euclid founded his theory of area for circles and polygons on Eudoxus' theory of proportion and thus (implicitly) on the axiom of Archimedes. The Greeks and Descartes dealt only with geometric objects. The Greeks regarded multiplication as an operation from line segments to plane figures. Descartes interpreted it as an operation from line segments to line segments. In the late 19th century, multiplication became an operation on points (that is 'numbers' in the coordinatizing field). Hilbert showed any plane satisfying his axioms HP5 (below) interprets a field and recovered Euclid's results about polygons via a first-order theory. The *biinterpretability* between various geometric theories and associated theories of fields is the key to the analysis here.

We begin by distinguishing several topics in plane geometry[1] that represent distinct data sets in Detlefsen's sense. In cases where *certain axioms are explicit, they are included in the data set*. Each set includes its predecessors. Then we provide specific axiomatizations of the various areas. Our division of the data sets is somewhat arbitrary and is made with the subsequent axiomatizations in mind.

Euclid I, polygonal geometry: Book I (except I.22), Book II.1-II.13, Book III (except III.1 and III.17), Book VI.)

Euclid II, circle geometry: I.22, II.14, III.1, III.17 and Book IV.

Archimedes, arc length and π: XII.2, (area of circle proportional to square of the diameter), approximation of π, circumference of circle proportional to radius, Archimedes' axiom.

Descartes, higher degree polynomials: nth roots; coordinate geometry

[1] In the first instance we draw from Euclid: Books I-IV, VI and XII.1, 2 clearly concern plane geometry; XI, the rest of XII and XIII deal with solid geometry; V and X deal with a general notion proportion and with incommensurability. Thus, below we put each proposition Books I-IV, VI, XII.1,2 in a group and consider certain geometrical aspects of Books V and X.

Hilbert, continuity: The Dedekind plane

In Part I, we formulated our formal system in a two-sorted vocabulary τ chosen to make the Euclidean axioms (either as in Euclid or Hilbert) easily translatable into first-order logic. This vocabulary includes unary predicates for points and lines, a binary incidence relation, a ternary collinearity relation, a quaternary relation for line congruence and a 6-ary relation for angle congruence. The *circle-circle intersection postulate* asserts: if the interiors of two circles (neither contained in the other) have a common point, the circles intersect in two points.

The following axiom sets[2] are defined to organize these data sets.

1. first-order axioms

   HP, HP5: We write HP for Hilbert's incidence, betweenness[3], and congruence axioms. We write HP5 for HP plus the parallel postulate. A *Pythagorean field* is any field associated[4] with a model of HP5; such fields are characterized by closure under $\sqrt{(1+a^2)}$.

   EG: The *axioms for Euclidean geometry*, denoted EG[5], consist of HP5 and in addition the circle-circle intersection postulate. A *Euclidean plane* is a model of EG; the associated *Euclidean field* is closed under $\sqrt{a}$ for $a > 0$.

   $\mathcal{E}^2$: Tarski's axiom system [Tarski 1959] for a plane over a real closed field (RCF[6]).

   EG_π and $\mathcal{E}^2_\pi$: Two new systems extending EG and $\mathcal{E}^2$ to discuss π.

2. Hilbert's continuity axioms, infinitary and second-order

   AA: The sentence in $L_{\omega_1,\omega}$ expressing the Archimedean axiom.

   Dedekind: Dedekind's second-order axiom that there is a point in each irrational cut in the line.

Notation 1.1. Closing a plane under ruler and compass constructions corresponds to closing the coordinatizing ordered field under square roots of positive numbers to give a Euclidean field[7]. As in Example 4.2.2.2 of Part I , F_s (surd field) denotes the minimal field whose geometry is closed under ruler and compass construction. Having named $0, 1$, each element of F_s is definable over the empty set[8].

[2] The names HP, HP5, and EG come from [Hartshorne 2000] and $\mathcal{E}^2$ from [Tarski 1959].

[3] These include Pasch's axiom (B4 of [Hartshorne 2000]) as we axiomatize *plane* geometry. Hartshorne's version of Pasch is that any line intersecting one side of triangle must intersect one of the other two.

[4] The field F is *associated* with a plane Π if Π is the Cartesian plane on F^2.

[5] In the vocabulary here, there is a natural translation of Euclid's axioms into first-order statements. The construction axioms have to be viewed as 'for all– there exist sentences. The axiom of Archimedes is of course not first-order. We write Euclid's axioms for those in the original [Euclid 1956] vrs (first-order) axioms for Euclidean geometry, EG.

[6] RCF abbreviates 'real closed field'; these are the ordered fields such that every positive element has a square root and every odd degree polynomial has at least one root.

[7] We call this process 'taking the Euclidean closure' or adding *constructible* numbers.

[8] That is for each point a constructible by ruler and compass there is a formula $\phi_a(x)$ such that $EG \vdash (\exists! x)\phi(x)$. in EG. That is, there is a unique solution to ϕ.

We referred to [Hartshorne 2000] to assert in Part I the sentences of Euclid I are provable in HP5 and the additional sentences of Euclid II are provable in EG. Here we consider the data sets of Archimedes, Descartes, and Dedekind and argue for the following claims.

1. Tarski's axioms $\mathcal{E}^2$ are a modest descriptive axiomatization of the Cartesian data set.
2. EG^2_π ($\mathcal{E}^2_\pi$) are modest descriptive axiomatizations of the extension by the Archimedean data set of Euclidean Geometry (Cartesian geometry).
3. Hilbert's axioms groups I-V give a modest descriptive axiomatization of the second-order geometrical statements concerning the plane $\Re^2$ but the system is immodest for even the Cartesian data set.

2 From Descartes to Tarski

Descartes and Archimedes represent distinct and indeed orthogonal directions in the project to make *geometric continuum* a precise notion. These directions can be distinguished as follows. Archimedes goes directly to transcendental numbers while Descartes investigates curves defined by polynomials. Of course, neither thought in these terms, although Descartes' resistance to squaring the circle shows an awareness of what became this distinction. We deviate from chronological order and discuss Descartes before Archimedes; as, in Section 3 we will extend both Euclidean and Cartesian geometry by adding π.

As we emphasized in describing the data sets, the most important aspects of the Cartesian data set are: 1) the explicit definition ([Descartes 1637], 1) of the multiplication of line segments to give a line segment, which breaks with Greek tradition[9]; and 2) on the same page to announce constructions for the extraction[10] of nth roots for all n.

Panza ([Panza 2011] 44) describes Euclid's plane geometry as an *open system*. But he writes, 'things are quite different with Descartes' geometry; this a *closed system*, equally well-framed as (Euclid's).' We follow Rodin's [Rodin 2017] exposition of the distinction between closed and open systems. It extends the traditional distinction between 'theorems' and 'problems' in Euclidean geometry in a precise way. Theorems have truth values; problems (constructions) introduce new objects. A closed system has a fixed domain of objects while the domain expands in an open system. According to Panza, Rodin and others Euclid's is an open system. In contrast, we treat both Euclidean and Cartesian geometries as Hilbert style closed systems. As Rodin

[9] His proof is still based on Eudoxus.

[10] This extraction cannot be done in EG, since EG is satisfied in the field which has solutions for all quadratic equations but not those of odd degree. See Section 12 of [Hartshorne 2000].

[Rodin 2017] points out the 'received notion' of axiomatics interprets a construction in terms of a$\forall\exists$-statements that asserts the construction can be made. Panza illustrates the openness by analyzing many types of 'mechanical constructions' in pre-Cartesian geometry[11]. According to Molland ([Molland 1976], 38) 'Descartes held the possibility of representing a curve by an equation (specification by property)' to be equivalent to its 'being constructible in terms of the determinate motion criterion (specification by genesis)'[12]. By adding the solutions of polynomial equations Tarski's geometry $\mathcal{E}^2$ (below), guarantees in advance the existence of Descartes' more general notion of construction. This extension is obscured by Hilbert overly generous continuity axiom

Descartes' proposal to organize geometry via the degree of polynomials ([Descartes 1637], 48) is reflected in the modern field of 'real' algebraic geometry, i.e., the study of polynomial equalities and inequalities in the theory of real closed ordered fields. To ground this geometry we adapt Tarski's 'elementary geometry'. Tarski's system differs from Descartes in several ways. He makes a significant conceptual step away from Descartes, whose constructions were on segments and who did not regard a line as a set of points. Tarski's axioms are given entirely formally in a one-sorted language with a ternary relation on points thus making explicit that a line is conceived as a set of points[13]. We will describe the theory in both algebraic and geometric terms using Hilbert's bi-interpretation of Euclidean geometry and Euclidean fields.[14] The algebraic formulation is central to our later developments. With this interpretation we can specify (in the metatheory) a minimal model of Tarski's theory, the plane over the *real algebraic numbers*[15]. It contains exactly (as we now understand) the objects Descartes viewed as solutions of those problems that it was 'possible to solve' (Chapter 6 of [Crippa 2014b]). In accordance with Descartes' rejection as non-geometric any method for quadrature of the circle, this model omits π.

Tarski's elementary geometry The theory $\mathcal{E}^2$ is axiomatized by the following sets of axioms 1) and 2a) using the bi-interpretation, while 2b) can be expressed in Tarski's vocabulary[16]).

1. Euclidean plane geometry[17] (HP5);
2. Either of the following two sets of axioms which are equivalent over HP5 (in a vocabulary naming two arbitrary points as $0, 1$):

[11] See Sections 1.2 and 3 of [Panza 2011] as well as [Bos 2001]. Rodin [Rodin 2014] further develops Panza 'open' notion as a 'constructive axiomatic' system.

[12] But as Crippa points out ([Crippa 2014a], 153), Descartes did not prove this equivalence and there is some controversy as to whether the 1876 work of Kempe solves the precise problem.

[13] Writing in 1832, Bolyai ([Gray 2004], appendix) wrote in his 'explanation of signs', 'The straight AB means the aggregate of all points situated in the same straight line with A and B.' This is the earliest indication I know of the transition to an extensional version of incidence. William Howard showed me this passage. See Bolzano quote in ([Rusnock 2000],53)).

[14] In our modern understanding of an axiom set the translation is routine, but anachronistic.

[15] That is, a real number that satisfies a polynomial with rational coefficients. A real number that satisfies no such polynomial is called *transcendental*.

[16] Translation to our official vocabulary requires quantifiers. We abuse Tarski's notation by letting $\mathcal{E}^2$ denote the theory in the vocabulary with constants $0, 1$.

[17] Note that circle-circle intersection is implied by the axioms in 2).

(a) An infinite set of axioms declaring the field is formally real and that every polynomial of odd-degree has a root.

(b) Tarski's axiom schema of continuity.

Just as restricting induction to first order formulas translates Peano's second order axioms to first order, Tarski translates Dedekind cuts to first order cuts. Require that for any two definable sets A and B, if beyond some point a all elements of A are below all elements of B, there there is a point b which is above all of A and below all of B. Tarski [Givant & Tarski 1999] formalizes the requirement with the *Axiom Schema of Continuity*:

$$(\exists a)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \to B(axy)] \to (\exists b)(\forall x)(\forall y)[\alpha(x) \wedge \beta(y) \to B(xby)],$$

where α, β are first-order formulas, the first of which does not contain any free occurrences of a, b, y and the second any free occurrences of a, b, x. Recalling that $B(x, z, y)$ represents 'z is between[18] x and y', the hypothesis asserts the solutions of the formulas α and β behave as the A, B above. This schema allows the solution of odd degree polynomials. So b) implies a). As the theory of real closed fields (ordered field satisfying a)) is complete[19], schema a) proves schema b).

In Detlefsen's terminology Tarski has laid out a *Gödel-complete* axiomatization, that is, the consequences of his axioms are a complete first-order theory of (in our terminology) Cartesian plane geometry. This completeness guarantees that if we keep the vocabulary and continue to accept the same data set no axiomatization[20] can account for more of the data. There are certainly open problems in Cartesian plane geometry [Klee & Wagon 1991]. But however they are solved, the proof will be formalizable in $\mathcal{E}^2$. Thus, in our view, the axioms are descriptively complete.

The axioms $\mathcal{E}^2$ assert, consistently with Descartes' conceptions and theorems, the solutions of certain equations. So they provide a *modest* complete descriptive axiomatization of the Cartesian data set. In the case at hand, however, there are more specific reasons for accepting the geometry over real closed fields as 'the best' descriptive axiomatization. It is the only one which is decidable and 'constructively justifiable'.

Remark 2.1 (Undecidability and Consistency)**.** Ziegler [Ziegler 1982] has shown that every nontrivial finitely axiomatized subtheory[21] of RCF is *not decidable*. Thus both to approximate more closely the Dedekind continuum and to obtain decidability we restrict to the theory of planes over RCF and so to Tarski's $\mathcal{E}^2$. The biinterpretability between RCF and the theory of all planes over real closed fields yields the decidability of $\mathcal{E}^2$ and a *finitary proof of its consistency*[22]. The crucial fact that makes decidability

[18] More precisely in terms of the linear order $B(xyz)$ means $x \leq y \leq z$.

[19] Tarski [Tarski 1959] proves that planes over real closed fields are exactly the models of $\mathcal{E}^2$.

[20] Of course, more perspicuous axiomatizations may be found. Or one may discover the entire subject is better viewed as an example in a more general context.

[21] A nontrivial subtheory is one satisfied in $\Re$.

[22] The geometric version of this result was conjectured by Tarski in [Tarski 1959]: The theory RCF is complete and recursively axiomatized so decidable. For the context of Ziegler result and Tarski's quantifier elimination in computer science see [Makowsky 2013].

possible is that the natural numbers are *not first-order definable* in the real field.

As we know, the preeminent contribution of Descartes to geometry is coordinate geometry. Tarski (following Hilbert) provides a converse; his interpretation of the plane into the coordinatizing line [Tarski 1998] unifies the study of the 'geometric continuum' with axiomatizations of 'geometry'. Three post-Descartes innovations are largely neglected in these papers: a) higher dimensional geometry, b) projective geometry c) definability by analytic functions. Item a) is a largely nineteenth century innovation which impacts Descartes's analytic geometry by introducing equations in more than three variables. We have used Tarski's axioms for plane geometry from [Tarski 1959]. However, they extend by a family of axioms for higher dimensions [Givant & Tarski 1999] to ground modern real algebraic geometry. This natural extension demonstrates the fecundity of Cartesian geometry. Descartes used polynomials in at most two variables. But once the field is defined, the semantic extension to spaces of arbitrary finite dimension, i.e. polynomials in any finite number of variables, is immediate. Thus, every n-space is controlled by the field so the plane geometry determines the geometry of any finite dimension. Although the Cartesian data set concerns polynomials of very few variables and arbitrary degree, all of real algebraic geometry is latent. Projective geometry, b), is essentially bi-interpretable with affine geometry. So both of these threads are more or less orthogonal to our development here which concerns the structure of the line (and moves smoothly to higher dimensional or projective geometry).

A provocative remark in [Dieudonné 1970] symbolizes c). He asserted the *only* correct usage of 'analytic geometry' is as the study of solution sets of *analytic functions* on real n-space for any n. 'It is absolutely intolerable to use *analytical geometry* for linear algebra with coordinates, still called analytical geometry in elementary textbooks. Analytical geometry in this sense has never existed. There are only people who do linear algebra badly by taking coordinates ... Everyone knows that analytical geometry is the theory of analytical spaces.' That there never was such a subject is surely hyperbole and [Dieudonné 1982] makes pretty clear that his sense of analytic geometry is a twentieth century creation. But Hilbert did lay the grounds for analytic geometry and mathematical analysis on Dedekind's reals, denoted $\Re$.

3 Archimedes: π and the circumference and area of circles

We begin with our rationale for placing various facts in the Archimedean data set[23]. Three propositions encapsulate the issue: Euclid VI.1 (area of a triangle), Euclid XII.2

[23]This classification is not in any sense chronological, as Archimedes attributes the method of exhaustion to Eudoxus who precedes Euclid. Post-Heath scholarship by Becker, Knorr, and Menn [Menn 2017] have identified four theories of proportion in the generations just before Euclid. [Menn 2017] led us to the three prototypic propositions.

(area of a circle), and Archimedes proof that the circumference of a circle is proportional to the diameter. Hilbert showed that VI.1 is provable already in the first order HP5 (Part I). While Euclid implicitly relies on the Archimedean axiom, Archimedes makes it explicit in a recognizably modern form. Euclid does *not* discuss the circumference of a circle. To deal with that issue, Archimedes develops his notion of arc length. By beginning to calculate approximations of π, Archimedes is moving towards the treatment of π as a number. Consequently, we distinguish VI.1 (Euclid I) from the Archimedean axiom and the theorems on measurement of a circle, and place the latter in the Archimedean data set. The validation in the theories EG_π and $\mathcal{E}^2_\pi$ set out below of the formulas $A = \pi r^2$ and $C = \pi d$ answer questions of Hilbert and Dedekind not questions of Euclid though possibly of Archimedes. But, we think the theory EG_π is closer to the Greek origins than Hilbert's second-order axioms are.

Certainly EG_π goes beyond Euclidean geometry by identifying a straight line segment with the same length as the circumference of a circle (as Dedekind's or Birkhoff's postulates, discussed below, demand). This demand contrasts with earlier views such as Eutocius (4th century), 'Even if it seemed not yet possible to produce a straight line equal to the circumference of the circle, nevertheless, the fact that there exists some straight line by nature equal to it is deemed by no one to be a matter of investigation[24].' Although Eutocius asserts the existence of a line of the same length as a curve but finds constructing it unimportant, Aristotle has a stronger view. Summarizing his discussion of Aristotle, Crippa ([Crippa 2014a], 34-35) points out that Aristotle takes the impossibility of such equality as the hypothesis of an argument on motion and Crippa cites Averroes as holding 'that there cannot be a straight line equal to a circular arc'.

It is widely understood[25] that Dedekind's analysis is radically different from that of Eudoxus. A principle reason for this, discussed in Section 3, is that Eudoxus applies his method to specific situations; Dedekind demands that every cut be filled. Secondly, Dedekind develops addition and multiplication on the cuts. Thus, *Dedekinds's postulate should not be regarded as part of either Euclidean data set*. But EG_π makes a much more restrained demand; as in Eudoxus, a specific problem is solved.

The geometry over a Euclidean field (every positive number has a square root) may have no straight line segment of length π. E.g., the model over the surd field (Notation 1.1) does not contain π. Neither does the field of real algebraic numbers so $\mathcal{E}^2$ does not resolve the issue. We want to find a theory which proves the circumference

[24]Taken from his commentary on Archimedes in *Archimedes Opera Omnia cum commentariis Eutociis*, vol. 3, p. 266. Quoted in: Davide Crippa (Sphere, UMR 7219, Universit Paris Diderot) Reflexive knowledge in mathematics: the case of impossibility.

[25]Stekeler-Weithofer [Stekeler-Weithofer 1992] writes, "It is just a big mistake to claim that Eudoxus's proportions were equivalent to Dedekind cuts". Feferman [Feferman 2008] avers, "The main thing to be emphasized about the conception of the continuum as it appears in Euclidean geometry is that the general concept of set is not part of the basic picture, and that Dedekind style continuity considerations of the sort discussed below are at odds with that picture". Stein, though, gives a long (but to me unconvincing) argument for at least the compatibility of Dedekind's postulate with Greek thought "reasons . . . plausible, even if not conclusive- for believing the Greek geometers would have accepted Dedekinds's postulate, just as they did that of Archimedes, once it had been stated". [Stein 1990]

and area formulas for circles. Our approach is to extend the theory EG so as to guarantee that there is a point in every model which behaves as π does. For Archimedes and Euclid, sequences constructed in the study of magnitudes in the *Elements* are of geometric objects, not numbers. But, in a modern account, as we saw already while discussing areas of polygons in Part I, we must identify the proportionality constant and verify that it represents a point in any model of the theory[26]. Thus this goal diverges from a 'Greek' data set and indeed is complementary to the axiomatization of Cartesian geometry by Tarski's $\mathcal{E}^2$.

Euclid's third postulate, 'describe a circle with given center and radius', entails that a circle is uniquely determined by its radius and center. In contrast, Hilbert simply defines the notion of circle and proves the uniqueness. (See Lemma 11.1 of [Hartshorne 2000].) In either case we have the basic correspondence between angles and arcs: two segments of a circle are congruent if they cut the same central angle. We established in Part I that for each model of EG and any line of the model, the surd field F_s is embeddable in the field definable on that line. On this basis we can interpret the Greek theory of limits by way of cuts in the ordered surd field F_s. The following extensions, EG_π and $\mathcal{E}^2_\pi$, of the systems EG and $\mathcal{E}^2$ guarantee the existence of π as such a cut.

Axioms for π: Add to the vocabulary a new constant symbol π. Let i_n (c_n) be the perimeter of a regular $3 * 2^n$-gon inscribed[27] (circumscribed) in a circle of radius 1. Let $\Sigma(\pi)$ be the collection of sentences (i.e. a type[28])

$$i_n < 2\pi < c_n$$

for $n < \omega$. Now, we can define the new theories.

1. EG_π denotes the deductive closure of the following set of axioms in the vocabulary τ augmented by constant symbols $0, 1, \pi$.

   (a) the axioms EG of a Euclidean plane;

   (b) $\Sigma(\pi)$.

2. $\mathcal{E}^2_\pi$ is formed by adding $\Sigma(\pi)$ to $\mathcal{E}^2$ and taking the deductive closure.

Dicta on constants: Here we named a further single constant π. But the effect is very different than naming 0 and 1 (Compare the Dicta on constants just after Theorem 4.2.1 of Part I.) The new axioms specify the place of π in the ordering of the definable points of the model. So the data set is seriously extended.

[26]For this reason, Archimedes needs only his postulate while Hilbert would also need Dedekind's postulate to prove the circumference formula.

[27]I thank Craig Smorynski for pointing out that is not so obvious that that the perimeter of an inscribed n-gon is monotonic in n and reminding me that Archimedes avoided the problem by starting with a hexagon and doubling the number of sides at each step.

[28]Let $A \subset M \models T$. A *type* over A is a set of formulas $\phi(\mathbf{x}, \boldsymbol{a})$ where $\mathbf{x}$, ($\boldsymbol{a}$) is a finite sequence of variables (constants from A) that is consistent with T. Taking T as EG, a type over all F_s is a type over $\{0, 1\}$ since each element of F_s is definable over $\{0, 1\}$ in EG.

Theorem 3.1. *EG_π is a consistent but not finitely axiomatizable*[29] *incomplete theory.*

Proof. A model of EG_π is given by closing $F_s \cup \{\pi\} \subseteq \Re$ to a Euclidean field. To see the theory is not finitely axiomatizable, for any finite subset $\Sigma_0(\pi)$ of $\Sigma(\pi)$ choose a real algebraic number p satisfying Σ_0 when p is substituted for π; close $F_s \cup \{p\} \subseteq \Re$ to a Euclidean field to get a model of $EG \cup \Sigma_0$ which is not a model of EG_π. $\Box_{3.1}$

Dicta on Definitions or Postulates: We now extend the ordering on segments by adding the lengths of 'bent lines' and arcs of circles to the domain. Two approaches[30] to this step are:

a) introduce an explicit but inductive definition; or

b) add a new predicate to the vocabulary and new axioms specifying its behavior. This alternative reflects in a way the trope that Hilbert's axioms are *implicit definitions*.

We take approach a) in Definitions 3.2, 3.3 etc. using our established geometric vocabulary. Crucially, the following definition of bent lines (and thus the perimeter of certain polygons) is not a single formal definition but a schema of formulas ϕ_n defining an approximation for each n.

Definition 3.2. *Let $n \geq 2$. By a* bent line[31] *$b = Y_1 \dots Y_n$ we mean a sequence of straight line segments Y_iY_{i+1}, for $1 \leq i \leq n-1$, such that each end point of one is the initial point of the next.*

We specify the length *of a bent line $b = Y_1 \dots Y_n$, denoted by $[b]$, as the length given by the straight line segment composed of the sum of the segments of b. Now we say an* approximant *$Y_1, \dots Y_{n+1}$ to the arc $X_1 \dots X_n$ of a circle with center P, is a bent line satisfying:*

1. *$X_1, \dots X_n, Y_1, \dots Y_{n+1}$ are points such that the X_i are on the circle and each Y_i is in the exterior of the circle.*
2. *Each of Y_iY_{i+1} ($1 \leq i \leq n$), line segment.*
3. *For $1 \leq i \leq n$, Y_iY_{i+1} is tangent to the circle at X_i;*

We obtain the circumference of a circle by requiring $X_n = X_1$.

[29] Ziegler ([Ziegler 1982], Remark 2.1) shows that EG is undecidable. Since for any T and type $p(x)$ consistent with T, the decidability of $T \cup \{p(c)\}$ implies the decidability of T, EG_π is also undecidable.

[30] We could define $<$ on the extended domain or, in style b), we could add an $<^*$ to the vocabulary and postulate that $<^*$ extends $<$ and satisfies the properties of the definition.

[31] This is less general than Archimedes (page 2 of [Archimedes 1897]) who allows segments of arbitrary curves 'that are concave in the same direction'.

Definition 3.3. *Let $\mathcal{S}$ be the set (of congruence classes of) straight line segments. Let $\mathcal{C}_r$ be the set (of equivalence classes under congruence) of arcs on circles of a given radius r. Now we extend the linear order on $\mathcal{S}$ to a linear order $<_r$ on $\mathcal{S} \cup \mathcal{C}_r$ as follows. For $s \in \mathcal{S}$ and $c \in \mathcal{C}_r$*

1. *The segment $s <_r c$ if and only if there is a chord XY of a circular segment $AB \in c$ such that $XY \in s$.*
2. *The segment $s >_r c$ if and only if there is an approximant $b = X_1 \ldots X_n$ to c with length $[b] = s$ and with $[X_1 \ldots X_n] >_r c$.*

It is easy to see that this order is well-defined as each chord of an arc is shorter than the arc and the arc is shorter than any approximant to it. Now, we encode a second approximation of π, using the areas I_n, C_n of the approximating polygons rather than their perimeters i_n, c_n. There are two aspects to transferring the defintion from circumference to area: 1) modifying the development of the area function of polygons described in Section 4.5 of Part I, by extending the notion of figure to include sectors of circles and 2) formalizing a notion of equal area, including a schema for approximation of circles[32] by finite polygons. We omit those details analogous to 3.2–3.3. We carried out the harder case of circumference to emphasize the innovation of Archimedes in defining arc length; unlike area it is not true that the perimeter of a polygon containing a second is larger than the perimeter of the enclosed polygon.

Lemma 3.4. *Let I_n and C_n denote the area of the regular 3×2^n-gon inscribed or circumscribing the unit circle. Then EG_π proves*[33] *each of the sentences $I_n < \pi < C_n$ for $n < \omega$.*

Proof. The intervals (I_n, C_n) define the cut for π in the surd field F_s and the intervals (i_n, c_n) define the cut for 2π and it is a fact about the surd field that one half of any realization of the second cut is a realization of the first. $\square_{3.4}$

To argue that π, as implicitly defined by the theory EG_π, serves its geometric purpose, we add new unary function symbols C and A mapping our fixed line to itself and satisfying a scheme asserting that the functions these symbols refer to do, in fact, produce the required limits. The definitions are identical except for substituting the area for the perimeter of the approximating polygons. This strategy mimics that in an introductory calculus course of describing the properties of area and proving that the Riemann integral satisfies them.

Definition 3.5. *A unary function $C(r)$ (($A(r)$) mapping $\mathcal{S}$, the set of equivalence classes (under congruence) of straight line segments, into itself that satisfies the conditions below is called* a circumference function *(*area function).

[32]By dealing with a special case, we suppressed Archimedes' anticipation of the notion of bounded variation.

[33]Note that we have not attempted to justify the convergence of the i_n, c_n, I_n, C_n in the formal system EG_π. We are relying on mathematical proof, not a formal deduction in first order logic; we explain this distinction in item 4 of Section 4.3.

1. $C(r)$ ($A(r)$) is less than the perimeter (area) of a regular 3×2^n-gon circumscribing circle of radius r.

2. $C(r)$ ($A(r)$) is greater than the perimeter (area) of a regular 3×2^n-gon inscribed in a circle of radius r.

We can extend EG_π to include definitions of $C(r)$ and $A(r)$. The theory $EG_{\pi,A,C}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory EG_π, obtained by the explicit definitions: $A(r) = \pi r^2$ and $C(r) = 2\pi r$.

In any model of $EG_{\pi,A,C}$ for each r there is an $s \in \mathcal{S}$ whose length[34] $C(r) = 2\pi r$ is less than the perimeters of all circumscribed polygons and greater than those of the inscribed polygons. We can verify that by choosing n large enough we can make i_n and c_n as close together as we like (more precisely, for given m, make them differ by $< 1/m$). In phrasing this sentence I follow Heath's description[35] of Archimedes' statements, 'But he follows the cautious method to which the Greeks always adhered; he never says that a given curve or surface is the *limiting form* of the inscribed or circumscribed figure; all that he asserts is that we can approach the curve or surface as nearly as we please.'

Invoking Lemma 3.4, since the $2I_n(2C_n)$ converge to the limit of the $i_n(c_n)$, they determine the same cut, that of 2π:

Theorem 3.6. *In $EG^2_{\pi,A,C}$, $C(r) = 2\pi r$ is a circumference function and $A(r) = \pi r^2$ is an area function.*

In an Archimedean field there is a unique interpretation of π and thus a unique choice for a circumference function with respect to the vocabulary without the constant π. By adding the constant π to the vocabulary we get a formula which satisfies the conditions in every model. But in a non-Archimedean model, any point in the monad[36] of $2\pi r$ would equally well fit our condition for being the circumference.

To sum up, we have extended our descriptively complete axiomatization from the polygonal geometry of Hilbert's first-order axioms (HP5) to Euclid's results on circles and beyond. Euclid doesn't deal with arc length at all and we have assigned straight line segments to both the circumference and area of a circle. It follows that our development would not qualify as a *modest* axiomatization of Greek geometry but only of the modern understanding of these formulas. However, this distinction is not a problem for the notion of descriptive axiomatization. The facts are given as sentences. The formulas for circumference and area are not the same sentences as the Euclid/Archimedes statements in terms of proportions, but the Greek versions are implied by the modern equational formulations.

We now want to make a similar extension of $\mathcal{E}^2$. Dedekind ([Dedekind 1963],

[34] A similar argument works for area and $A(r)$.

[35] Archimedes, Men of Science [Heath 2011], Chapter 4.

[36] The *monad* of a is the collection of points that are an infinitessimal distance from a.

37-38) observes that the field of real algebraic numbers is 'discontinuous everywhere' but 'all constructions that occur in Euclid's *Elements* can ... be just as accurately effected as in a perfectly continuous space'. Strictly speaking, for *constructions* this is correct. But the proportionality constant π between a circle and its circumference is absent, so, it can't be the case that both a straight line segment of the same length as the circumference and the diameter are in the model[37]. We want to find a middle ground between the constructible entities of Euclidean geometry and Dedekind's postulation that all transcendentals exist. That is, we propose a theory which proves the circumference and area formulas for circles and countable models of the geometry over RCF, one, where 'arc length behaves properly'.

In contrast to Dedekind and Hilbert, Descartes eschews the idea that there can be a ratio between a straight line segment and a curve. [Crippa 2014b] writes, "Descartes excludes the exact knowability of the ratio between straight and curvilinear segments"; then he quotes Descartes:

> ... la proportion, qui est entre les droites et les courbes, n'est pas connue, et mesme ie croy ne le pouvant pas estre par les hommes, on ne pourroit rien conclure de là qui fust exact et assuré[38].

Hilbert[39] asserts that there are many geometries satisfying his axioms I-IV and V1 but only one, 'namely the Cartesian geometry' that also satisfies V2. Thus the conception of 'Cartesian geometry' changed radically from Descartes to Hilbert; even the symbol π was not introduced until 1706 (by Jones). One wonders whether it had changed by the time Hilbert wrote. That is, had readers at the turn of the 20th century already internalized a notion of Cartesian geometry which entailed Dedekind completeness and that was, at best, formulated in the 19th century (Bolzano-Cantor-Weierstrass-Dedekind)?

We now define a theory $\mathcal{E}^2_\pi$ analogous to EG_π that does not depend on the Dedekind axiom but can be obtained in a first-order way. Given Descartes' proscription of π, the new system will be immodest with respect to the Cartesian data set. But we will argue at the end of this section that both of our axioms for π are closer to Greek conceptions than the Dedekind Axiom. At this point we need some modern model theory to guarantee the *completeness* of the theory we are defining.

A first-order theory T for a vocabulary including a binary relation $<$ is *o-minimal* if every model of T is linearly ordered by $<$ and every 1-ary formula is equivalent in T to a Boolean combination of equalities and inequalities [Dries 1999]. Anachronistically, the o-minimality of the reals is a main conclusion of Tarski in [Tarski 1931]. We can now show.

[37]Thus, Birkhoff's protractor postulate (below) is violated.

[38]Descartes, Oeuvres, Vol. 6, p. 412. This is Crippa's translation of Descartes' archaic French. Crippa also quotes Averroes as emphatically denying the possibility of such a ratio and notes that Vieta held similar views.

[39]See pages 429-430 of [Hallett & Majer 2004].

Theorem 3.7. *Form $\mathcal{E}^2_\pi$ by adjoining $\Sigma(\pi)$ to $\mathcal{E}^2$. $\mathcal{E}^2_\pi$ is first-order complete for the vocabulary τ augmented by constant symbols $0, 1, \pi$.*

Proof. We have established that there is definable ordered field whose domain is the line through the points $0, 1$. By Tarski, the theory of this real closed field is complete. The field is bi-interpretable with the plane [Tarski 1998] so the theory of the geometry T is complete as well. Further by [Tarski 1931], the field is o-minimal. Therefore, the type over the empty set of any point on the line is determined by its position in the linear ordering of the subfield F_s (Notation 1.1). Each i_n, c_n is an element of the field F_s. The position of 2π in the linear order on the line through 01 is given by Σ. Thus $T \cup \Sigma(\pi)$ is a complete theory. $\square_{3.1}$

As we added π to EG, we now extend the theory $\mathcal{E}^2_\pi$.

Definition 3.8. *The theory $\mathcal{E}^2_{\pi,A,C}$ is the extension of the $\tau \cup \{0, 1, \pi\}$-theory $\mathcal{E}^2_{\pi,A}$ obtained by adding the explicit definitions: $A(r) = \pi r^2$ and $C(r) = 2\pi r$.*

Theorem 3.9. *The theory $\mathcal{E}^2_{\pi,A,C}$ is a complete, decidable extension of $EG_{\pi,A}$ that is coordinatized by an o-minimal field. Moreover, in $\mathcal{E}^2_{\pi,A,C}$, $C(r) = 2\pi r$ is a circumference function and $A(r) = \pi r^2$ is an area function.*

Proof. We are adding definable functions to $\mathcal{E}^2_\pi$ so o-minimality and completeness are preserved. The theory is recursively axiomatized and complete so decidable. The formulas continue to compute area and circumference correctly (as in Theorem 3.6) since they extend $EG_{\pi,A,C}$. $\square_{3.9}$

The assertion that π is transcendental is a theorem of the first order theory $\mathcal{E}^2_\pi$. As, Lindemann proved that π does not satisfy a polynomial of degree n for any n. Thus for any polynomial $p(x)$ over the rationals $p(\pi) \neq 0$ is a consequence of the complete type[40] generated by $\Sigma(\pi)$ and so is a theorem of $\mathcal{E}^2_\pi$. We explore this type of argument in point 4 of Section 4.3.

We now extend the known fact that the theory of real closed fields is 'finitistically justified' (in the list of such results on page 378 of [Simpson 2009]) to $\mathcal{E}^2_{\pi,A,C}$. For convenience, we lay out the proof with reference to results[41] recorded in [Simpson 2009].

The theory $\mathcal{E}^2$ is bi-interpretable with the theory of real closed fields. And thus it (as well as $\mathcal{E}^2_{\pi,A,C}$) is finitistically consistent, in fact, provably consistent in primitive recursive arithmetic (PRA). By Theorem II.4.2 of [Simpson 2009], RCA_0

[40] Recall that $\Sigma(x)$ is a consistent collection of formulas in one free variable, which by Tarski's quantifier elimination are Boolean combinations of polynomials.

[41] We use RCOF here for what we have called RCF before. Model theoretically adding the definable ordering of a formally real field is a convenience. Here we want to be consistent with the terminology in [Simpson 2009]. Note that Friedman [Friedman 1999] strengthens the results for PRA to exponential function arithmetic (EFA). Friedman reports Tarski had observed the constructive consistency proof much earlier. The theories discussed here, in increasing proof strength are EFA, PRA, RCA_0 and WKL_0.

proves the system $(Q, +, \times, <)$ is an ordered field and by II.9.7 of [Simpson 2009], it has a unique real closure. Thus the existence of a real closed ordered field and so $Con(RCOF)$ is provable in RCA_0. (Note that the construction will imbed the surd field F_s.)

Lemma IV.3.3 of [Friedman et al. 1983] asserts the provability of the completeness theorem (and hence compactness) for countable first-order theories from WKL_0. Since every finite subset of $\Sigma(\pi)$ is satisfiable in any RCOF, it follows that the existence of a model of $\mathcal{E}^2_\pi$ is provable in WKL_0. Since WKL_0 is π^0_2-conservative over PRA, we conclude PRA proves the consistency $\mathcal{E}^2_\pi$. As $\mathcal{E}^2_{\pi,C,A}$ is an extension by explicit definitions, its consistency is also provable in PRA, as required.

It might be objected that such minor changes as adding to $\mathcal{E}$ the name of the constant π, or adding the definable functions C and A undermines the earlier claim that $\mathcal{E}^2$ is descriptively complete for Cartesian geometry. But π is added because the modern view of 'number' requires it and increases the data set to include propositions about π which are inaccessible to $\mathcal{E}^2$.

We have so far tried to find the proportionality constant only in specific situations. In the remainder of the section, we introduce a model-theoretic scheme the systematize the solution of families of 'quadrature' problems. Crippa describes Leibniz' distinguishing two types of quadrature,

> ... universal quadrature of the circle, namely the problem of finding a general formula, or a rule in order to determine an arbitrary sector of the circle or an arbitrary arc; and on the other [hand] he defines the problem of the particular quadrature, ..., namely the problem of finding the length of a given arc or the area of a sector, or the whole circle ... (page 424 of [Crippa 2014a])

Birkhoff [Birkhoff, George 1932] ignores such a distinction with the *protractor postulate* of his system[42].

> POSTULATE III. The half-lines ℓ, m, through any point O can be put into $(1, 1)$ correspondence with the real numbers $a(\mathrm{mod} 2\pi)$, so that, if $A \neq O$ and $B \neq O$ are points of ℓ and m respectively, the difference $a_m - a_\ell(\mathrm{mod} 2\pi)$ is $\angle AOB$. Furthermore, if the point B varies continuously in a line r not containing the vertex 0, the number a_m varies continuously also[43].

This axiom is analogous to Birkhoff's 'ruler postulate' which assigns each segment a real number length. Thus, he takes the real numbers as an unexamined

[42] This is the axiom system used in virtually all U.S. high schools since the 1960's.
[43] I slightly modified the last sentence from Birkhoff, in lieu of reproducing the diagram.

background object; at one swoop he has introduced addition and multiplication, and assumed the Archimedean and completeness axioms. So even 'neutral' geometries studied on this basis are actually greatly restricted[44]. He argues that his axioms define a categorial system isomorphic to $\Re^2$. So his system (including an axiomatization of the real field that he hasn't specified) is bi-interpretable with Hilbert's.

However, the protractor postulate conflates three distinct problems: i) the rectifiability of arcs, the assertion that each arc of a circle has the same length as a straight line segment; ii) the claim there is an algorithm for finding such a segment; and iii) the measurement of angles, that is assigning a measure to an angle as the arc length of the arc it determines.

The next task is to find a more modest version of Birkhoff's postulate, namely, a first-order theory with countable models which assign to each angle a measure between 0 and 2π. Recall that we have a field structure on the line through the points $0, 1$ and the number π on that line, so we can make a further explicit definition.

A *measurement of angles* function is a map μ from congruence classes of angles into $[0, 2\pi)$ such that if $\angle ABC$ and $\angle CBD$ are disjoint angles sharing the side BC, $\mu(\angle ABD) = \mu(\angle ABC) + \mu(\angle CBD)$.

If we omitted the additivity property this would be trivial: Given an angle $\angle ABC$ less than a straight angle, let C' be the intersection of a perpendicular to BC through A with BC and let $\mu(\angle ABC) = 2\pi \cdot \sin(\angle ABC) = \frac{2\pi \cdot BC'}{AB}$. (It is easy to extend to larger angles.)

Here we use approach b) of the Dicta on Definitions rather than the explicit definition approach a) used for $C(r)$ and $A(r)$. We define a new theory with a function symbol μ which is 'implicitly defined' by the following axioms.

Definition 3.10. *The theory $\mathcal{E}^2_{\pi,A,C,\mu}$ is obtained by adding to $\mathcal{E}^2_{\pi,A,C}$, the assertion that μ is a continuous additive map from congruence classes of angles to $(0, 2\pi]$.*

It is straightforward to express that μ is continuous. If we omitted that requirement in Definition 3.10, $\mathcal{E}^2_{\pi,A,C,\mu}$ would be incomplete since μ would be continuous in some models, but not in some non-Archimedean models. Thus, we require continuity.

Showing consistency of $\mathcal{E}^2_{\pi,A,C,\mu}$ is easy; we can define (in the mathematical sense, not as a formally definable function) in $\mathcal{E}^2_{\pi,A,C}$ such a function μ^*, say, the restricted arc-cosine[45]. Hence, the axioms are consistent and this solves the rectifiability problem. But, merely assuming the existence of a μ does not solve our problem ii) as

[44] That is, they must be metric geometries.

[45] In fact, by coding a point on the unit circle by its x-coordinate and setting $\mu((x_1, y_1), (x_2, y_2)) = \cos^{-1}(x_1 - x_2)$ one gets such a function which definable in the theory of the real field expanded by the cosine function restricted to $(0, 2\pi]$ and in $E_{\pi,C,A}$ by Theorem 4.3.1 of Part 1. This theory is known to be o-minimal [Dries 1999]. But there is no known axiomatization and David Marker tells me it is unlikely to be decidable without assuming the Schanuel conjecture.

we have no idea how to compute μ and a recursive axiomatization is a real mathematical problem.

Some countable models of geometry

Blanchette [Blanchette 2014] distinguishes two approaches to logic, deductivist and model-centric and argues that Hilbert represents the deductivist school and Dedekind the model-centric. Essentially, the second proposes that theories are designed to try to describe an intuition of a particular structure. We now consider a third direction; are there 'canonical' models of the various theories we have been considering?

By modern tradition, the continuum is the real numbers and geometry is the plane over it. Is there a smaller model which reflects the geometric intuitions discussed here? For Euclid II, there is a natural candidate, the Euclidean plane over the surd field F_s. Remarkably, this does not conflict with Euclid XII.2 (the area of a circle is proportional to the square of the diameter). The model is Archimedean and π is not in the model. But Euclid only requires a proportionality which defines a type $\Sigma(x)$, not a realization π of $\Sigma(x)$. Plane geometry over the real algebraic numbers plays the same role for $\mathcal{E}^2$. Both are categorical in $L_{\omega_1,\omega}$. In the second case, the axiomatization is particularly nice: the Archimedean axiom and 'every field element is algebraic'.

We have developed a method of assigning measures to angles. Now we argue that the methods of this section better reflect the Greek view than Dedekind's approach does. Mueller ([Mueller 2006], 236) makes an important point distinguishing the Euclid/Eudoxus use from Dedekind's use of cuts. In broad outline, the following quotation describes the methodology here.

> One might say that in applications of the method of exhaustion the limit is given and the problem is to determine a certain kind of sequence converging to it, ...Since, in the *Elements* the limit always has a simple description, the construction of the sequence can be done within the bounds of elementary geometry; and the question of constructing a sequence for any given arbitrary limit never arises.

But what if we want to demand the realization of various transcendentals? Mueller's description suggests the principle that we should only realize cuts in the field order that are *recursive* over a finite subset. We might call these *Eudoxian transcendentals*. So a candidate would be a recursively saturated model[46] of $\mathcal{E}^2$. Remarkably, almost magically[47], this model would also satisfy $\mathcal{E}^2_{\pi,A,C,\mu}$. A recursively saturated model is necessarily non-Archimedean. There are however many different countable recursively saturated models depending on which transcendentals are realized

[46] A model is recursively saturated if every recursive type over a finite set is realized [Barwise 1975].

[47] The magic is called resplendency. Every recursively saturated model is resplendent [Barwise 1975] where M is resplendent if any formula $\exists A\phi(A, \boldsymbol{c})$ that is satisfied in an elementary extension of M is satisfied by some A' on M. Examples are the formulas defining C, A, μ.

Arguably there is a more canonical candidate for a natural model which admits the 'Eudoxian transcendentals'; take the smallest elementary submodel of $\Re$ closed[48] under A, C, μ that contains the real algebraic numbers and all realizations of recursive cuts in F_s. The Scott sentence[49] of this sentence is a categorical sentence in $L_{\omega_1,\omega}$. The models in this paragraph are all countable; we cannot do this with the Hilbert model of the plane over the real numbers; it has no countable $L_{\omega_1,\omega}$-elementary submodel.

We turn to the question of modesty. Mueller's distinction can be expressed in another way. Eudoxus provides a technique to solve certain problems, which are specified in each application. In contrast, Dedekind's postulate solves $2^{\aleph_0}$ problems at one swoop. Each of the theories $\mathcal{E}^2_{\pi}$, $\mathcal{E}^2_{\pi,A,C}$, $\mathcal{E}^2_{\pi,A,C,\mu}$ and the later search for their canonical models reflect this distinction. Each solves at most a countable number of recursively stated problems.

In summary, we regard the replacement of 'congruence class of segment', by 'length represented by an element of the field' as a *modest* reinterpretation of Greek geometry. But this treatment of length becomes *immodest* relative even to Descartes when this length is a transcendental. And *most immodest* of all is to demand lengths for arbitrary transcendentals.

4 And back to Hilbert

In this section we examine Hilbert's 'continuity axioms'. We study the syntactic form of various axioms, their consequences, and their role in clarifying the notion of the continuum. The Archimedean Axiom is minimizing; each cut is realized by at most one point so each model has cardinality at most $2^{\aleph_0}$. The Veronese postulate (Footnote 52) or Hilbert's *Vollständigkeitaxiom* is maximizing; each cut is realized; in the absence of the Archimedean axiom the set of realizations could have arbitrary cardinality.

4.1 The role of the Axiom of Archimedes in the *Grundlagen*

A primary aim expressed in Hilbert's introduction is 'to bring out as clearly as possible the significance of the groups of axioms.' Much of his book is devoted to this metamathematical investigation. In particular this includes Sections 9-12 (from [Hilbert 1971]) concerning the consistency and independence of the axioms. Further examples[50], in Sections 31-34, show that without the congruence axioms, the Axiom of Archimedes is necessary to prove what Hilbert labels as Pascal's (Pappus) theorem.

[48] Interpret A, C, μ on $\Re$ in the standard way.

[49] For any countable structure M there is a 'Scott' sentence ϕ_M such that all countable models of ϕ_M are isomorphic to M; see chapter 1 of [Keisler 1971].

[50] I thank the referee for pointing to the next two examples and emphasizing Hilbert's more general goals of understanding the connections among organizing principles. The reference to Dehn was dropped in later editions of the *Grundlagen*.

In the conclusion to [Hilbert 1962], Hilbert notes Dehn's work on the necessary role of the Archimedean Axiom in establishing over neutral geometry the relation between the number of parallel lines through a point and the sum of the angles of a triangle. These are all metatheoretical results. In contrast, the use of the Archimedean Axiom in Sections 19 and 21 to prove equidecomposable is the same as equicomplementable (equal content) (in 2 dimensions) is certainly a proof in the system. But an unnecessary one. As we argued in Section 4.4 of Part I, Hilbert could just have easily defined 'same area' as 'equicomplementable' (as is a natural reading of Euclid).

Thus, we find no *geometrical* theorems in the Grundlagen that essentially depend on the Axiom of Archimedes. Rather, Hilbert's use of the Axiom of Archimedes is i) to investigate the relations among the various principles and ii) in conjunction with the *Vollständigkeitsaxiom*, identify the field defined in the geometry with the independently existing real numbers as conceived by Dedekind. With respect to the problem studied here, I contend that these results do not affect the conclusion that Hilbert's full axiom set is an immodest axiomatization[51] of Euclid I or Euclid II or of the Cartesian data set since those data sets *contain* and are implied by the appropriate first-order axioms.

4.2 Hilbert and Dedekind on Continuity

In this section we compare various formulations of the completeness axiom. Hilbert wrote:

> Axiom of Completeness (Vollständigkeitsaxiom): To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid. [Hilbert 1971]

In this article we have used the following adaptation of Dedekind's postulate for geometry (DG):

> DG: Any cut in the linear ordering imposed on any line by the betweenness relation is realized.

While this formulation is convenient for our purposes, it misses an essential aspect of Hilbert's system; in a *context with a group*, DG implies the Archimedean Axiom, while Hilbert was aiming for an independent set of axioms. Hilbert's axiom

[51] It might seem I could claim immodesty for Archimedes as well, in view of my first order axioms for π. But that would be a cheat. I restricted that data set to Archimedes on the circle, while Archimedes proposed a general notion of arc length and studied many other transcendental curves.

does not imply Archimedes'. A variant VER[52] on Dedekind's postulate that does not imply the Archimedean Axiom was proposed by Veronese in [Veronese 1889]. If VER replaces DG, those axioms would also satisfy the independence criterion.

Hilbert's completeness axiom in [Hilbert 1971] asserting any model of the rest of the theory is maximal, is inherently model-theoretic. The later line-completeness [Hilbert 1962] is a technical variant[53]. Giovannini's account [Giovannini 2013], which relies on [Hallett & Majer 2004] includes a number of points already made here and three more. First, Hilbert's completeness axiom is not about deductive completeness (despite having such consequences), but about maximality of every *model* (page 145). Secondly (last line of 153) Hilbert expressly rejects Cantor's intersection of closed intervals axiom because it relies on a sequence of intervals and 'sequence is not a geometrical notion'. A third intriguing note is an argument due to Baldus in 1928 that the parallel axiom is an essential ingredient in the categoricity of Hilbert's axioms[54].

Here are two reasons for choosing Dedekind's (or Veronese's) version. One is that Dedekind's formulation, since it is about the geometry, not about its axiomatization, directly gives the kind of information about the existence of transcendental numbers that we observe in this paper. Even more basic is that one *cannot* formulate Hilbert's version as a sentence Φ_H in second-order logic[55] of geometry with the intended interpretation $(\Re^2, \mathbf{G}) \models \Phi_H$. The axiom requires quantification over subsets of an extension of the model which putatively satisfies it. Here is a second-order statement[56] Θ, where ψ denotes the conjunction of Hilbert's first four axiom groups and the axiom of Archimedes.

$$(\forall X)(\forall Y)\forall \mathbf{R})[[X \subseteq Y \wedge (X, \mathbf{R}\restriction X) \models \psi \wedge (Y, \mathbf{R}) \models \psi] \rightarrow X = Y]$$

whose validity expresses Hilbert's V.2 but which is a sentence in pure second order

[52] The axiom VER (see [Cantú 1999]) asserts that for a partition of a linearly ordered field into two intervals L, U (with no maximum in the lower L or minimum in the upper U) and a third set in between with at most one point, there is a point between L and U just if for every $e > 0$, there are $a \in L, b \in U$ such that $b - a < e$. Veronese derives Dedekind's postulate from his axiom and Archimedes in [Veronese 1889] and the independence in [Veronese 1891]. In [Levi-Civita 1892] Levi-Civita shows there is a non-Archimedean ordered field that is Cauchy complete. I thank Philip Ehrlich for the references and recommend section 12 of the comprehensive [Ehrlich 2006]. See also the insightful reviews [Pambuccian 2014a] and [Pambuccian 2014b], where it is observed that Vahlen [Vahlen 1907] also proved this axiom does not imply Archimedes.

[53] Since any point is in the definable closure of any line and any one point not on the line, one can't extend any line without extending the model. Since adding either the Dedekind postulate and or Hilbert completeness gives a categorical theory satisfied by a geometry whose line is order isomorphic to $\Re$ the two axioms are equivalent (over HP5 + Archimedes Axiom).

[54] Hartshorne (sections 40-43 of [Hartshorne 2000] gives a modern account of Hilbert's argument that replacing the parallel postulate by the axiom of limiting parallels gives a geometry that is determined by the underlying (definable) field. With Hilbert's V.2 this gives a categoricical axiomatization for hyperbolic geometry.

[55] Of course, this analysis is anachronistic; the clear distinction between first and second-order logic did not exist in 1900. By $\mathbf{G}$, we mean the natural interpretation in $\Re^2$ of the geometric predicates from Section 1.

[56] I am leaving out many details, $\mathbf{R}$ is a sequence of relations giving the vocabulary of geometry and the sentence 'says' they are relations on Y; the coding of the satisfaction predicate is suppressed.

logic rather than in the vocabulary for geometry. Väänänen investigates this anomaly by distinguishing (on page 94 of [Väänänen 2012]) between $(\Re^2, \mathbf{G}) \models \Phi$, for some Φ and the validity of Θ. He expounds in [Väänänen 2014] a new notion, 'Sort Logic', which provides a logic with a sentence Φ'_H which, by allowing a sort for an extension, formalizes Hilbert's V.2 with a more normal notion of truth in a structure.

In [Väänänen 2012, Väänänen & Wang 2017], Väänänen discusses the categoricity of natural structures such as real geometry when axiomatized in second-order logic. He develops the striking phenomena of 'internal categoricity[57]', which arguably refutes the view that second order categoricity 'depends on the set theory'. Suppose the second-order categoricity of a structure A is formalized by the existence of sentence Ψ_A such that $A \models \Psi_A$ and any two models of Ψ_A are isomorphic. If this second clause in provable in a standard deductive system for second-order logic, then it is valid in the Henkin semantics, not just the full semantics.

Philip Ehrlich has made several important discoveries concerning the connections between the two 'continuity axioms' in Hilbert and develops the role of maximality. First, he observes (page 172) of [Ehrlich 1995] that Hilbert had already pointed out that his completeness axiom would be inconsistent if the maximality were only with respect to the first-order axioms. Secondly, he [Ehrlich 1995, Ehrlich 1997] systematizes and investigates the philosophical significance of Hahn's notion of Archimedean completeness. Here the structure (ordered group or field) is not required to be Archimedean; the maximality condition requires that there is an extension which fails to extend an Archimedean equivalence class[58]. This notion provides a tool (not yet explored) for investigating the non-Archimedean models studied in Section 3.

In a sense, our development is the opposite of Ehrlich's in *The absolute arithmetic continuum and the unification of all numbers great and small* [Ehrlich 2012]. Rather than trying to unify all numbers great and small in a *class* model we are interested in the minimal collection of numbers that allow the development of a geometry that proved a *modest* axiomatization of the data sets considered.

4.3 Against the Dedekind Posulate for Geometry

Our fundamental claim is that (slight variants on) Hilbert's first-order axioms provide a modest descriptively complete axiomatization of most of Greek geometry.

One goal of Hilbert's continuity axioms was to obtain categoricity. But categoricity is not part of the data set but rather an external limitative principle. The notion that there was 'one' geometry (i.e. categoricity) was implicit in Euclid. But it is not a geometrical statement. Indeed, Hilbert described his metamathematical formulation of the completeness axiom (page 23 of [Hilbert 1962]) as, 'not of a purely geometrical

[57]See [Väänänen & Wang 2017] and Chapters 3 and 11.2 of [Baldwin 2017b].

[58]In an ordered group, a and b are *Archimedes-equivalent* if there are natural numbers m, n such that $m|a| > |b|$ and $n|b| > |a|$.

nature'. We argued in Section 3 of [Baldwin 2014] against the notion of categoricity as an independent desiderata for an axiom system. We noted there that various authors have proved under $V = L$, any countable or Borel structure can be given a categorical axiomatization and that there are no strong structural consequences of the mere fact of second order categoricity. However, there we emphasized the significance of axiomatizations, such as Hilbert's, that reveal underlying principles concerning such iconic structures as geometry and the natural numbers. Here we go further, and suggest that even for an iconic structure there may be advantages to a first-order axiomatization that trump the loss of categoricity.

We argue now that the Dedekind postulate is inappropriate (in particular immodest) in any attempt to axiomatize the Euclidean or Cartesian or Archimedean data sets for several reasons:

1. The requirement that there be a straight-line segment measuring any circular arc contradicts the intent of various pre-Cartesian geometers and of Descartes. It clearly extends the collection of geometric entities appearing in either Euclid or Descartes.

2. The Dedekind postulate is not needed to establish the properly geometrical propositions in the data set. Hilbert[59] closes the discussion of continuity with 'However, in what is to follow, no use will be made of the "axiom of completeness".' Why then did he include the axiom? Earlier in the same paragraph[60], he writes that 'it allows the introduction of limiting points' and enables one 'to establish a one-one correspondence between the points of a segment and the system of real numbers'. Archimedes Axiom makes the correspondence injective and the *Vollständigkeitsaxiom* makes it surjective. We have noted here that the grounding of real algebraic geometry (the study of systems of polynomial equations in a real closed field) is fully accomplished by Tarski's axiomatization. And we have provided a first-order extension to deal with the basic properties of the circle. Since Dedekind, Weierstrass, and others pursued the 'arithmetization of analysis' precisely to ground the theory of limits, identifying the geometrical line as the Dedekind line reaches beyond the needs of geometry.

3. Proofs from Dedekind's postulate obscure the true geometric reason for certain theorems. Hartshorne writes:

   > ... there are two reasons to avoid using Dedekind's axiom. First, it belongs to the modern development of the real number systems and notions of continuity, which is not in the spirit of Euclid's geometry. Second, it is too strong. By essentially introducing the real numbers into our geometry, it masks many of the more subtle distinctions and obscures questions such as constructibility that we will discuss

[59]Page 26 of [Hilbert 1971].

[60]For a thorough historical description, see the section *The Vollständigkeitsaxiom*, on pages 426-435 of [Hallett & Majer 2004]. We focus on the issues most relevant to this paper.

in Chapter 6. So we include the axiom only to acknowledge that it is there, but with no intention of using it. ([Hartshorne 2000], 177)

4. The use of second-order axioms undermines a key proof method – informal (semantic) proof – the ability to use higher order methods to demonstrate that there is a first order proof. A crucial advantage of a first-order axiomatization is that it licenses the kind of argument described in Hilbert and Ackerman[61]:

   > *Derivation of Consequences from Given Premises; Relation to Universally Valid Formulas*
   > So far we have used the predicate calculus only for deducing valid formulas. The premises of our deductions, viz Axioms a) through f), were themselves of a purely logical nature. Now we shall illustrate by a few examples the general methods of formal derivation in the predicate calculus . . . It is now a question of deriving the consequences from any premises whatsoever, no longer of a purely logical nature.
   >
   > The method explained in this section of formal derivation from premises which are not universally valid logical formulas has its main application in the setting up of the primitive sentences or axioms for any particular field of knowledge and the derivation of the remaining theorems from them as consequences . . . We will examine, at the end of this section, the question of whether every statement which would intuitively be regarded as a consequence of the axioms can be obtained from them by means of the formal method of derivation.

We noted that Hilbert proved that a Desarguesian plane embeds in 3 space by this sort of argument in Section 2.4 of [Baldwin 2013] and we exploited this technique in Section 3 to provide axioms for the calculation of the circumference and area of a circle[62].

In this article we expounded a procedure [Hartshorne 2000] to define the field operations in an arbitrary model of HP5. We argued that the first-order axioms of EG suffice for the geometrical data sets Euclid I and II, not only in their original formulation but by finding proportionality constants for the area formulas of polygon geometry. By adding axioms to require that the field is real closed we obtain a complete first-order theory that encompasses many of Descartes innovations. The plane over the real algebraic numbers satisfies this theory; thus, there is no guarantee that there is a line segment of length π. Using the o-minimality of real closed fields, we can guarantee there is such a segment by adding a constant for π and requiring it to realize the proper cut in the rationals. However, guaranteeing the uniqueness of such a realization requires the $L_{\omega_1,\omega}$ Archimedean axiom.

[61] Chapter 3, §11 Translation taken from [Blanchette 2014].

[62] Väänänen (in conversation) made a variant of it apply to those sentences in second-order logic that are internally categorical. He shows certain second-order propositions can be derived from the formal system of second-order logic by employing 3rd (and higher) order arguments to provide semantic proofs.

Hilbert and the other axiomatizers of 100 years ago wanted more; they wanted to secure the foundations of mathematical analysis. In full generality, this surely depends on second-order properties. But there are a number of directions of work on 'definable analysis[63]' that provide *first order* approaches to analysis. The study of o-minimal theories makes major strides. One direction of research in o-minimality has been to prove the expansion of the real numbers by particular functions (e.g. the Γ-function on the positive reals [Speissinger & van den Dries 2000]). Peterzil and Starchenko study the foundations of calculus in [Peterzil & Starchenko 2000]. They approach complex analysis through o-minimality of the real part in [Peterzil & Starchenko 2010]. The impact of o-minimality on number theory was recognized by a Karp prize (Peterzil, Pila, Wilkie, Starchenko) in 2014. And a non-logician [Range 2014], suggests using methods of Descartes to teach Calculus. A key feature of the interaction of o-minimal theories with real algebraic geometry has been the absence[64] of Dedekind's postulate for most arguments [Bochnak et al. 1998].

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[63] See Section 6.3 of [Baldwin 2017b].

[64] For an interesting perspective on the historical background of the banishment of infinitesimals in analysis see [Borovik & Katz 2012].

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