Completeness and Categoricity (in power): Formalization without Foundationalism

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Abstract

We propose a criterion to regard a property of a theory (in first or second order logic) as virtuous: the property must have explanatory power. Explanatory power is interpreted in pragmatic terms as having mathematical consequences for the theory. We then rehearse some unpublished results of Marek, Magidor, H. Friedman and Solovay to argue that for second order logic, 'categoricity' has little virtue. For first order logic, categoricity is trivial. But 'categoricity in power' has enormous structural consequences for any of the theories satisfying it. This virtue extends to other theories according to properties defining the stability hierarchy. We describe arguments using the properties, which essentially involve formalizing mathematics, to obtain results in 'core' mathematics. Further these methods (i.e. the stability hierarchy) provide an organization for much mathematics which more than fulfils a dream of Bourbaki.

1 What is the role of categoricity?

In correspondence in 2008, Michael Detlefsen raised a number of questions about the role of categoricity. We discuss two of them here renumbering, for convenience, in this paper.

> Question I (philosophical question)[1]: (A) Which view is the more plausible—that theories are the better the more nearly they are categor-

*I realized while writing that the title was a subconscious homage to the splendid historical work on Completeness and Categoricity by Awodey and Reck[3].

[1]These were questions III.A and III.B in the original letter. I thank Professor Detlefsen for permission to quote this correspondence.

ical, or that theories are the better the more they give rise to significant non-isomorphic interpretations?

(B) Is there a single answer to the preceding question? Or is it rather the case that categoricity is a virtue in some theories but not in others? If so, how do we tell these apart, and how to we justify the claim that categoricity is or would be a virtue just in the former?

I begin the main part of the paper by analyzing Question I.B. What does it mean for a property of a theory to be virtuous? That is, before one can decide the virtue of categoricity, one must clarify what is meant by virtue. In current model theoretic parlance 'categoricity' generally means 'categorical in some uncountable power' while in philosophy the traditional meaning: 'a theory has exactly one model' is retained. We clarify this terminology in a first brief section.

After providing criteria for a 'good or virtuous property' in Section 2, we argue in Section 3 that for second order logic categoricity is interesting for a few sentences describing particular structures. But this interest arises from the importance of those structures and not from any intrinsic consequence of 2nd order categoricity for arbitrary theories. Section 4 contains the main argument. Formalization impacts mathematical practice most directly not because of its foundational aspect but by the direct application of formal methods. In particular, the study of first order theories is a significant mathematical tool. And identifying virtuous properties of first order theories is a key part of that tool. In Subsection 4.1, we expound in detail the distinction (made for example in [15]) between the axiomatic method and formal methods. We argue that the study of formal theories (in particular by stability theoretic means) a) provides a scheme for organizing mathematics and b) is an effective mathematical tool. Sections 4.1 through 4.4 contains the argument for a). In Subsection 4.2, we argue that while categoricity is trivial for first order theories, *categoricity in power* is a notion with significant explanatory power. The mere fact a theory T is categorical in power tells us a great deal about the models T. Subsection 4.3 provides several example of completeness of a theory as mathematical tool. Further categoricity in power leads to a general dimension theory for models of first order theories with broad applications to organize and to do mainstream mathematics. We elaborate on this in Subsection 4.4. In Subsection 4.5 we provide some examples of the uses of model theory in mathematics stressing the connections to the properties of theories described above. In Subsection 4.6 we give a more extended treatment of one of the topics in Subsection 4.5, groups of finite Morley rank. We summarize the study of categoricity in $L_{\omega_1,\omega}$ in Subsection 5.1. In Subsection 5.2, we expound recent work of Hyttinnen, Kangas, and Väänänen that invoke the first order analysis to obtain striking results on categoricity in infinitary second order logic. Section 6 summarizes the argument.

2 Framing the Question

{quest}

We address Question I along two axes, 'Why is the theory studied and what is the logic?'. But first we analyze the underlying issue. What are the criteria by which one property of a theory is judged better than another? After fixing terminology in Subsection 2.1, we provide in Subsection 2.2 our criteria for what makes a property of theories virtuous.

2.1 Terminology

{term}

I fix the following terminology.

A theory T is a collection of sentences in some logic $\mathcal{L}$. (We will consider first order, second order, $L_{\omega_1,\omega}$, $L^2_{\kappa,\omega}$ and $L_{\omega_1,\omega}(Q)$.) I assume the existence of a semantics for each logic is defined in ZFC.

For simplicity, we will assume that T is consistent (has at least one model), has only infinite models, and is in a countable vocabulary.

T is *categorical* if it has exactly one model (up to isomorphism).

T is *categorical in power* κ if it has exactly one model in cardinality κ. T is *totally categorical* if it is categorical in every infinite power.

Definition 2.1.1. *1. A* deductive system *is complete if for every* ϕ

$$\vdash \phi \text{ if and only if } \models \phi.$$

2. A theory T *in a logic* $\mathcal{L}$ *is (semantically) complete if for every sentence* $\phi \in \mathcal{L}$

$$T \models \phi \text{ or } \models \neg\phi.$$

3. A theory T *in a logic* $\mathcal{L}$ *is (deductively) complete if for every sentence* $\phi \in \mathcal{L}$

$$T \vdash \phi \text{ or } \vdash \neg\phi.$$

Note that under these definitions, every categorical theory is complete. Further every theory in a logic which admits upward and downward Löwenheim-Skölem theorem for *theories* that is categorical in some infinite cardinality is complete. First order logic is the only one of our examples that satisfies this condition without any qualification.

Note that for any structure M and any logic $\mathcal{L}$, $\mathrm{Th}_{\mathcal{L}}(M) = \{\phi \in \mathcal{L} : M \models \phi\}$ is a complete theory.

2.2 Criteria for evaluating properties of theories

{criteria}

I take the word virtue in Question I to mean: the property of theories has explanatory value. For categoricity to have explanatory value it would have to be that some properties of theories are explained by them being categorical. And this should be more than just having only one model. We will note in Subsection 3 that the mere fact of categoricity has few consequences.

A property could be explanatory because it has useful equivalents. For example a first order theory T is model complete if every submodel N of a model M is an elementary submodel $N \prec M$. This is equivalent to every formula $\psi(\mathbf{x})$ is equivalent over T to a formula $(\exists \mathbf{y})\phi(\mathbf{x}, \mathbf{y})$ where ϕ is quantifier free. This version is of enormous help in analyzing the definable subsets of a model of T. Or as we'll see it could be because the property (e.g. categoricity in power) gives rich structural information about the models of T.

Usually we will interpret explanation[2] in a pragmatic vein. What are the consequences for the theory or for models of the theory that follow from this property. We will be particularly interested in 'structural properties', information about how the models of the theory are constructed from simpler structures. So a property of theories is virtuous if it has mathematical consequences beyond the mere fact asserted. In this context Shelah's notion of a *dividing line* provides those properties which are 'most virtuous'.

> A dividing line is not just a good property, it is one for which we have some things to say on both sides: the classes having the property and the ones failing it. In our context normally all the classes on one side, the "high" one, will be provably "chaotic" by the non-structure side of our theory, and all the classes on the other side, the "low" one will have a positive theory. The class of models of true arithmetic is a prototypical example for a class in the "high" side and the class of algebraically closed field the prototypical non-trivial example in the "low" side.[3]

We will argue in general that categoricity is not very virtuous. It's importance is as a qualifying sign that a theory describing a particular structure has succeeded. But for first order theories, *categoricity in power* is an important property of a theory. Further we will argue that restricting the number of models in a cardinal is a sign that the models of the theory have a strong structure theory.

[2]Our view of explanation differs from that of Kitcher or Steiner; this is a matter for further study. For now, just note that in general Kitcher tries to evaluate the explanatory value of an entire system and Steiner discusses the explanatory value of a particular proof. In contrast, we look at the family of all properties of a theory and try to compare the relative explanatory value of different properties.

[3]See page 3 of [56]. Shelah elaborates this theme in Section 2, 'For the logically challenged' of the same chapter.

3 Categoricity of Second Order Theories

{2nd}

As we argue in Section 4.1, Löwenheim-Skolem conditions undermine the significance of categoricity in logics weaker than second order. In this section we argue against categoricity per se as a significant property of second order theories, while acknowledging the importance of noticing certain *axiomatizations* are categorical.

We need to distinguish here between the categoricity of an axiomatization[4] and the categoricity of a theory. One aim of axiomatization is to describe a particular, fundamental structure. There are really very few such structures. In addition to the reals and $\mathbb{N}$ one could add $\mathbb{Z}, \mathbb{C}, \mathbb{Q}$ and of course real geometry. In the 20th century such structures as the p-adic numbers enter the canon. Given the ease described below of obtaining complete second order theories, categoricity is a necessary condition for calling a second order axiomatization of a theory successful. But it is not sufficient[5]. The goal of an axiomatization is to illuminate the central intuitions about the structure. The real linear order could be given a categorical axiomatization by describing the construction of the rationals from the natural numbers and then the reals as Cauchy sequences of rationals. As pointed out in [62], this construction takes place in $V_{\omega+7}$. A more direct categorical axiomatization of the real order is as a complete linear order with a countable dense subset. This axiomatization highlights the properties needed for the foundations of calculus[59]. Thus *the interest in categoricity is not really that the theory is categorical but in the particular axiomatization that expresses the intuitions about the target structure*.

From the perspective of providing a *unique* description of our intuitions, even a categorical second order axiomatization (of say the reals) is subject to attack from radically different perspectives (e.g. constructive mathematics or Ehrlich's absolute continuum[18]).

It is evident that *categoricity* of $\mathrm{Th}(M)$ implies $\mathrm{Th}(M)$ is semantically complete for any logic; the converse fails if the logic has only a set of theories as there are a proper class of structures. Many *second order* theories are categorical. Consider the following little known results[6]:

Marek-Magidor (V=L) The second order theory of a countable structure is categorical.

H. Friedman (V=L) The second order theory of a Borel structure is categorical.

[4]Following modern model theoretic practice, I say a class is $\mathcal{L}$-axiomatizable if it is the class of models of a set of $\mathcal{L}$-sentences. If I want to say recursively axiomatized I add this adjective.

[5]This perspective is highlighted by Huntington's initial name for categoricity, sufficiency, (See page 16 of [3]).

[6]These results appeared in a paper by Marek[37], its review by Magidor, and a thread on Fom (`http://cs.nyu.edu/pipermail/fom/2006-May/010544.html`). They were summarised by Ali Enayat at `http://mathoverflow.net/questions/72635/categoricity-in-second-order-logic/72659#72659`. Solovay's forcing argument for independence is at `http://cs.nyu.edu/pipermail#fom/2006-May/010549.html`.

Solovay (V=L) A recursively axiomatizable complete 2nd order theory is categorical.

Solovay It is consistent with ZFC that there is a complete finitely axiomatizable second order theory that is not categorical.

The first two results concern the 2nd-order theory of a structure. They show that any structure which is easily described (countable or Borel) has a categorical theory. Awodey and Reck [3] point out that Carnap provided (as he realized) a false proof that every finitely axiomatized complete 2nd order theory is categorical. Solovay's second result above shows this question cannot have a positive answer in ZFC.

To summarize these results, if a second order theory is complete and easily described (recursively axiomatized) or has an intended model which is an easily described structure (countable or Borel) then it is categorical. The fact that the most fundamental structures were categorical[7] may partially explain why it took so long for the distinction between complete and categorical to arise. As reported in [3], Fraenkel [21] had distinguished these notions in a context of higher order logic without establishing that they are really distinct[8]. {fn1}

One might argue that category, that is provable in ZFC, is hard to achieve. But for this argument to have much weight, one would have to get around two facts. 1) Consistently, categoricity is easy to achieve. Even in ZFC, there are many examples of categorical structures: various ordinals, the least inaccessible, the Hanf number of second order logic etc., etc. 2) Second order categoricity tells us nothing about the internal 'algebraic' properties of the structure. So the fact that a second order theory is categorical provides little information.

Bourbaki (page 230 of [15]) wrote[9],

> Many of the latter (mathematicians) have been unwilling for a long time to see in axiomatics anything other else than a futile logical hairsplitting not capable of fructifying any theory whatever. This critical attitude can probably be accounted for by a purely historical accident. The first axiomatic treatments and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories, *i.e.* theories which are entirely determined by their complete systems of axioms; for this reason they could not be applied to any theory except the one from which they had been abstracted (quite contrary to what we have seen, for instance, for the theory of groups). If the

[7] We follow current model theoretic practice and label a structure with any property that is satisfied by its complete first theory. We extend this practice by saying e.g. M is 2nd-order categorical when the 2nd order theory of M is categorical.

Indeed the recent results cited above show that under $V = L$ each of the fundamental structures had to be categorical.

[8] More precisely Awodey and Reck point out that in the 2nd edition of Fraenkel's book (1923) he had distinguished between categoricity and deductive completeness. In the 3rd (1928) edition he also clarifies the distinction between syntactic and semantic completeness.

[9] Parenthetical references added to shorten quotation.

same had been true of all other structures, the reproach of sterility brought against the axiomatic method, would have been fully justified.

Bourbaki misses two critical observations about the examples of Dedekind, Peano, and Hilbert: the axiomatizations are second order and they were explicitly intended to describe certain vital structures. Nevertheless from their standpoint of investigating the impact on mathematics at large, Bourbaki reasonably ascribes sterility to such specific axiomatizations; the goal of the axiomatizers was to understand the given structure not generalization. Even an insightful categorical axiomatization is expected only to explain the given structure, not to organize arguments about other structures. The other arguments mentioned above providing many other categorical second order structures don't even have this benefit. Their real significance is to understanding second order logic, telling some kinds of structures it can code (at least under $V = L$)[10]. Even though (at least in L) there are many 2nd-order categorical structures, this fact tells us nothing much about such structures. They have fairly simple descriptions, but not in a way that the reflects the properties of the structure. There is no uniform consequence of the statement that the second order theory T is categorical beyond 'it has only one model'. We will see the situation is far different for first order logic and categoricity in power.

4 First order logic

{effform}

In this section we will see that there are a family of properties of first order theories which are virtuous in the sense of Subsection 2.2. Most were generated by Shelah's search for dividing lines in the spectrum of first order theories. In the first section, we distinguish between two uses of formalization. We introduce the notion of obtaining structural information about the models of a theory T by global properties[11] of T with the motivating example of categoricity in power (Subsection 4.2). Subsection 4.3 discusses how completeness yields connections among the models of such a complete theory and thus complete theories become the natural class of theories to study. In addition the classification imposed by these properties has a small finite number classes yielding a hierarchy of theories. We introduce in Subsection 4.4 various formal (sytactic) properties that explain common properties of classes of complete first order theories and so are virtuous in the sense of Subsection 2.2. Finally in Subsection 4.5, we explore how the properties described earlier in the Section are exploited in current mathematical research.

[10]The 'idea' of the arguments presented on fom (See footnote 3) is that for well-ordered structures one can express in 2nd order logic that a model is minimal (no initial segment is a model) provided that the axiomatization can be properly coded. The coding can be done in L. A similar approach (by Scott) proving that semantic completeness does imply categoricity for pure second order logic is Proposition 3 of [4].

[11]Here a global property of a theory such as categoricity or a place in the stability hierarchy is distinguished from local properties of the models of T.

4.1 Formal methods as a mathematical tool

{formal}

In [31] and [9], Juliette Kennedy and I have discussed the development of some 'formalism-free' approaches in logic and especially in model theory. The goal of this section is to show how formal symbolic logic plays an increasingly important role in core mathematical investigations and provides schemes for organizing mathematics aimed not at finding foundations but at organization around mathematical ideas that link apparently diverse areas of mathematics[12]. We will see in Section 4.2 that these ideas develop from appropriate weakenings of categoricity and that they provided an unexpected fulfillment of some hopes of Bourbaki. We give our definition of formalization in detail to come to grips with the thoughtful paper of Bourbaki [15] entitled 'The Architecture of Mathematics'.

Definition 4.1.1 (Formalization)**.** *A* full formalization *involves the following components.*

1. *Vocabulary: specification of primitive notions.*
2. *Logic:*
   (a) *Specify a class*[13] *of well formed formulas.*
   (b) *Specify truth of a formula from this class in a structure.*
   (c) *Specify the notion of a formal deduction for these sentences.*
3. *Axioms: Specify the basic properties of the situation in question by sentences of the logic.*

What does this definition have to do with core mathematics? A leading representation theorist David Kazhdan, in the first chapter, logic, of his lecture notes on motivic integration [30], writes:

> One difficulty facing one who is trying to learn Model theory is disappearance of the natural distinction between the formalism and the substance. For example the *fundamental existence theorem* says that the syntactic analysis of a theory [the existence or non- existence of a contradiction] is equivalent to the semantic analysis of a theory [the existence or non-existence of a model].

At first glance this statement struck me as a bit strange. The fundamental point of model theory is the distinction between the syntactic and the semantic. On reflection[14], it seems that Kazhdan is making a crucial point which I elaborate as follows.

[12] Formalization is of course a crucial tool for foundational studies in the traditional global sense. We are expounding another use of that tool.

[13] For most logics there are only a set of formulas, but some infinitary languages have a proper class of formulas.

[14] I thank Udi Hrushovski, Juliette Kennedy, and David Marker for illuminating correspondence on this issue.

The separation of syntax and semantics is a relatively recent development. Dedekind understands it late in the 19th century [15]. But it is really clearly formulated as a tool only in Hilbert's 1917-18 lectures [58]. And immediately Hilbert smudges the line in one direction by seeing the 'formal objects' as mathematical. As Sieg (page 75 of [57] writes,

> But it was only in his paper of 1904, that Hilbert proposed a radically new, although still vague, approach to the consistency problem for mathematical theories. During the early 1920's he turned the issue into an elementary arithmetical problem...

Hilbert has begun the study of metamathematics, considering the formal language and its deductive relations as mathematical objects. Thus syntactical analysis is regarded as a study of mathematical objects (substance)[16].

The first great role of formalization aims to provide a global foundation for mathematics. The Hilbert program treats the syntax as a mathematical 'substance'.

But Kazhdan is commenting on a smudge in the other direction; to prove the completeness theorem, Gödel constructs a model (a mathematical object) from the syntactical formulas[17].

When one views the completeness theorem solely from the standpoint of logic, the construction of 'models' from syntactic objects to make a statement about syntactic objects is less jarring. The surprise is when a real mathematical object arises from the syntactic paraphernalia. Marker e-mailed, 'I've found when lecturing that a similar stumbling block comes when giving the model theoretic proof of the Nullstellensatz (page 88 of [39]) or Hilbert's 17th Problem when the variables in the polynomial become the witnessing elements in a field extension.'

In [30], Kazhdan is not concerned with the global foundations of mathematics; he is concerned with laying a foundation for the study of motivic integration. This is an example of the second great application of formalization: *By specifying the primitive concepts involved in a particular area of mathematics and postulating the crucial insights of that field (usually thought of as defining the concepts implicitly in Hilbert's sense), one can turn the resources of 'formalization' on the analysis of 'normal' mathematical problems.*

These resources include the completeness theorem, quantifier elimination, techniques of interpreting theories, and the entire apparatus of stability theory.

[15] See footnote 11 of [57]

[16] Even earlier this translation is seen even in the title of Post's 1920 thesis [45].

[17] In fact this blurring is frequent in the standpoint of the Schröder school of algebraic logic. Badesa makes this point in [6]; his argument is very clearly summarized in [2]. The combinatorics of the proofs of Löwenheim and Skolem are very close to those of Gödel. But Gödel makes the distinction between the syntactic and semantic clear since the warrant for the existence of the syntactic configuration is that it does not formally imply a contradiction.

Bourbaki [15] distinguishes between 'logical formalism' and the 'axiomatic method', which, as they are too modest to say, is best exemplified by the Bourbaki treatise[18] 'We emphasize that it (logical formalism) is but one aspect of this (the axiomatic) method, indeed the least interesting one[19]'. In part, this remark is a reaction to the great pedantry of early 20th century logic as the language of mathematics was made rigorous. It also is a reaction to the use of logic only for foundational purposes in the precise sense of finding a universal grounding for mathematics[20]. Bourbaki is reacting against a foundationalism which sacrifices meaning for verifiability. The coding of mathematics into set theory performs a useful function of providing a basis; unfortunately, the ideas are often lost in the translation. In contrast, the second role of formalization described above provides a means for analysis of ideas in different areas of mathematics.

In his remarks at the Vienna Gödel centenary symposium in 2006, Angus Macintyre wrote[33],

> That the 1931 paper had a broad impact on popular culture is clear. In contrast, the impact on mathematics beyond mathematical logic has been so restricted that it is feasible to survey the areas of mathematics where ideas coming from Gödel have some relevance.

This sentence unintentionally makes a false identification[21]. Macintyre's paper surveys the areas of mathematics where he sees the ideas coming from 'the 31 paper' have some relevance. But incompleteness is not the only contribution of Gödel. In Subsection 4.5 we barely touch the tip of the iceberg of results across mathematics that develop from the Gödel completeness theorem and the use of formalization as we describe above.

It is perhaps not surprising that in 1939, Dieudonne sees only minimal value of formalization, 'le principal mérite de la méthode formaliste sera d'avoir dissipé les obscurités qui pesaieant encore sur la pensée mathématique'; the first true application of the compactness theorem in mathematics occurs only in Malcev's 1941 paper [35][22].

Bourbaki [15] hints at an 'architecture' of mathematics by describing three great 'types of structures': algebraic structures, order structures, and topological structures. As we describe below in Subsection 4.4, the methods of stability theory provide a much more detailed and useful taxonomy which provides links between areas that

[18] Mathias [40] has earlier made a more detailed and more emphatic but similar critique to ours of Bourbaki's foundations.

[19] Parenthetical remarks added. Page 223 of [15].

[20] I use the term global foundations for this study.

[21] Macintyre has confirmed via email that he intended only to survey the influences of the incompleteness results.

[22] Malcev writes, 'The general approach to local theorems does not, of course, give the solutions to any difficult algebraic theorems. In many cases, however, it makes the algebraic proofs redundant.' Malcev goes on to point out that he significantly generalizes one earlier argument and gives a uniform proof for all cardinalities of an earlier result of Baer that held only for countable groups.

were not addressed by the Bourbaki standpoint.

4.2 Categoricity in Power

{cat}

Categoricity is trivial for first order logic. All and only finite structures are categorical in $L_{\omega,\omega}$. The interesting notion is 'Categoricity in (uncountable) Power'. The upward and downward Löwenheim-Skolem theorems show that for first order theories, categoricity in power *implies* completeness of $\mathrm{Th}(M)$. Ryll-Nardjewski[47], characterized first order theories that are $\aleph_0$-categorical. Unlike the second order case, this theorem contains a lot of information. We don't detail all the technical definitions in the following description; they can be found in introductory texts in model theory such as [39].

Theorem 4.2.1. *The following are equivalent.*

1. *T is $\aleph_0$-categorical.*
2. *T has only countably many finite n-types for each n.*
3. *T has only finitely many inequivalent n-ary formulas for each n.*
4. *T has a countable model that is both prime and saturated.*

But, there are still wildly different kinds of theories that are $\aleph_0$-categorical. The theory of an (infinite dimensional) vector space over a finite field differs enormously from the theory of an atomless Boolean algebra, the random graph or a dense linear order. But in the 1960's there was no way to make this difference precise. One distinction stands out; only the vector space is categorical in an uncountable power.

We discuss below the role of 'admitting a structure theory'. Theories that are categorical in an uncountable power have the simplest kind of structure theory and their study led to a more general analysis. The following theorem summarizes the basic landscape ([41, 10, 69]). {catchar}

Theorem 4.2.2. *(Morley/ Baldwin-Lachlan/Zilber) The following are equivalent:*

1. *T is categorical in one uncountable cardinal.*
2. *T is categorical in all uncountable cardinals.*
3. *T is ω-stable and has no two cardinal models.*
4. *Each model of T is prime over a strongly minimal set.*
5. *Each model of T can be decomposed by finite 'ladders' of strongly minimal sets*[23].

[23] Zilber shows certain automorphism groups (the linking groups of the strongly minimal sets) are first order definable; this leads to the definability of the field in certain groups of finite Morley rank (See Subsection 4.6.

First order theories which are *totally categorical* have a much stronger structure theory. Zilber's quest to prove that no totally categorical theory is finitely axiomatizable[68, 17] not only gave a detailed description of the models of such theories but sparked 'geometric stability theory'. Further, to eliminate the classification of the finite simple groups from the proof Zilber gave new proofs of the classification of two transitive groups[20, 66, 67].

Because of Morley's theorem we can say $\aleph_1$-categorical for a theory which is categorical in one (and therefore) all uncountable cardinals. In addition [10] shows an $\aleph_1$-categorical theory has either 1 or $\aleph_0$ models in $\aleph_0$. Since by 4 and 5 of Theorem 4.2.2, strongly minimal sets are the building blocks of uncountably categorical theories, we should describe them. {smgeo}

Definition 4.2.3. *Let T be a first order theory. A definable subset X of a model is T is* strongly minimal *if every definable subset $\phi(x, \boldsymbol{a})$ of X is finite or cofinite (uniformly in $\boldsymbol{a}$).*

A theory T is strongly minimal if the set defined by $x = x$ is strongly minimal in T.

The notion of a *combinatorial geometry* generalizes such examples as vector space closure or algebraic closure in fields. An essential contribution of model theory is to find such geometries in many different contexts. {geodef}

Definition 4.2.4. *A* pregeometry *is a set G together with a dependence relation*

$$cl : \mathcal{P}(G) \to \mathcal{P}(G)$$

satisfying the following axioms.

A1. $cl(X) = \bigcup\{cl(X') : X' \subseteq_{fin} X\}$

A2. $X \subseteq cl(X)$

A3. *If $a \in cl(Xb)$ and $a \notin cl(X)$, then $b \in cl(Xa)$.*

A4. $cl(cl(X)) = cl(X)$

If points are closed the structure is called a geometry.

Note that the preceding is a mathematical (formalism-free) concept. The next definition and theorem provide formal (syntactic conditions) on a theory for its models to be combinatorial geometries under an appropriate notion of closure.

Definition 4.2.5. *$a \in \mathrm{acl}(B)$ (algebraic closure) if for some $\phi(a, \mathbf{y})$ and some $\boldsymbol{b} \in B$, $\phi(a, \boldsymbol{b})$ and $\phi(x, \boldsymbol{b})$ has only finitely many solutions.*

{smgeoth }

Theorem 4.2.6. *A complete theory T is strongly minimal if and only if it has infinite models and*

1. *algebraic closure induces a pregeometry on models of T;*
2. *any bijection between acl-bases for models of T extends to an isomorphism of the models*

The second condition is often rendered as, 'the pre-geometry is homogeneous'; it is equivalent to say all independent sets of the same cardinality realize the same type.

Thus the syntactic condition about the number of solutions of formulas leads to the existence of a geometry and a dimension for each model of the theory.

The complex field or an infinite vector space over any field is strongly minimal. By Theorem 4.2.3 all strongly minimal theories have a dimension similar to that of vector spaces or the transcendence dimension in field theory. The dimension of a model of an arbitrary $\aleph_1$-categorical theory is the dimension of the strongly minimal set over which it is prime by Theorem 4.2.2.2. Thus a theory is $\aleph_1$-categorical if and only if each model is determined by a single dimension[24]. Getting closer to the Bourbaki ideal, Zilber conjectured that all strongly minimal sets had a trivial geometry or were 'vector-space like'(modular) or 'field-like' (nonmodular). (See [68] for a summary.) In his Gödel lecture[72], Zilber sketches his motivations and outlines the program. Sections 5 and 6 of Pillay's survey, Model Theory[42], gives an accessible and more detailed account of this program, which is a refinement of Shelah's general classification program for first order theories. Hrushovski [26] refuted this conjecture but Hrushovski and Zilber [27] began launched a program to rescue the conjecture (and better attune model theory to algebraic geometry). They analyzed the counterexamples that were at first sight pathologies (more of Bourbaki's monsters) by showing exactly how close 'ample Zariski structures' are to being algebraically closed fields. Zilber [72] lays out a detailed account of the further development of this second program.

Moreover, this dimension theory extends to more general classes than $\aleph_1$-categorical. For ω-stable theories a dimension (Morley rank) can be defined on all definable subsets similar to (and specializing to in the case of algebraically closed fields) the notion of dimension in algebraic geometry.

If a theory is viewed as axiomatizing the properties of specific structure (e.g. the complex field) categoricity in power is the best approximation that first order logic can make to categoricity. But, it turns out to have far more profound implications than categoricity for studying the original structure. If the axioms are universal existential

[24]For uncountable models this dimension is the same as the cardinality of the model; [10] show that for countable models either every model has dimension $\aleph_0$ or there are models of infinitely many finite dimensions.

then the theory is model complete (and under slightly more technical conditions admits elimination of quantifiers). Thus the complexity of definable sets is determined by global properties of the models. What can be very technical proofs of quantifier elimination by induction on quantifiers are replaced by more direct arguments for categoricity.

4.3 Complete Theories

{comp}

Before turning to complete theories, we note that the notion of a theory provides a general method for studying 'families of mathematical structures'. The most obvious example is that algebraic geometers want to study 'the same' variety over different fields. This is crisply described as the solution in the field k of the equations defining the variety. Similarly the Chevalley groups can be seen as the matrix groups, given by a specific definition interpreted as solutions in each field. For finite fields, this gives 'most' of the non-exceptional finite simple groups [34]. These are examples of affine schemes. In introducing the notion of an affine group scheme, Waterhouse [63] begins with a page and a half of examples and defines a group functor. He then writes,

> The crucial *additional*[25] property of our functors is that elements of $G(R)$ are given by finding solutions in R of some family of equations with coefficients in k. ... Affine group schemes are exactly the group functors constructed by solutions of equations. But such a definition would be technically awkward, since quite different collections of equations can have essentially the same solutions.

He follows with another page of proof and then a paragraph with a slightly imprecise (the source of the coefficients is not specified) definition of affine group scheme over a field k. Note that (for Waterhouse) a k-algebra is a commutative ring R with unit that extends k. Now, for someone with a basic understanding of logic, an affine group scheme over R is a collection of equations ϕ over k that define a group under some binary operations defined by equations ψ. The group functor F sends R to the subgroup defined in R defined by those equations[26]. The key point is that, not only is the formal version more perspicuous, it underlines the fundamental notion. Definability is not an 'addition'; The group functor aspect is a consequence of the equational definition.

We give several examples that illustrate the mathematical power of the notion of complete theory, demonstrating that completeness is a virtuous property. As we said

[25] My emphasis. See next paragraph.

[26] We are working with the incomplete theory T in the language of rings with names for the elements of k. Two finite systems of equations over k, $\sigma(\mathbf{x})$ and $\tau(\mathbf{x})$, are equivalent if $T \vdash (\forall \mathbf{x})\sigma(\mathbf{x}) = \tau(\mathbf{x})$. The functorial aspects of F are immediate from the preservation of positive formulas under homomorphism. Note that I am taking full advantage of k being a field by being able to embed k in each k-algebra. And to be fair to Waterhouse's longer exposition he is introducing further terminology that is useful in the development.

above, axiomatic theories arise from two distinct motivations. One is to understand a single significant structure such as $(N, +, \cdot)$ or $(R, +, \cdot)$. The other is to find the common characteristics of a number of structures; theories of the second sort include groups, rings, fields etc. In the second case, there is little gained simply from knowing a class is axiomatized by first order sentences. In general, the various completions of the theory simply provide too many alternatives. But for complete theories, the models are sufficiently similar so information can be transferred from one to another.

Detlefsen asked.

> Question II (philosophical question)[27]: Given that categoricity can rarely be achieved, are there alternative conditions that are more widely achievable and that give at least a substantial part of the benefit that categoricity would? Can completeness be shown to be such a condition? If so, can we give a relatively precise statement and demonstration of the part of the value of categoricity that it preserves?

We argue that the notion of completeness provides a formidable mathematical tool. One that is not sterile but allows for the comparison in a systematic way structures that are closely related but not isomorphic. So we argue that *completeness* gives substantial benefits that approximate the benefits of categoricity in power (but are completely impossible for categorical theories.)

Kazhdan [30] illuminates the key reason to study complete theories:

> On the other hand, the Model theory is concentrated on gap between an abstract definition and a concrete construction. Let T be a complete theory. On the first glance one should not distinguish between different models of T, since all the results which are true in one model of T are true in any other model. One of main observations of the Model theory says that our decision to ignore the existence of differences between models is too hasty. Different models of complete theories are of different flavors and support different intuitions. So an attack on a problem often starts which a choice of an appropriate model. Such an approach lead to many non-trivial techniques for constructions of models which all are based on the compactness theorem which is almost the same as the fundamental existence theorem.
>
> On the other hand the novelty creates difficulties for an outsider who is trying to reformulate the concepts in familiar terms and to ignore the differences between models.

The next two examples use the concept of a complete theory to transfer results between structures in way that was impossible or *ad hoc* without the formalism.

[27]This was question IV in the Detlefsen letter.

Example 4.3.1 (Ax-Grothedieck Theorem)**.** The Ax-Grothendieck theorem [5, 23] asserts an injective polynomial map on an affine algebraic variety over $\mathbb{C}$ is surjective. The model theoretic proof[28] [5]observes the condition is axiomatized by a family of 'for all -there exist' first order sentences ϕ_i (one for each pair of a map and a variety). Such sentences are preserved under direct limit and the ϕ_i are trivially true on all finite fields. So they hold for the algebraic closure of F_p for each p (as it is a direct limit of finite fields). Note that $T = \mathrm{Th}(\mathbb{C})$, the theory of algebraically closed fields of characteristic 0, is axiomatized by a schema Σ asserting each polynomial has a root and stating for each p that the characteristic is not p. Since each ϕ_i is consistent with every finite subset of Σ, it is consistent with Σ and so proved by Σ, since the consequences T of Σ form a complete theory.

Note that surjective implies bijective is false. A model theorist might immediately sense the failures since injective is universal and passes to substructure while surjective is $\forall\exists$ and so does not pass to substructures; an algebraist would immediately note that the map $x \mapsto x^2$ is a counterexample in, for example, the complex numbers.

Tao ([60]) gives an algebraic proof. He makes extensive use of the nullstellensatz and notably misses the simpler direct limit argument to go from the finite fields to the algebraic closure of F_p; he gracefully acknowledges this simplification in reply to a comment. But it does reflect the different perspective of logic on such a problem.

Example 4.3.2 (Division Algebras)**.** A: Real division algebras: Any finite-dimensional real division algebra must be of dimension 1, 2, 4, or 8. This is proved (by Hopf, Kervaire, Milnor) using methods of algebraic topology.

B: Division algebras over real closed fields Any finite-dimensional division algebra over a real closed field must be of dimension 1, 2, 4, or 8.

B. follows immediately from A by the completeness of the theory of real closed fields. No other proof is known.

In a rough sense undecidability seems to disappear when a structure is 'completed' to answer natural questions (e.g. adding inverses and then roots to the natural numbers). Here is a specific measure of that idea.

Example 4.3.3 (Ruler and Compass Geometry)**.** *Beeson [13] notes that the theory of 'constructible geometry' (i.e. the geometry of ruler and compass) is undecidable. This result is an application of Ziegler's proof [65] that any* finitely axiomatizable theory *in the vocabulary* $(+, \cdot, 0, 1)$ *of which the real field is a model is undecidable. Thus the complete theory is tractable while none of its finitely axiomatized subtheories are.*

In the next example, we study an infinite family of complete theories, algebraically closed fields of various characteristics. The Lefschetz principle was long

[28]There is a non-model theoretic proof in the spirit of Ax which replaces model completeness by multiple references to the Nullstellensatz [29]. Ax was apparently unaware of Grothendiecks proof. He cites other work by Grothendieck, and not this. And he says "This fact seems to have been noticed only in special case (e.g. for affine space over the reals by Bialynicki-Birula and Rosenlicht."

known informally by algebraic geometers and appeared in Lefschetz' Algebraic Geometry. But the proof of the precise version described below is due to Seidenberg[49].

Example 4.3.4 (Lefschetz Principle)**.** Let ϕ be a sentence in the language $\mathcal{L}_r = \{0, 1, +, -, \cdot\}$ for rings, where $0, 1$ are constants and $+, -, \cdot$ are binary functions.

The following are equivalent:

1. ϕ is true in every algebraically closed field of characteristic 0.
2. ϕ is true in some algebraically closed field of characteristic 0.
3. ϕ is true in algebraically closed fields of characteristic p for arbitrarily large primes p.
4. ϕ is true in algebraically closed fields of characteristic p for sufficiently large primes p.

Proof. This follows from the completeness of algebraically closed fields of characteristic zero and Gödel's completeness theorem.

There are extensions to infinitary logic[19]. Again math overflow`http://mathoverflow.net/questions/90551/what-does-the-lefschetz-principle-in-algebraic-geometry-mean-exactly` provides a nice overview and pushback from algebraic geometers who prefer viewing the principle as a heuristic to understanding undergraduate logic.

It frequently turns out that important information about a structure is only implicit in the structure but can be manifested by taking a saturated elementary extension of the theory. In particular, within the saturated models the syntactic types over a model can be realized as orbits of automorphism of the ambient saturated model. The germs of this idea are seen in the following example.

Example 4.3.5 (Algebraic geometry)**.** Algebraic geometry is the study of definable subsets of algebraically closed fields. This isn't quite true: 'definable by positive quantifier free formulas'. More precisely, this describes 'Weil' style algebraic geometry. Here are two equivalent statements of the same result.

Theorem 4.3.6. *Chevalley-Tarski Theorem*

Chevalley: The projection of a constructible set is constructible.

Tarski: The theory of algebraically closed fields admits elimination of quantifiers.

The version of this theorem for the reals is also known as the Tarski-Seidenberg[29] theorem [48].

[29] Tarski announced the quantifier eliminability of the real closed fields in [61]. He apparently became

The notion of a generic point on a variety X defined over a field k is a rather amorphous for much of the twentieth century. In the model theoretic approach a generic point a is a point in an extension field of k. More precisely, if $\overline{k}$ is the algebraic closure of k, a is a realization in an elementary extension of $\overline{k}$ of a non-forking extension of the type of minimal Morley rank and contained in X.

4.4 Virtuous properties as an organizing principle

{fol}

The stability hierarchy is a collection of properties of theories as envisioned in Subsection 2.2 that organize complete first order theories (that is structures) into families with similar mathematically important properties. Bourbaki (page 228 [15]) has some beginning notions of combining the 'great mother-structures' (group, order, topology). They write 'the organizing principle will be the concept of a hierarchy of structures, going from the simple to complex, from the general to the particular.' But this is a vague vision. We now sketch a realization of a more sophisticated organization of parts of mathematics.

There are two key components: 1) the formulation of a general scheme for a structure of each model of a complete first order theory; 2) a classification that determines whether a given theory admits a structure theorem in the sense of 1).

The fundamental tool of this organization is the study of properties of *definable* sets. Depending on the situation, there are several reasons why the subclass of definable sets is adequate to this task. In algebraic geometry (both real and complex) it turns out that mathematicians are basically only studying (some) of the definable sets in the first place. In the other direction, the Wedderburn theory for non-commutative rings is on its face second order because of the study of ideals. But, for stable rings, there are enough principal ideals to carry out the arguments obtaining the structure theorems for stable rings [11].

Shelah's stability theory[54] provides a method to categorize theories into two major classes (the *main gap*): admit a structure theory (*classifiable*) and creative/chaotic. If a theory admits a structure theory, then all models of any cardinality are controlled by countable submodels by a mechanism which is the same for all such theories. In particular, this implies that the number of models in cardinality $\aleph_\alpha$ is bounded by $\beth_\beta(\alpha)$ (where $\beta < |T|^+$). In contrast, the number of models in $\aleph_\alpha$ of a chaotic theory is $2^{\aleph_\alpha}$; essentially new methods of creating models are always needed. In the last 25 years, tools in the same spirit of definability allow the investigation of the definable subsets of creative theories; these include the study of simple, o-minimal and theories without the independence property. While the counting of the number of models in each cardinality is the test question for this program, the greatest benefit lies in the development of tools for investigating the fine structure of models of classifiable

aware that his argument extended to the complex numbers when Robinson proved the quantifier eliminability of algebraically closed fields in [46]. There are rumors that Chevalley was well aware of Tarski's proof for the reals.

theories.

The general idea of a structure theory is to isolate 'definable' subsets of models of a theory that admit a dimension theory analogous to that in vector spaces. And then to show that all models are controlled by a family of such dimensions. Theories that are categorical in power are the simplest case. There is a single dimension and the control is very direct.

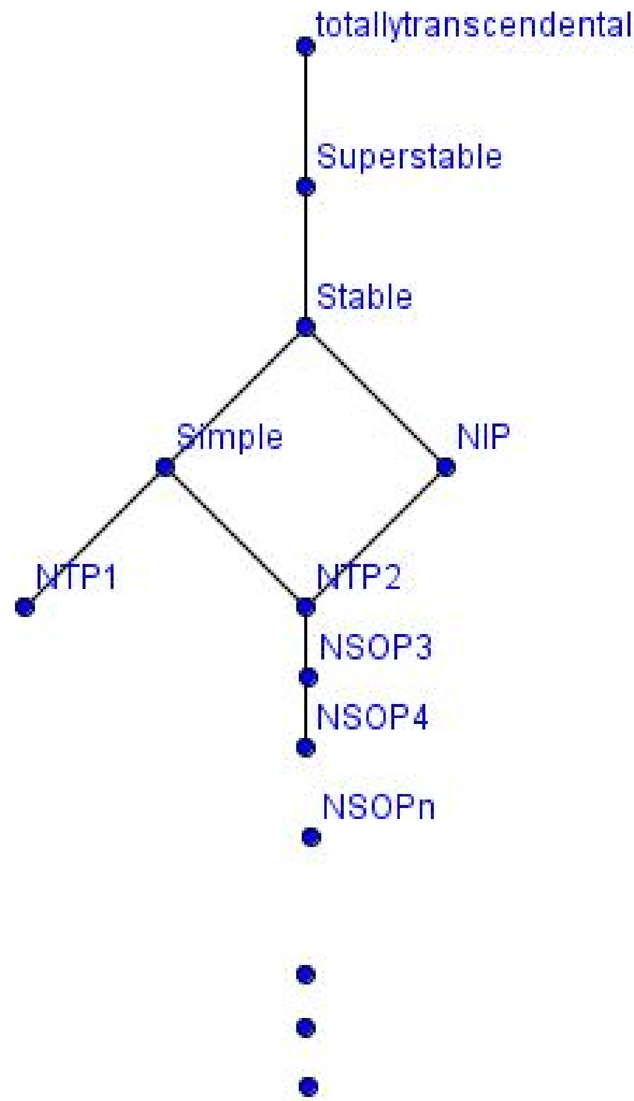
The Stability Hierarchy: Every complete first order theory falls into one of the following 4 classes.

1. ω-stable
2. superstable but not ω-stable
3. stable but not superstable
4. unstable

A common reading of Shelah is that the further down the above list a theory falls the more it lies on the side of chaos: 'chaos' means 'many models'. But Shelah has pointed out that this reads his program upside down, 'The aim is classification, finding dividing lines and their consequences. This should come with test problems. The number of models is an excellent test problem and few models is the strongest non-chaos.' But Shelah has suggested, beginning in late 70's, test problems for the study of theories with many non-isomorphic models, in particular, of unstable theories without the strict order property: existence of saturated extensions [51], the Keisler order (Chapter 6 of [50]) and the existence of universal models [32]. And for the theories with the strict order property [55] in SOP_4.

The study of o-minimality, simple theories and recent advances in studying nip (not the independence property[30]) show that 'tame' is a broader category than stable. Among the canonical structures, the complex field is a prototype for good behavior and the real field (and even the real exponential field) are o-minimal and so admit a great deal of structure. Only arithmetic of these structures is has so far resisted these methods of understanding. This is witnessed by its having both the independence property and being linearly ordered. Of course set theory is equally unruly; a pairing function is incompatible with a dimension theory. Recent work has studied the territory of unstable theories. The following diagram illustrates some of the continuing analysis of unstable theories.

[30]also called dependent



This hierarchy is essentially orthogonal to decidability. There are continuum many strongly minimal theories so most are not decidable. The random graph has the independence property but is recursively axiomatized and $\aleph_0$ categorical so decidable. Similarly, the theory of atomless Boolean algebras has both the strict order property and the independence property but is decidable.

The proof of the main gap relies on discovering several more dividing lines. We omit the technical definitions of the *dimensional order property (DOP)*, the *omit-*

ting types order property (OTOP) and the *shallow/deep* dichotomy. Such properties lead to the existence of 2^κ models of cardinality κ[31]. If none of these properties hold the number of models is bounded well below 2^κ and there is good decomposition theorem for the models. We discuss this in more detail in section 5.2. Hart, Hrushovski, Laskowski produced a full account of the spectrum problem for countable theories, including the greater intricacy for small infinite cardinalities in [25].

In the last few paragraphs we have glimpsed the ways in which complete theories provide a framework for analysis. In Subsection 4.5, we discuss some of the profound implications of the hierarchy of complete theories for work in core mathematics. Thus I argue that the study of complete theories a) focuses attention on the fundamental concepts of specific mathematical disciplines and b) even provides techniques for solving problems in these disciplines.

4.5 Formal Methods as a tool in mathematics

{apps}

In the last few paragraphs we have glimpsed the ways in which complete theories provide a framework for analysis. In Subsection 4.5, we discuss some of the profound implications of this development in the study of real algebraic geometry and diophantine geometry. Thus I would argue that the study of complete theories a) focuses attention on the fundamental concepts of specific mathematical disciplines and b) even provides techniques for solving problems in these disciplines.

The next list contains examples of theorems of core mathematics that are proved using at least the spirit of the formal tools of stability theory. We analyze in very general terms below the kind of use that is involved. The crucial point is that the stability hierarchy is defined by syntactic conditions. For example, a formula $\phi(\mathbf{x}, \mathbf{y})$ has the *order property* in a model M if there are $\boldsymbol{a}_i, \boldsymbol{b}_i \in M$ such that

$$M \models \phi(\boldsymbol{a}_i, \boldsymbol{b}_j) \text{ iff } i < j.$$

T is stable if no formula has the order property in any model of T. But existentially quantifying out the $\boldsymbol{a}_i, \boldsymbol{b}_i$, ϕ is unstable in T just if for every n the sentence $\exists x_1, \ldots x_n \exists y_1, \ldots y_n \bigwedge_{i<j} \phi(x_i, y_i) \wedge \bigwedge_{j \geq i} \neg\phi(x_i, y_i)$ is in T. This last is clearly a syntactic condition. The (local) dimension theory of a stable theory which leads to the structural results follows from this syntactic condition.

In an echo of the Bourbaki assertion of the importance of groups, in the presence of a group the stability conditions translate to chain conditions on 'definable subgroups'[32]. In an ω-stable (superstable) theory there is no descending chain of definable subgroups (with infinite index at each stage). This principle is now seen to apply to the

[31] The non-structure arguments are not tightly linked to first order logic. They correspond to a theorem scheme for building many non-isomorphic models as Skolem hulls of sets of (linearly ordered or tree ordered) indiscernibles. The Skolem hull can accomodate some infinitary languages.

[32] In rings these subgroups become ideals.

different algebraic structures and gives a unified explanation for finding various kinds of radicals. (See [8] for a very early account of this phenomena.)

1. Shelah: uniqueness of differential closures
2. Zilber's classification of 2-transitive groups
3. Hrushovski Mordell-Lang for function fields, interaction of '1-based' with arithmetic algebraic geometry.
4. Sela: All free groups on more than two generators are elementarily equivalent.
5. o-minimality, Hardy's problem
6. Denef-van den Dries: rationality of Poincaire series by induction on quantifiers
7. motivic integration: Denef/Cluckers/Hrushovski/Kazhdan/Loeser
8. MacPherson-Steinhorn: asypmtotic classes, understanding the classification of simple groups

Let us try to categorize these arguments. Shelah's proof is a direct application of his theorem of the uniqueness of prime models over sets for ω-stable theories. With some actual investigation of differential equations, he shows they need not be minimal. It might be objected that 'differentially closed field' is a logical notion introduced by Abraham Robinson. This is belied by the active research integrating model theoretic tools.

Zilber's classification is a purely mathematical result he needed to solve a logical problem. Are there any totally categorical finitely axiomatizable first order theories with only infinite models? In fact, solving this problem led to the discovery that classical groups are definably embedded in e.g. all sufficiently complicated first order theories that are categorical in power.

Hrushovski's proof involves both direct applications of Shelah's notions of orthogonality and p-regularity and such notions from geometric stability theory as one-based but integrating these tools with the arithmetic algebraic geometry.

The notion of o-minimality arose with work of van den Dries on the analysis of real exponentiation. But Pillay and Steinhorn recognized the notion as a generalization of strong minimality. Strong minimality characterizes the definable subsets (no matter how extensive the ambient vocabulary) as easily describeable using only equality (finite or cofinite). A theory is o-minimal if every definable subset is easily described in terms of a linear order of the model (finite union of intervals). Wilkie [64] proved that the real exponential field is o-minimal. A number of examples of further o-minimal structures were discovered, many expansions of the real field. [38]. Although this notion is explicitly defined in terms of formal definability, in real algebraic geometry as in algebraic geometry in general, the only relations considered are definable. This has led to enormous intergration between real algebraic geometry and model theory.

Denef's original work is more in the school of just using the formality of quantifier induction systematically; tools of desingularization can be avoided by a careful induction on quantifier rank. But the development of the theory in the 21st century connects with the ideas of cell decomposition arising in the study of o-minimality and with issues arising from the study of p-adically closed fields as NIP theories.

I have tagged the Sela work with Tarski's famous conjecture since that is the easiest for logicians to understand. But there are intimate connections and ramifications for combinatorial group theory.

Macpherson and Steinhorn [34] define an *asymptotic class* as a class of finite models in which the number of solutions of a formula $\phi(\mathbf{x}, \boldsymbol{a})$ in a finite model M can be uniformly approximated as $\mu M^{d/N}$ where N is a parameter of the class and μ, d are uniformly defined depending on $\boldsymbol{a}$. This generalizes classical results on finding the number of solutions of diophantine equations in finite fields. But it also provides a scheme to try to explain the families of finite simple groups, (in terms of their definability).

4.6 Groups of finite Morley rank

{FMR}

In this subsection, we sketch how the resources of formalization, in particular interpretation and the stability apparatus contribute to the study of a class of groups that were in fact defined by model theorists. This example illustrates the ways that formalization interacts with core mathematics and exhibits the power of formalization to introduce a generalizing principle. In this case the formalization (considering groups of finite Morley rank) provides a framework which includes both finite groups and algebraic groups over algebraically closed fields, thereby illuminating the role of finiteness conditions in each case. A group of finite Morley rank (FMR) is a structure which admits a group operation and is ω-stable with finite rank[33]. The driving Conjecture 4.6.1, posed by model theorists, links a model theoretic concept with algebraic geometry. The 25 year project to solve the conjecture has developed as an amalgam of basic stability theoretic tools with many different tools from finite and, recently, combinatorial group theory.

The basic scheme for understanding of the structure of groups relies on the Jordan-Hölder theorem: Each finite group can be written (uniquely up to the order of the decomposition) as $G = G_0 > G_1 \ldots G_n = 1$, where G_{i+1} is normal in G_i and the quotient groups G_i/G_{i+1} are simple. Thus identifying the finite simple groups is

[33] I am not giving a detailed historical survey here so I do not give every attribution or reference. The book of Poizat [43, 44] provides the general setting as in the late 80's. Borovik and Nesin [14] gives a good overview of the finite rank case in the mid 90's. Cherlin's webpage [16] lays out the Borovik program in broad strokes with references. The most recent summary is [1].

a key step to understanding all finite groups. (We also need to know the nature of the extension at each level.)

Macintyre proved in the early 70's that an $\aleph_1$-categorical, indeed any ω-stable field is algebraically closed. An algebraic group is variety G over a field k equipped with a group operation from $G \times G \to G$ that is a morphism (in the sense of algebraic geometry[34]. The definition of an algebraic group (over an algebraically closed field) yields immediately that it is interpretable[35] in an algebraically closed field and so has finite Morley rank. The algebraic definition of the dimension of an (in fact definable) subset yields the same value as the Morley rank.

Work in the late 1970's, showed similar properties of algebraic groups over algebraically closed field and groups of finite Morley rank in low rank. Cherlin showed the conditions rank 1 implies abelian; rank 2 implies solvable extend from algebraic groups to FMR groups. Zilber showed that a solvable connected (see below) FMR group which is not nilpotent interprets an algebraically closed field. From this it ensues that every FMR group 'involves' an algebraically closed field; the issue is, 'how close is the involvement?'. Although algebraic groups over algebraically closed fields have FMR, Groups of finite Morley rank (FMR) are clearly more general. The Prüfer group $\mathbb{Z}_\infty$ is ω-stable but not algebraic and FMR groups are closed under direct sum while algebraic groups are not. But the role of rank/dimension in each of the cases and the identification of the field in the group led to the idea that groups of finite Morley rank were some kind of natural closure of the algebraic groups. In particular, the basic building blocks are the same. {czconj}

Conjecture 4.6.1 (Cherlin/Zilber)**.** *A simple group of finite Morley rank is algebraic.*

The classification of the finite simple groups identifies most of them as falling into families of algebraic groups over finite fields, the Chevalley groups. Families such as the Chevalley groups are a natural notion in model theory. They are the solution of the same definition of a matrix group as the field changes. A further impetus for this study is an analogy between finite groups and FMR groups. In the proofs, the descending chain conditions on all subgroups for finite subgroups is replaced by the descending chain condition on definable subgroups. This allows an algebraic definition of 'connected' replacing the topological definition in the study of algebraic groups. Induction on the cardinality of the group is replaced by induction on its Morley rank. The use of definability now provides a common framework for the study of algebraic and finite groups[36]. The quest for Conjecture 4.6.1 has led to intricate analysis of all three generations of the proof of the classification of finite simple groups. In particular the main strategy of the proof is an induction on the 'minimal counterexample' and the

[34] A by-product of the study under discussion is that an equivalent definition is: group defined in an algebraically closed field.

[35] We don't spell out here the definition of an interpretation; any of the general sources in model theory mentioned above do so.

[36] Borovik had independently introduced the notion of ranked group (one which admits a collection of subsets containing the finite set and closed under (Boolean operation, projection, quotient). Poizat showed the class of such groups is exactly the FMR groups.

possibilities for this counterexample are sorted analogously to the finite case.

The following (slightly shortened) passage from the introduction to Chapter 3, Interpretation, in [14] shows the deep ties between the logical notion of 'interpretation' and algebra.

> The notion of interpretation in model theory corresponds to a number of familiar phenomena in algebra which are often considered distinct: coordinatization, structure theory, and constructions like direct product and homomorphic image. For example a Desarguesian projective plane is coordinatized by a division ring; Artinian semisimple rings are finite direct products of matrix rings over divisions rings; many theorems of finite group theory have as their conclusion that a certain abstract group belongs to a standard family of matrix groups over .... All of these examples have a common feature: certain structures of one kind are somehow encoded in terms of structures of another kind. All of these examples have a further feature which plays no role in algebra but which is crucial for us: in each case the encoded structures can be recovered from the encoding structures definably.

The last sentence is one reason why a FMR group is allowed to have relations beyond the group operation. Since the underlying field structure is often recoverable, it should be permitted in the language.

Another role of the formalization is seen in the ability to focus on the key idea of a proposition. The standard statement of the Borel Tits theorem takes half a page and gives a laundry list of the possible kinds of maps (albeit considering the fields of definition of the groups). Zilber (fully proved by Poizat see 4.17 [43, 44].) gives the following elegant statement.

Theorem 4.6.2 (Borel-Tits a la Zilber/Poizat)**.** *Every pure group isomorphism between two simple algebraic groups over algebraically closed fields K and L respectively can be written as the composition of a map induced by a field isomorphism between K and L followed by a quasi-rational function over L.*

Both the explanation of the role of interpretation in [14] and the statement of the Borel-Tits theorem illustrate the role of formalization in providing context and clarity to mathematical results. While the development of this particular project takes place in the context of ω-stable theories, the role of classes further down the Shelah hierarchy appears in other examples.

5 Infinitary Logic

We begin by describing the role of categoricity in power the logic $L_{\omega_1,\omega}$. This logic is 'first order' in the sense that only quantification over individuals is allowed. But countable conjunctions are permitted. Then we return to the beginning with a slight twist. We consider second order logic with infinite conjunctions of various lengths.

5.1 $L_{\omega_1,\omega}$

{inf}

In this subsection we survey the status of categoricity and categoricity in power for sentences of $L_{\omega_1,\omega}$. The main results are in [52, 53]; we give a systematic development in [7].

Since there are $2^{\aleph_0}$ inequivalent sentences and $2^{2^{\aleph_0}}$ theories but a proper class of structures some theories must fail to be categorical. In contrast to first order there are countable structures that are categorical for $L_{\omega_1,\omega}$. By the downward Löwenheim-Skolem theorem, no uncountable structure can be categorical for a sentence of $L_{\omega_1,\omega}$. But the $L_{\omega_1,\omega}$-*theory* of the reals is categorical.

A countable structure is categorical iff it has no proper $L_{\omega_1,\omega}$-elementary submodel. For sentences in $L_{\omega_1,\omega}$, categoricity in power $\aleph_1$*implies* the existence of a complete sentence satisfied by the model of cardinality $\aleph_1$. It is open whether this implication holds for $\aleph_2$-categoricity in $L_{\omega_1,\omega}$.

The best generalization of Morley's theorem to $L_{\omega_1,\omega}$ is due to Shelah[52, 53]. Shelah shows that one can more profitably study this subject by focusing on classes of the form $EC(T, Atomic)$, the class of atomic[37] models of complete countable first order theory.

The class of models of a complete sentence of $L_{\omega_1,\omega}$ is in 1-1 correspondnce with an $EC(T, Atomic)$-class (Chapter 6 of [7]). Regarding it as the class of atomic models of a first order theory is a key simplification. $EC(T, Atomic)$ is one defined An $EC(T, Atomic)$-class is *excellent* if for every finite n it is possible to find a unique amalgamation of n independent countable models in the class.

Theorem 5.1.1 (ZFC: Shelah 1983)**.** *If $\boldsymbol{K}$ is an* excellent $EC(T, Atomic)$*-class then if it is categorical in one uncountable cardinal, it is categorical in all uncountable cardinals.*

Theorem 5.1.2 (Shelah 1983)**.** *Assume $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for finite n. If an $EC(T, Atomic)$-class $\boldsymbol{K}$ is categorical in $\aleph_n$, for all $n < \omega$, then it is excellent.*

Thus for $L_{\omega_1,\omega}$ the study of categoricity in power is in a relatively complete state; the outstanding question from a philosophical standpoint is whether the *very*

[37]Each finite sequence realizes a complete type over the empty set.

weak generalized continuum hypothesis (VWGCH: for all n, $2^{\aleph_n} < 2^{\aleph_{n+1}}$) is actually needed. There are only a few papers aimed at finding an extension to the stability hierarchy in this framework ([22, 12]).

There are important mathematical structures, e.g. complex exponentiation which exhibit the Gödel phenomena and so cannot be analyzed by stability techniques *in first order logic*. However, Zilber[71, 70, 7] has conjectured a means for such an analysis in the logic $L_{\omega_1,\omega}(Q)$.

5.2 Deja vu: Categoricity in infinitary second order logic

{dejavu}

We rehearse here some recent results of Hyttinnen, Kangas, and Väänänen [28][38] that identify in a systematic way a proper class of categorical structures. Consider the logic $L^2_{\kappa,\omega}$ which allows first and second order quantification and conjunctions of length κ. In this family of logics, there are a class of sentences so the cardinality argument for the existence of non-categorical structures fails. In fact, every structure of cardinality κ is categorical in $L^2_{\kappa^+,\omega}$. The goal is to identify those structures of cardinality κ that are categorical $L^2_{\kappa,\omega}$ (κ not $\kappa+$). Since this work draws on the first order stability hierarchy discussed in Section 4.4, we begin with more detail on the main gap.

Any standard text in stability theory shows that if T is stable then via the notion of non-forking an independence relation generalizing the combinatorial geometries discussed in Subsection 4.2 can be defined on all models of T. In general the closure relation fails to be a geometry because $\mathrm{cl}(\mathrm{cl}(X) \neq \mathrm{cl}(X)$. But on the set of realizations of a so-called regular type it is. Thus in a model M and for any regular type p with domain in M, we can define the dimension of the realizations of p in M.

If T is not stable or even not superstable, T has 2^κ models in every uncountable κ [54]). We discussed the role of DOP and OTOP in Subsection 4.4. If either DOP fails T has the maximal number of models in each uncountable cardinal. If the dimensional order property and the omitting types order property do not hold (NDOP and NOTOP), each model M of cardinality κ can be decomposed as a tree of countable submodels indexed by some tree I; essentially 'deep' means this tree is not well-founded. This root of this tree, M_0 is a prime[39] model. Each M_s has a set of up to κ extension extensions $M_{s\frown i}$ which are independent over M_s; M is prime over $\bigcup_{s\in I} M_s$. The systematic representation of a model as prime over a tree of (independent) submodels is a fundamentally new mathematical notion. The theory is shallow if there is a uniform bound over all models on the rank of the tree; essentially 'deep' means this tree is not well-founded. If a theory satisfies NDOP, NOTOP and is shallow the theory is called *classifiable*.

For a classifiable theory the number of models in $\aleph_\alpha$ is bounded by $\beth_\beta(\alpha)$ where β is a bound on the rank of the decomposing tree for all models of T (indepen-

[38]The authors use 'characterizable' for what we call 'categorical'.

[39]That is, under an appropriate notion of submodel, it can embedded in every model of T.

dently of cardinality). [54] claims that each model of such a theory is characterized by a sentence in a certain 'dimension logic'. Unfortunately there are technical difficulties in the definition of this logic. However, the new paper [28] shows how to find such a categorical sentence in $L^2_{\kappa^+,\omega}$. Thus they obtain (in a slightly less general form):

Theorem 5.2.1 (Hyttinnen, Kangas, and Väänänen)**.** *Assume GCH. The countable complete theory T is classifiable if and only if for every model M of T with $|M| \geq \beth_{\omega_1}$, the $L^2_{\kappa^+,\omega}$ theory of M is categorical.*

The deduction from classifiable is a highly technical argument that the decomposition of the models (and the dimensions of the types involved) can be defined in $L^2_{\kappa^+,\omega}$. Conversely, if a theory is not classifiable (on the chaotic side of the main gap), it has 2^κ models in κ. But there are only $2^{<\kappa}$ sentences in $L^2_{\kappa^+,\omega}$ so there must be a sentence which is not categorical in the logic $L^2_{\kappa^+,\omega}$.

So using the virtuous properties developed in first order logic, the authors are able to uniformly identify a large family of structures with cardinality κ that are categorical in $L^2_{\kappa,\omega}$. But categoricity is used in Huntington's role of 'sufficiency'. It is again a test of an axiomatization. In contrast to the *ad hoc* search for the axioms of the fundamental structure, since there is a scheme (implicit in Shelah's structure theorems) for obtaining the axiomatization, categoricity of a theory is an informative property. But while the axiomatizations of the fundamental structures informed us about the principles underlying proofs in the underlying number theory and real analysis, these axiomatizations inform us about the structure of the models of the underlying classifiable first order theory.

6 Conclusion and Further Directions

{conc}

We have argued that the criteria for evaluating the significance of a property of theories (in some logic) is the explanatory power of the property. Specifically, do the theories or more importantly the models of the theories which have this property display other significant similarities?

From this standpoint we have argued that categoricity is not very interesting for second order logic and trivial for first order logic. But for first order logic, categoricity in power is very significant because all categorical theories are seen to possess a dimension theory similar to prototypical examples such as vector spaces. The stability hierarchy provides both a classification of first order theories which calibrates their ability to support nice structure theories and the details of such a structure theory. The key to the structure is the definition of local dimensions extending the basic phenomena in theories which are categorical in power.

We see this analysis as demonstrating the use of formal methods in mathematics. In the Subsection 4.5 we gave more examples of the use of a formal language as a tool for proving mathematical results. Thus, this paper is a counterpoint to our [9] where we discussed certain ‘formalism-free’ developments in model theory. In this paper we stressed one of the dominant themes of model theory: the role of formal language in understanding mathematical questions. More than the use of formalism in seeking global foundations for mathematics[40], these applications have real effect in mathematics.

Here are some further directions.

1. Connect the notion of ‘explanation’ here with the work of Steiner, Kitcher, Mancosu,Hafner [36].
2. Connect the analysis of rcf in [24] with o-minimality and classification of algebras
3. Discuss interpretability of theories as a key tool for the generality.
4. Give more examples of how the connections across fields work.

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