The Hanf number for Extendability is the first measurable cardinal

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Exploring Cantor's Paradise

David Hilbert

"No one shall drive us from the paradise which Cantor has created for us."

William Shakespeare

There are more things in heaven and earth, Horatio, Than are dreamt of in your philosophy.

Thesis

Cardinality is intimately related with structural as well as combinatorial properties.

Infinitary logic allows us to explore this relation.

The proof involves:

1. Combinatorics around ultrafilters
2. The distinction between ‘independence in vector spaces’ and ‘independence in Boolean Algebra’
3. Generalizations of the ‘Fraïssé’ constuction
4. Structural properties of Boolean Algebras

Main Result

Theorem: There is a *complete sentence* ϕ of $L_{\omega_1,\omega}$ such that ϕ has maximal models in a set of cardinals λ that is cofinal in the first measurable μ while ϕ has no maximal models in any $\chi \geq \mu$.

Outline of Argument

- I. $\lambda < \mu_0$ implies there is a BA with witnessed (incompleteness) in λ
- II. There is P_0-maximal witnessed BA in λ
  1. Characterize P_0-maximal
  2. Find nicely free P_0-maximal model M_*.
- III. Find the complete sentence ϕ
- IV. Correcting M_* to a model of ϕ
  1. If $M \in \mathbb{M}_2$ then $M \models \phi$.
  2. There is an $M \in \mathbb{M}_2$ which satisfies all tasks.

Hanf Numbers

Hanf's principle

If a certain property P can hold for only set-many objects then it is eventually false.

Hanf's principle

If a certain property P can hold for only set-many objects then it is eventually false.
Hanf refines this twice.

1. If $\mathcal{K}$ a *set* of collections of structures $\boldsymbol{K}$ and $\phi_P(X, y)$ is a formula of set theory such $\phi(\boldsymbol{K}, \lambda)$ means some member of $\boldsymbol{K}$ with cardinality λ satisfies P.

$$\mu_{\boldsymbol{K}} = \sup\{\lambda : P(\boldsymbol{K}, \lambda) \text{ holds if there is such a sup }\}$$

   Hanf number $HN(P)$ of P = $\sup_{\boldsymbol{K}} \mu_{\boldsymbol{K}}$.
   Thus, if P holds somewhere above $HN(P)$ it holds for arbitrarily large cardinals.

2. If the property P is closed down for sufficiently large members of each $\boldsymbol{K}$, then 'arbitrarily large' can be replaced by 'on a tail' (i.e. eventually).

Examples

Large cardinals: Boney- Unger -Shelah

The Hanf number for 'all aec's are tame' is a compact cardinal with various decorations.

small cardinals: B, Hjorth Koerwein, Kolesnikov,Laskowski, Lambdie-Hanson, Shelah, Souldatos

Erratic behavior for amalgamation, disjoint amalgamation, maximal models, joint embedding.
All below $\beth_{\omega_1}$. (BKS disjoint amalg).

The big gap

Theorem. B-Boney

The Hanf number for Amalgamation is at most the first strongly compact cardinal

The best lower bound known is $\beth_{\omega_1}$. (BKS disjoint amalg)

Maximality, JEP, AP, Arbitarily Large

A maximal model plus (global) JEP or AP implies a bound on the cardinality of models.

Test question: non-maximality

Let $\boldsymbol{K}_0$ be the collection of models of a complete sentence in $L_{\omega_1,\omega}$ in a countable vocabulary.

to avoid negatives:
$\boldsymbol{K}_0$ is *universally extendible in* λ if every model in λ is extendible – has a proper $L_{\omega_1,\omega}$ extension.

Theorem. B-Shelah

The Hanf number for universal extendibility (complete sentences) is the first measurable cardinal μ_0 if it exists.

Clearly, every model with cardinality at least μ_0 has a proper $L_{\omega_1,\omega}$-extension.

Complete vs Incomplete

Complete sentence of $L_{\omega_1,\omega}$

Definition: complete sentence ϕ of $L_{\omega_1,\omega}$

1. For every $\psi \in L_{\omega_1,\omega}$, $\phi \to \psi$ or $\phi \to \neg\psi$.
2. (Equivalently) Every model of ϕ realizes only countably many distinct $L_{\omega_1,\omega}$-types.

countable vocabularies:

Morley: Hanf number of existence in $L_{\omega_1,\omega}$ is $\beth_{\omega_1}$

Hjorth: Hanf number of existence in $L_{\omega_1,\omega}$: complete sentence) is $\aleph_{\omega_1}$. Much harder.

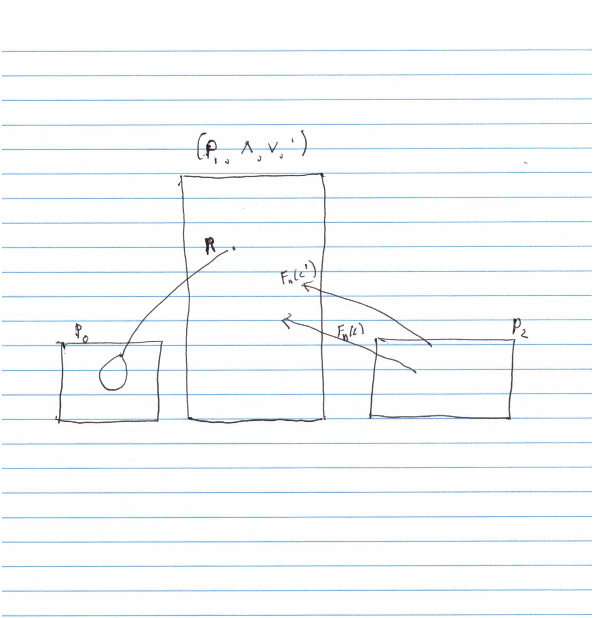
An incomplete example: arbitrarily large maximal models below μ_0 -first measurable cardinal

Consider a class ***K*** of 4-sorted structures describing a Boolean algebra of sets.

1. P_0 is a set.
2. P_1 is a Boolean algebra of subsets (given by an extensional binary R) of P_1.
3. P_2 is an index set for functions $F_n(c)$ ($n < \omega$) such that $F_n(c)$ enumerates a countable sequence from P_1.
   As c varies each countable sequence is enumerated. (Need $\lambda^\omega = \lambda$).
4. If a sequence $F_n(c) \subseteq P_1$ has the finite intersection property then the intersection is non-empty.

Let $\psi \in L_{\mathcal{A}} \subsetneq_{\omega_1,\omega}$ axiomatize ***K***.

The incomplete Example



Non-principal, witnessed

The underlying motif

Suppose M is extended to N by adding an element $a*$ to P_0^M.
Then

$$\{b \in P_1^M : E(a*, b)\}$$

is a non-principal $\aleph_1$-complete ultrafilter on P_1^M.

Proof:

1. ultrafilter: clear
2. non-principal
   Every $a \in P_0^M$ fails $a^* \not\leq a$.
3. $\aleph_1$-complete using $L_{\omega_1,\omega}$.

Why maximal?

M is a $L_{\mathcal{A}}$-maximal model of $\boldsymbol{K} = \mathrm{mod}(\psi)$ if

1. $\lambda <$ first measurable
2. $|P_0^M| = \lambda$.
3. $P_1^M = \mathcal{P}(P_0^M)$
4. The $F_n(c)$ for $c \in P_2^M$ enumerate ${}^{\omega}(P_1^M)$

M can only be extended by adding an element $a*$ to P_0^M. But then

$$\{b \in P_1^M : E(a*, b)\}$$

is a non-principal $\aleph_1$-complete ultrafilter on λ.

But ψ is not complete. There are $2^{\aleph_0}$ 2-types over the empty set, given, for each $X \subset \omega$, via (c, d) realizes p_X iff $X = \{n : F_n(c) \cap F_n(d) \neq \emptyset\}$.

Witnessed Boolean algebras

Theorem I: Witnessed Boolean algebras

Definition

For a Boolean algebra $\mathbb{B} \subset \mathcal{P}(\lambda)$ a set $\mathcal{A}$ of λ ω-sequences from $\mathbb{B}$ *witnesses the incompleteness of non-principal ultrafilters on* $\mathbb{B}$ if there is a set $\mathcal{A} \subseteq {}^{\omega}\mathbb{B}$ such that:

i) for each sequence $\overline{A} = \langle A_n : n < \omega \rangle$, any $\alpha < \lambda$ is in only finitely many of the A_n.

ii) $\mathbb{B}$ includes the finite subsets of λ; but every nonprincipal ultrafilter D on λ intersects some $\overline{A} \in \mathcal{A}$ infinitely often.

Theorem I

[ZFC] Assume for some μ, $\lambda = 2^{\mu}$ and λ is less than the *first measurable*, then then there is a uniformly $\aleph_1$-incomplete with $|\mathbb{B}| = \lambda$. $\boxplus(\lambda)$ in the paper

Finding witnessed Boolean algebras

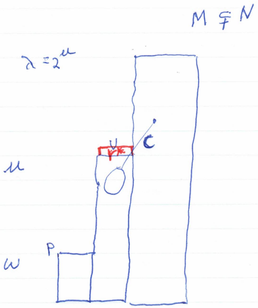
Vocabulary

Fix the vocabulary τ with unary predicates P, U, a binary predicate C, and a binary function F.

Construction

1. Let $\langle C_\alpha : \alpha < \lambda \rangle$ list *without repetitions* $\mathcal{P}(\mu)$ such that $C_0 = \emptyset$ and also let $\langle f_\alpha : \mu \leq \alpha < \lambda \rangle$ list ${}^\mu\omega$.
2. Define the τ-structure M by:
   1. The universe of M is λ; $P^M = \omega$; $U^M = \mu$;
   2. $C(x, y)$ is binary relation on $U \times M$ defined by $C(x, \alpha)$ if and only $x \in C_\alpha$.
   3. Let $F_2^M(\alpha, \beta)$ map $M \times U^M \to P^M$ by $F_2^M(\alpha, \beta) = f_\alpha(\beta)$ for $\alpha < \lambda$, $\beta < \mu$;
   4. $F_2^M(\alpha, \beta) = 0$ for $\alpha < \lambda$ and $\beta \in [\mu, \lambda)$.

$UF(M) = \emptyset$: diagram



Lemma Proof: I

Lemma:

If $\lambda < \mu_0$ and $2^{mu} = \lambda$, there is a τ structure M, $|M| = \lambda$ and every proper elementary extension N of M extends P^M.

proof sketch: 1st Step: Since $C^M(x, y)$ enumerates all subsets of $U^M = U^N$ any proper extension must extend U.

$\aleph_1$-incomplete ultrafilters

Fact: (folklore? Hachtman)

Let $D \subseteq \mathcal{P}(X)$ then tfae

1. for each partition $Y \subseteq \mathcal{P}(X)$ of X into at most countably many sets, $|D \cap Y| = 1$.
2. D is a countably complete ultrafilter.

Proof. Sample argument for hard direction. Suppose 1), by considering $\{W, W^-\}$ for $W \subset X$, exactly one of W and W^-, must be in D. But then D must be closed up since for $W_1 \subseteq W_2$ with $W_1 \in D$, the partition $\{W_1, W_2 - W_1, W_2^-\}$ shows $W_2^- \notin D$ and so $W_2 \in D$. If $W_1, W_2 \in D$, consider the partitions $\{W_1 \cap W_2, W_1 - (W_2 \cap W_1, W_1^-\}$ and $\{W_1 \cap W_2, W_1 - (W_2 \cap W_2, W_2^-\}$. Since both W_1^- and W_2^- are not in D; exactly one of the other 3 can be in and it must be the intersection.

Lemma Proof: II

2nd step

If $U^M \subsetneq U^N$ and $P^M = P^N$, then there is a countably complete non-principal ultrafilter on μ, contradicting that μ is not measurable.

The sequence $\langle f_\alpha \colon \mu \leq \alpha < \lambda \rangle$ is a list of all non-trivial partitions of μ into at most countably many pieces.

Let $\nu^* \in U^N - U^M$. For $\alpha \in N$, denote $F_2^N(\alpha, \nu^*)$ by n_α.
Since $P^M = P^N$, $n_\alpha \in M$.
By elementarity, for $\alpha \in M, \eta \in U^M$, $F_2^N(\alpha, \eta) = F_2^M(\alpha, \eta) = f_\alpha(\eta)$. Now, let

$$D = \{x \subseteq U^M \colon x \neq \emptyset \wedge (\exists \alpha \in M)\; x \supseteq f_\alpha^{-1}(n_\alpha)\}.$$

Verify $|D \cap Y| = 1$ for any partition Y of X.

The $\aleph_1$-incomplete Boolean algebra

Claim

If $\mathbb{B}$ is the Boolean algebra of definable formulas in the M just defined, there is an $\mathcal{A}$ such that $(\mathbb{B}, \mathcal{A})$ is witnesses $\aleph_1$-incompleteness.

Proof. i) We can choose $\mathcal{A}$ as families $\mathcal{A}_n^{\phi} \subseteq M$ whose Skolem functions map into P^M (ω) to have the finite intersection property. (Not immediate)

The $\aleph_1$-incomplete Boolean algebra II

ii) $\mathbb{B}$ includes the finite subsets of λ; but every nonprincipal ultrafilter D on λ intersects some $\overline{A} \in \mathcal{A}$ infinitely often.
Let D be an arbitrary non-principal ultrafilter on λ and let $\phi(v, \mathbf{y})$ vary over first order τ-formulas such that $\mathbf{y}$ and $\boldsymbol{a}$ have the same length.
Define the type $p_D(x) = p(x)$ as:

$$p(x) = \{\phi(x, \boldsymbol{a}) \wedge P(\sigma_\phi(\alpha, \boldsymbol{a}))\colon \{\alpha \in M\colon M \models \phi(\alpha, \boldsymbol{a})\} \in D\}.$$

Since D is an ultrafilter, p is a complete type over M.
Let d realize p in $N \succ M$. WOLOG, let N be the Skolem hull of $M \cup \{d\}$. Since D is non-principal, so is p; thus, $N \neq M$. Since P must increase, we can choose a witness $c \in P^N - P^M$. Since, N is the Skolem hull of $M \cup \{d\}$ there is a Skolem term $\sigma(w, \mathbf{y}) = \sigma_\phi(w, \mathbf{y})$ and $\boldsymbol{a} \in M$ such that $c = \sigma^N(d, \boldsymbol{a})$. Since $c \notin M$, for each $n \in P^M$, $N \models \bigwedge_{k<n} c \neq k$ so $N \models \bigwedge_{k<n} \sigma(d, \boldsymbol{a}) \neq k$ so $\bigwedge_{k<n} \sigma(x, \boldsymbol{a}) \neq k$ is in p.
That is, for each σ_ϕ, $A_{\sigma_\phi(w, \boldsymbol{a})}$ is in D.

Templates for complete sentences

Schemata for getting complete sentences

Template

1. Fix a collection $(\boldsymbol{K}_0, \leq)$ of countably many ‘finite’ structures.
2. Let $(\boldsymbol{K}_1, \leq)$ (often $\hat{\boldsymbol{K}}$) the collection of direct limits of structures in $\boldsymbol{K}_0$.

If $(\boldsymbol{K}_0, \leq)$ has the amalgamation property and joint embedding then it has a generic model M – universal and homogenous with respect to $(\boldsymbol{K}_0, \leq)$.

What does ‘finite’ mean?

‘Finite’ may mean:

1. *uniformly locally finite*: finite structures; finite relational language. First order $\aleph_0$-categoricity; Theory of generic has arb large models and full amalgamation.

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   1. (Hjorth): Build by a non-uniform induction models up to some $\aleph_\alpha$. disjoint amalgamation of f.g. over a large base
   2. (B-Friedman-Koerwien-Laskowski) If there is a counterexample to Vaught’s conjecture there is one where every model in $\aleph_1$ is maximal (sharpening Hjorth)
   3. (B-Koerwien-Laskowski); prove n-dimensional amalgamation of models up to $\aleph_n$. (2-ap in $\aleph_{n-2}$) No model in $\aleph_{n+1}$.
3. *finitely generated* –The new technique here.

K_{-1}: The basic class of structures

K_{-1}:The Boolean algebra

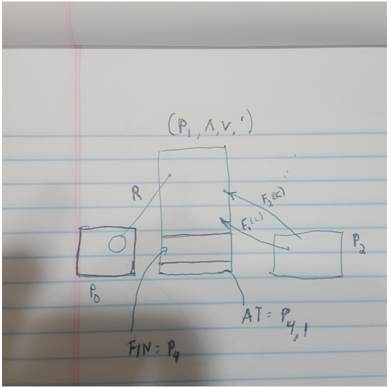
We define a class of (pseudo) Boolean set algebras with functions witnessing countable incompleteness.

Vocabulary

τ is a vocabulary with unary predicates P_0, P_1, P_2, P_4, binary R, $\wedge, \vee, \leq$ unary functions $^-$, G_1, constants 0,1 and unary functions F_n, for $n < \omega$.

1. P_0 is a set of elements.
2. P_1 is the domain of a boolean algebra.
3. R is a binary relation making P_1 code subsets of P_1
4. $P_{4,1}$ denotes the set of atoms of P_1 and P_4 the *ideal* they generate.
5. G_1 is a bijection from P_0 onto $P_{4,1}$.

Rough idea of structure



K_{-1}: Witnessing incompleteness

The F_n

1. F_n maps the index set P_2 into the Boolean algebra P_1.
2. (*countable incompleteness*) If $a \in P^M_{4,1}$ and $c \in P^M_2$ then $(\forall^\infty n)\ a \not\leq_M F^M_n(c)$. As, $a \wedge F^M_n(c) = 0$. Since a is an atom, this implies $\bigwedge_{n\in\omega}\{x\colon (G_1(x) \in F^M_n(c)\} = 0$.
3. P^M_1 is generated as a Boolean algebra by $P^M_4 \cup \{F^M_n(c)\colon c \in P^M_2, n \in \omega\} \cup X$ where X is a finite subset of P^M_1.

A P_0-maximal model in $\boldsymbol{K}_{-1}$

Theorem II.1: Characterizing P_0-maximality

Definition: P_0-maximal

We say $M \in K_{-1}$ is P_0*-maximal* (in K_{-1}) if $M \subseteq N$ and $N \in K_{-1}$ implies $P_0^M = P_0^N$.

Definition: [uf(M)]

For $M \in K_{-1}$, let $\mathrm{uf}(M)$ be the set of ultrafilters D of the Boolean Algebra P_1^M such that $D \cap P_{4,1}^M = \emptyset$ and for each $c \in P_2^M$ only finitely many of the $F_n^M(c)$ are in D.

Theorem II.1

An $M \in \boldsymbol{K}_{-1}$ is P_0-maximal if and only if $\mathrm{uf(M)} = \emptyset$.

Boolean Algebra Interlude I

Definitions and Facts

1. A BA is atomic if every element is a join of atoms or equivalently if every non-zero element is above at least one atom.
   The second version is clearly first order.
2. A BA is atomless if there are no atoms.
3. Every 'Boolean algebra of Sets' is atomic.
4. Let I be an ideal in a Boolean algebra B.
   $b - c \in I$ implies $b/I \leq c/I$

Free Boolean algebras

1. A Boolean algebra is free on generators $\{b_i : i < \kappa\}$ if $\sigma(b_{i_1}, b_{i_k}) = 0$ implies every Boolean algebra satisfies $\forall x_1, \ldots x_k)\sigma(x_1, \ldots x_k) = 0$.
2. An infinite free Boolean algebra is atomless.
3. A countably infinite atomless Boolean algebra is free.

Characterizing P_0-maximality: Proof

Suppose M is not P_0-maximal and $M \subset N$ with $N \in \boldsymbol{K}_{-1}$ and $d^* \in P_0^N - P_0^M$. Then $\{b \in M : R^N(d^*, b)\}$ is a non-principal ultrafilter D_0 of the Boolean algebra P_1^M.
Easy check that $D_0 \in \mathrm{uf}(\mathrm{M})$.

Conversely, if $D \in \mathrm{uf}(\mathrm{M})$.
Extend to N by adding an element $d \in P_0^N$ with

$$R^N(d, b) \leftrightarrow b \in D.$$

Let P_1^N be the Boolean algebra generated by $P_1^M \cup \{G_1(d)\}$ modulo the ideal generated by $\{G_1^N(d) - b : b \in D\}$.
Thus, in the quotient $G_1(d) \leq b$.
Let $P_2^N = P_2^M$ and $F_n^N(c) = F_n^M(c)$. Since $D \in \mathrm{uf}(\mathrm{M})$, P_1^N is witnessed.
It is easy to check that $N \in \boldsymbol{K}_{-1}$.

Nicely Free

Definition: Nearly Free

$M \in K_{-1}$ is *nearly free* when $|P_1^M| = \lambda$
and $\mathbf{b} = \langle b_\alpha : \alpha < \lambda \rangle$ satisfies

(a) $b_\alpha \in P_1^M - P_4^M$;
(b) $\langle b_\alpha / P_4^M : \alpha < \lambda \rangle$ generate P_1^M / P_4^M freely;

Definition: Nicely Free

$M \in K_{-1}$ is *nicely free* when $|P_1^M| = \lambda$ when M it is nearly free and there is a set $Y \subset P_2^M$ of cardinality λ and a sequence $\langle u_c : c \in Y \rangle$ of pairwise disjoint sets of distinct ordinals such that,
for $c \in Y$, setting

$$u_c = \{F_n^M(c) : n < \omega\},$$

$\langle u_c : c \in Y \rangle$ partitions a subset of the basis (mod atoms) $\langle b_\alpha : \alpha < \lambda \rangle$.

Theorem II.2: Maximal model in $\boldsymbol{K}_{-1}$

Theorem II.2

If for some μ, $\lambda = 2^{\mu}$ and λ is less than the first measurable cardinal then there is a P_0-maximal model M_* in $\boldsymbol{K}_{-1}$ such that

1. $|P_i^{M_*}| = \lambda$ (for $i = 0, 1, 2$),
2. $P_1^{M_*}$ is an atomic Boolean algebra,
3. $\mathrm{uf}(\mathrm{M}_*) = \emptyset$,
4. M_* is nicely free.

Theorem II.2: Maximal model in $\boldsymbol{K}_{-1}$ Construction: 0)

Construct a sequence of models $\langle (M_\epsilon, D_\epsilon, f_\epsilon, : \epsilon \leq \omega + 1\rangle$.
Guarantee at each finite step: M_ϵ is:

1. nearly free (extending previous basis)
2. $D_\epsilon \in \mathrm{uf}(\mathrm{M}_\epsilon)$
3. For last condition recall:

Definition

A Boolean algebra $\mathbb{B} \subset \mathcal{P}(\lambda)$ is Uniformly $\aleph_1$-incomplete if there is a set $\mathcal{A} \subseteq {}^{\omega}\mathbb{B}$ such that:

i) $\mathcal{A}$ is a family of λ countable sequences, each with the finite intersection property.

ii) $\mathbb{B}$ includes the finite subsets of λ; but every non-principal ultrafilter D on λ intersects some $\overline{A} \in \mathcal{A}$ infinitely often.

Theorem II.2: Maximal model in $\boldsymbol{K}_{-1}$ Construction: i)

At stage 1) construct a nearly free Boolean algebra on λ elements and define a P_2^M of cardinality λ and define the $F_n(c)$ to map 1-1 into that basis.

Theorem II.2: Maximal model in $\boldsymbol{K}_{-1}$ Construction: ii)

$\epsilon = \zeta + 1 < \omega$: Given $\mathbb{B}$ and $\mathcal{A}$.

There is a 1-1 function f_ϵ from λ onto $P_{4,1}^{M_\epsilon}$ such that:

1. for every $X \in \mathbb{B}$ (from $\boxplus$) there is a $b = b_X \in P_1^{M_\epsilon}$ such that

$$\{\alpha < \lambda : f_\epsilon(\alpha) \leq_{M_\epsilon} b_X\} = \{\alpha < \lambda : \alpha \in X\};$$

2. for each $\overline{A} = \langle A_n : n < \omega \rangle \in \mathcal{A}$ there is a $c \in P_2^{M_\epsilon}$ such that for each n:

$$A_n = \{\alpha < \lambda : f_\epsilon(\alpha) \leq_{P_1^{M_\epsilon}} F_n^{M_\epsilon}(c)\}.$$

Theorem II.2: A proof technique: 0)

Quotients in Boolean Algebra

For $b, c \notin I$

1. $b \wedge c \in I$ implies b/I and c/I are disjoint.
2. $b \vartriangle c \in I$ implies $b/I = c/I$.
3. $b - c \in I$ implies $b/I \leq c/I$.

Theorem II.2: A proof technique: i)

case 3: $\epsilon = \zeta + 1 < \omega$

The element $b_{\zeta,\alpha}$ is the b_{A_α} from last slide.

1. choose as the new atoms introduced at this stage a set $B_\epsilon \subseteq \mathcal{P}(\lambda)$ with $B_\epsilon \cap M_\zeta = \emptyset$ and $|B_\epsilon| = \lambda$.
2. Let f_ϵ be a one-to-one function from λ onto $B_\epsilon \cup P^{M_\zeta}_{4,1}$.
3. Let $\langle X_\gamma : \gamma < \lambda \rangle$ list the elements of $\mathbb{B}$ from iii) of last slide.

Theorem II.2: A proof technique: ii)

Relevant Quotients

Fix a sequence $\{b_{\zeta,\alpha} : \alpha < \lambda\}$, which are distinct and not in $M_\zeta \cup B_\epsilon$, and let $\mathbb{B}'_\zeta$ be the Boolean Algebra generated freely by

$$P_1^{M_\zeta} \cup \{b_{\zeta,\alpha} : \alpha < \lambda\} \cup \{f_\epsilon(\alpha) : \alpha < \lambda\}.$$

Let I_ζ be the ideal of B'_ζ generated by

1. $\sigma(a_0, \ldots a_m)$ when $\sigma(x_0, \ldots x_m)$ is a Boolean term, $a_0, \ldots a_m \in P_1^{M_\zeta}$ and $P_1^{M_\zeta} \models \sigma(a_0, \ldots a_m) = 0$.
2. $f_\epsilon(\alpha) - b_{\zeta,\gamma}$ when $\alpha \in X_\gamma$ and $\alpha, \gamma < \lambda$.
3. $b_{\zeta,\gamma} \wedge f_\epsilon(\alpha)$ when $\alpha \in \lambda - X_\gamma$ and $\alpha, \gamma < \lambda$.
4. $f_\epsilon(\alpha) - b$ when $\alpha < \lambda$, $f_\epsilon(\alpha) \notin P_{4,1}^{M_\zeta}$ and $b \in D_\zeta$.

Theorem II.2: iii)

Let $P_1^{M_\epsilon} = \mathbb{B}_\epsilon$ be $\mathbb{B}'_\zeta / J_\zeta$ with quotient map, $j_\epsilon(b) = b/J_\zeta$.

1. Condition 1) of Proof method: ii) guarantees M_ϵ is nearly free (Condition of 1) of Construction i).
2. To satisfy Condition i) of Construction ii) choose $b_{X_\gamma} = b_{\zeta,\gamma}$ by conditions 2) and 3) in Proof method: ii).
3. Stage $\epsilon = \omega + 1$. For each fixed $\overline{A} \in \mathcal{A}$,define $F_n^\epsilon(C) = b_{\zeta,\gamma}$ where $X_\gamma = A_n$.

Another one page argument shows $\mathrm{uf}(\mathrm{M}_{\omega+1}) = \emptyset$.

Lecture II: Complete Sentence and the Corrections

Main Result: reprise

Theorem: There is a *complete sentence* ϕ of $L_{\omega_1,\omega}$ such that ϕ has maximal models in a set of cardinals λ that is cofinal in the first measurable μ while ϕ has no maximal models in any $\chi \geq \mu$.

Outline of Argument: reprise

1. $\lambda < \mu_0$ implies there is a BA with witnessed (incompleteness) in λ
2. ($\boldsymbol{K}_{-1}$) There is P_0-maximal witnessed BA in λ.
   1. Characterize P_0-maximal
   2. Find nicely free P_0-maximal model M_*.
3. Find the complete sentence ϕ.
4. Correcting M_* to a model of ϕ: If M modifies M_* so that
   1. goal: $M \in \boldsymbol{K}_1$ (but not $\boldsymbol{K}_1$-free).
   2. Task A: M is rich –existentially complete
   3. Task B: technical step showing $\mathrm{uf(M)} = \emptyset$.

   then $M \models \phi$ and is maximal.

Independence: BA

Definition

1. For $X \subseteq B$ and B a Boolean algebra, $\overline{X} = X_B = \langle X \rangle_B$ be the subalgebra of B generated by X.
2. A set Y is *independent* (or *free*) from X over an ideal $\mathcal{I}$ in a Boolean algebra B if and only if for any Boolean-polynomial $p(v_0, \ldots, v_k)$ (that is not identically 0), and any $a \in \langle X \rangle_B - \mathcal{I}$, and distinct $y_i \in Y$, $p(y_0, \ldots, y_k) \wedge a \notin \mathcal{I}$.

Let π map B to $B/\mathcal{I}$. If 'Y is independent from X over $\mathcal{I}$' then the image of Y is free from the image of X (over $\emptyset$) in $B/\mathcal{I}$.

And conversely.

The closure system of substructure closure gives an independence system but NOT a matroid.

Reprise: K_{-1}

Vocabulary

τ is a vocabulary with unary predicates P_0, P_1, P_2, P_4, binary R, $\wedge, \vee, \leq$ unary functions $^-$, G_1, constants 0,1 and unary functions F_n, for $n < \omega$.

K_{-1}

1. P_1 is the domain of a Boolean algebra
2. In each model $R(x, y)$ defines a Homomorphism from P_1 into the BA of subsets of P_0.
   G_1 is a bijection between $P_{4,1}$ (atoms of P_1) and P_0.
   $R(u, b)$ iff $G_1^{-1}(u) \leq b$.
3. P_2 is a set with no structure but for each n,
   $\{F_n(c) : c \in P_2\}$ is a set of elements of P_1.
   Cofinitely many of them along with P_2 and P_0 generate the model.

$K^1_{<\aleph_0}$

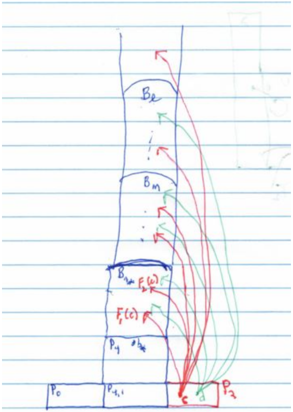
Finitely generated models in K_1

Each $M \in K^1_{<\aleph_0}$

1. is in K_{-1};
2. $P_1^M = \bigcup\{B_n : n \geq n_*\}$ where
   1. each B_n is a finite free Boolean algebra.
   2. B_{n_*} has a maximal element b_* which is the sup of the atoms of P_1^M;
   3. B_{n_*}/P_4^M is free.
3. P_2^M is finite and M is generated by $B_{n_*} \cup \{F_n(c) : c \in P_2^M, n < \omega\}$
4. For each c, $\{F_n(c) : n < \omega\}$ are independent over P_4^M.
5. The set $\{F_m^M(c) : m \geq n_*, c \in P_2^M\}$ (the enumeration is without repetition) is free from B_{n_*} over P_4^M.
6. $B_{n_*} \supsetneq P_4^M$ and $F_m^M(c) \wedge b_* = 0$ for $m \geq n_*$.

K_1 is the 'universal class determined by $K^1_{<\aleph_0}$, the closure under direct limits.

A model in $\boldsymbol{K}^1_{<\aleph_0}$



$\boldsymbol{K}_1$-Free Extension

Definition

When $M_1 \subseteq M_2$ are both in $\boldsymbol{K}_1$, we say M_2 is $\boldsymbol{K}_1$*-free over* M_1 and write $M_1 \subseteq_{fr} M_2$, witnessed by (I, H) when

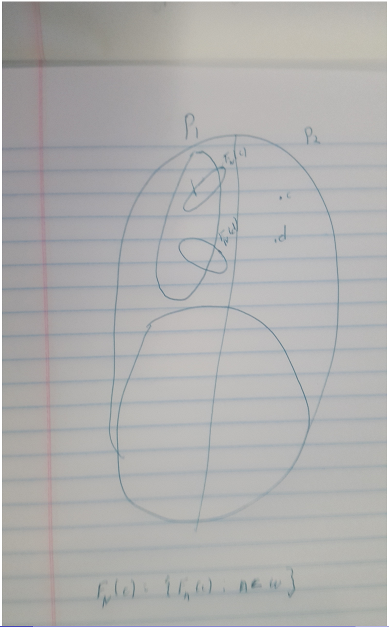
1. $I \subset P_1^{M_2} - (P_1^{M_1} \cup P_4^{M_2})$ such that i) $I \cup P_1^{M_1} \cup P_4^{M_2}$ generates $P_1^{M_2}$ and ii) I is independent from $P_1^{M_1}$ over $P_4^{M_2}$ in $P_1^{M_2}$.
2. There is a function H from $P_2^{M_2} \setminus P_2^{M_1}$ to $\mathbb{N}$ such that the $F_n(c)$ for $n \geq H(c)$ are distinct and

$$\{F_n^M(c) : c \in P_2^{M_2} \setminus P_2^{M_1} \text{ and } n \geq H(c)\} \subset I.$$

M is $\boldsymbol{K}_1$-free over the empty set or simply $\boldsymbol{K}_1$-free if M is a free extension of M_{min}.

All members of $\boldsymbol{K}^1_{<\aleph_0}$ are $\boldsymbol{K}_1$-free.

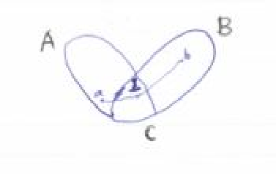
Free Extension Picture



Free Amalgamation of Boolean algebras

Notation

Let $C \subseteq A, B$ be Boolean algebras. The disjoint amalgamation $D = A \otimes_C B$ is characterized internally by the following condition.



For $a \in A - C, b \in B - C$: $a \leq b$ in D if and only if there is a $c \in C$ with $a < c < b$ (and symmetrically). D is generated as a Boolean algebra by $A \cup B$ where A and B are sub-Boolean algebras of D.

Free amalg of finite algebras destroys atoms: (If a is an atom of A and $b_1, \ldots b_n$ are the atoms of B, for at least one i, $A \otimes_C B \models 0 < a \wedge b_i < a$.)

Amalgamation result: $\boldsymbol{K}_1$-free

Theorem

If $B \in \boldsymbol{K}_1$ is a free extension of $A \in \boldsymbol{K}_{<\aleph_0}$ and $C \in \boldsymbol{K}_{<\aleph_0}$ is a free extension of A, there is an amalgam of B and C over A.

There are three key ingredients in the amalgamation proof:

1. N_1 and N_2 must be finitely generated;
2. Secondly, M_1 must be $\boldsymbol{K}_1$-free.
3. Hard part: Ensure that 'atomicity' is preserved in constructing extensions of Boolean algebra so the definitions of P_4 and $P_{4,1}$ are 'absolute' between models.

Amalgamation Proof Outline

Theorem

Suppose $M_1 \in \boldsymbol{K}_1$ is free and $N_1 \subset M_1$. Let $N_1 \subset N_2$ with both in $\boldsymbol{K}^1_{<\aleph_0}$. Choose a new set A in 1-1 correspondence with atoms of $N_2 - N_1$. Then there are an $M_2 \supset M_1$ amalgamating with N_2 over N_1 via g extending $f : A \rightarrowtail P^{N_2}_{4,1} - P^{N_2}_{4,1}$.

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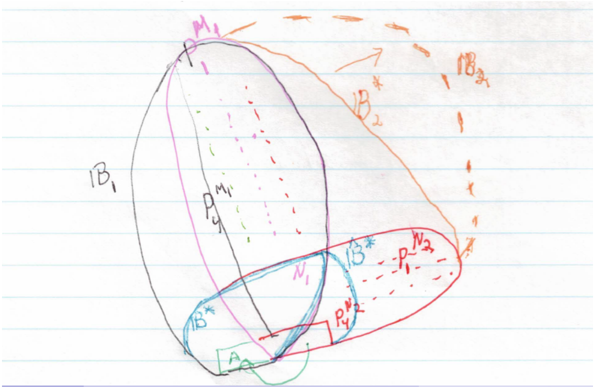
Step 1 construct a Boolean algebra $\mathbb{B}_1$ that is generated by $P^{M_1}_1 \cup A$ and so that the atoms of $\mathbb{B}_1$ are $P^{M_1}_{4,1} \cup A$.

Step 2 Find a sub-Boolean algebra $\mathbb{B}^*$ of $\mathbb{B}_1$ that is a suitable amalgamation base.

Step 3: Construct a Boolean algebra $\mathbb{B}_2$ which is a quotient of the pushout $\mathbb{B}_2'$ of $\mathbb{B}_1$ and $P^{N_2}_1$ over the sub-Boolean algebra $\mathbb{B}^*$ of $\mathbb{B}_1$ generated by $P^{N_1}_1$ and A. Moreover, $\mathbb{B}_2$ contains M_1 and $f(\check{\mathbb{B}}^*)$ and the atoms of $\mathbb{B}_2$ are $P^{\mathbb{B}_1}_{4,1} \cup A$.

Step 4 Check the auxiliary functions work as desired.

Diagram of Amalgamation Proof



The generic model and $\boldsymbol{K}_2$

Corollary 1

There is a countable generic model M for $\boldsymbol{K}_1$.

Moreover M is $\boldsymbol{K}_1$-free.

We denote its Scott sentence by ϕ_M and $\mathrm{mod}(\phi_M)$ by $\boldsymbol{K}_2$.

M is rich ($\boldsymbol{K}^1_{<\aleph_0}$-homogeneous). If $M, N \in \boldsymbol{K}_2$, $M \equiv_{\infty,\omega} N$ so they satisfy Φ_M.

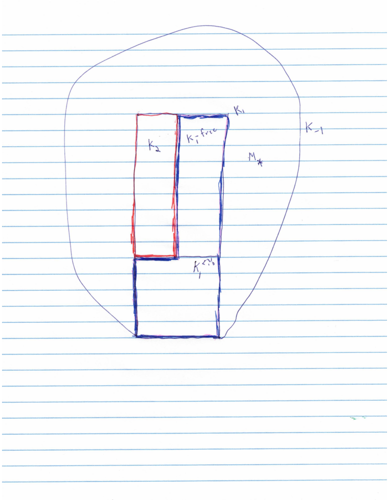
If $M \subset N$ and are both in $\boldsymbol{K}_2$, $M \prec_{\infty,\omega} N$.

Corollary 2

There is a similar complete sentence axiomatizing a class of atomic, nearly free, Boolean algebras.

Correcting M_* to a model in $\boldsymbol{K}_2$

Geography of the proof of the main result



What are the corrections?

1. The domains of the structures constructed in this section are subsets of M_*; the F_n are redefined so the new structures are substructures only of the reduct of M_* to $\tau - \{F_n : n < \omega\}$.
2. In all the M considered here $P_1^M = P_1^{M_*}$ and these Boolean algebras have the same set of ultrafilters.
   However, $\mathrm{uf(M)} \neq \mathrm{uf(M_*)}$ as the definition of uf depends on properties of the F_n.
3. The set $\{F_n^M(c) : c \in P_2^M\}$ is not required to be an independent subset in $\boldsymbol{K}_{-1}$. But it is in $\boldsymbol{K}_1 \subseteq \boldsymbol{K}_2$.
4. The final counterexample is in $\boldsymbol{K}_1$ but is not $\boldsymbol{K}_1$-free.

Fixing Notation

Notation

We define a family of trees of sequences:

1. For $\alpha < \lambda$, let $\mathcal{T}_\alpha = \{\langle\rangle\} \cup \{\alpha\widehat{\ }\eta; \eta \in {}^{<\omega}3\}$ and $\mathcal{T} = \bigcup_{\alpha<\lambda} \mathcal{T}_\alpha$.
2. $\lim(\mathcal{T}_\alpha)$ is the collection of paths through $\mathcal{T}_\alpha$.

Claim

Since M_* is nicely free, without loss of generality, we may assume:

1. The universe of M_* is λ and the 0 of $P_1^{M_*}$ is the ordinal 0.
2. We can choose sequences of elements of $P_1^{M_*}$, $\mathbf{b} = \langle b_\eta : \eta \in \mathcal{T}\rangle$ so that their images in the natural projection of $P_1^{M_*}$ on $P_1^{M_*}/P_4^{M_*}$ freely generate $P_1^{M_*}/P_4^{M_*}$.
3. For every $a \in P_{4,1}^{M_*}$ and the even ordinals $\alpha < \lambda$, there is an n such that for any $\nu \in \mathcal{T}_\alpha$, $\lg(\nu) \geq n$ implies $a \wedge b_\nu = 0$.

$\mathbb{M}_1$ defined

$\mathbb{M}_1$ Defined

Let $\mathcal{M}^1 = \mathcal{M}^1_\lambda$ be the set of $M \in \boldsymbol{K}_{-1}$ such that

1. the universe of M is contained in λ, the universe of M_*,
2. and for $i < 2$, (or $i = 4$ or $(4, 1)$) $P_i^M = P_i^{M_*}$, $M \restriction (P_0^M \cup P_1^M) = M_* \restriction (P_0^{M_*} \cup P_1^{M_*})$
3. while P_2^M will not equal $P_2^{M_*}$.

Two tasks and a goal

Task Satisfaction

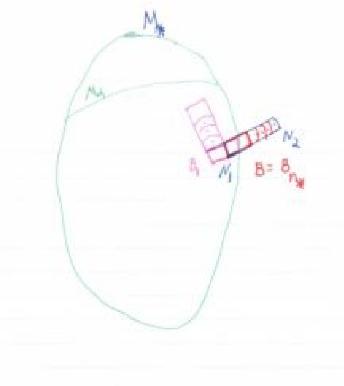
1. **tasks** We say $M \in \mathcal{M}_1$ *satisfies the task* **t** if either:
   - A) $\mathbf{t} = (N_1, N_2) \in \boldsymbol{T}_1$ (so $N_1 \subset M$) and there exists an embedding of N_2 into M over N_1.
   - B) $\mathbf{t} = c$, where $c \in P_2^{M_*}$, is in $\boldsymbol{T}_2$ and
     for every ultrafilter D on P_1^M,
     $(\exists^\infty n)F_n^{M_*}(c) \in D$, implies there is a $d_D \in P_2^M$ such that
     $(\exists^\infty n)F_n^M(d_D) \in D$
2. **goal** $M \in \boldsymbol{K}_1$.

Claim

If $M \in \mathcal{M}_1$ satisfies Task A and the goal then $M \in \boldsymbol{K}_2$. CLEAR
If M satisfies Task B, it is P_0-maximal. BELOW

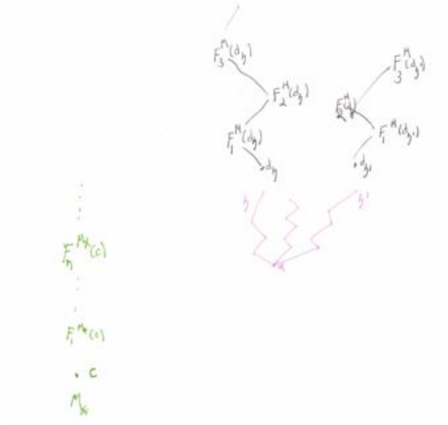
Satisfying Task A: Get Rich

$\mathbf{t} = (N_1, N_2) \in \boldsymbol{T}_1$ (so $N_1 \subset M$) and there exists an embedding of N_2 into M over N_1.



Task B

For each non-principal D such that $S_c^{M_*}(D) = \{n \colon F_n^{M_\alpha}(c) \in D\}$ is infinite,
we construct an $\eta = \eta_D$ and d_η such that $S_n^{M_*}(D) = \{n \colon F_n^M(d_\eta)) \in D\}$ is infinite



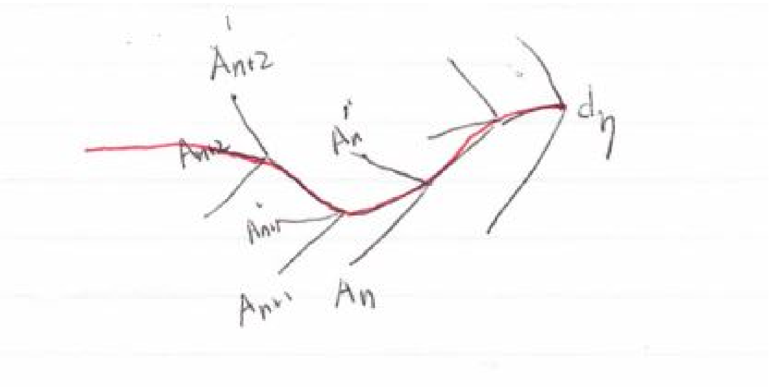
Boolean Algebra Interlude II

Ultrafilters and Boolean Algebras

1. $A \triangle B = A' \triangle B'$
2. Given A_1, A_2, A_3. The intersection of the $A_i \triangle A_j$ ($i \neq j$) is empty. So for any ultrafilter D at least one $A_i \triangle A_j$ is not in D.
3. But, applied to the complements, at least one $A_i \triangle A_j$ is in D.
4. If $a_1, a_2 \ldots$ are independent so are $a_1 \triangle a_2, a_2 \triangle a_3 \ldots$.

Thus there is a pair with both in or both out.

Finding the path: diagram



$$F_n^M(d_\eta) = A_n \triangle A'_n \triangle F_n^{M_*}(c)$$

Finding the path: text

Given an independent sequence $\{A_n : n < \omega\}$. Fix an $\alpha \in P_2^{M_*}$.

Renumber as $b_{\alpha\frown\nu}$ for $\nu \in {}^{<\omega}3$.
α is the $d \in P_2^M$ corresponding to d_η.
Choose $F_n^M(d_\eta)$ inductively.
At stage $n+1$:

1. Fix $i, j < 3$ such that both $b_{\eta\restriction n\frown i} = A_n, b_{\eta\restriction n\frown, j} = A'_n$ are both in or both are out.
2. Let $\eta^D(n+1)$ be the $k < 3$ not used.
3. $F_{n+1}^M(d_\eta) = b_{\eta\restriction n\frown i} \triangle b_{\eta\restriction n\frown j} \triangle F_n^{M_*}(c)$.

Maintaining ‘witnessed’

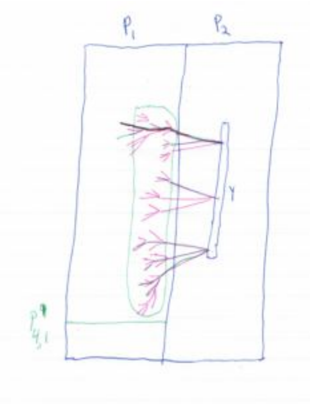
For both tasks we needed to show $(\bigwedge F_n^M(d_\eta)) = \emptyset$.
But this immediate from the following fact.

Boolean algebra: Maintaining ‘witnessed’

If a is an atom, $a \wedge b_0 = 0$ and $a \wedge b_1 = 0$, then $a \wedge (b_0 \bigtriangleup b_1) = 0$.

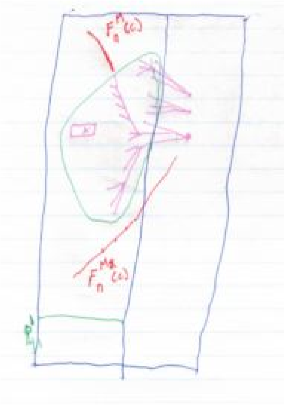
We constructed the $F_n^M(d)$ by taking the symmetric difference of generators of M_*.

Towards the Goal: k_Y and $\mathbb{B}^0_Y$



$\mathbb{B}^0_Y$ is the Boolean algebra generated by the atoms and the complement of the green bubble.
k_Y is the finite bound on length of excluded paths.

toward $\boldsymbol{K}_1$: k_{XY} and $\boldsymbol{F}^{\ell}_{XY}$: diagram



The green bubble and the atoms are the base; As ℓ increases $F_n^M(c_\ell)$ is added to $\boldsymbol{F}$.
Note the construction is independent from the $F_n^{M_*}(c)$'s.

toward $\boldsymbol{K}_1$: k_{XY} and $\boldsymbol{F}^{\ell}_{XY}$: text

The goal is to ensure $\{F_n^M(c_i) : i < |Y|, k_{XY} \leq n < \omega\}$ is independent.
$F_n^M(c_i) = b_{\eta_i \restriction n \frown i} \triangle b_{\eta_i \restriction n \frown j} \triangle F_n^{M_*}(b_\eta \restriction n)$
For this we need one more Boolean interlude.

Boolean Algebra Interlude IV

Independence

Suppose $\mathbb{B}_1 \subseteq \mathbb{B}_2$ are Boolean algebras with $a \in \mathbb{B}_1$, and $b_1 \neq c_1$ are in $\mathbb{B}_2$, and $\{b_1, c_1\}$ is independent over $\mathbb{B}_1$ in $\mathbb{B}_2$.

1. The element $(b_1 \bigtriangleup c_1) \bigtriangleup a \in \mathbb{B}_2$ is independent over $\mathbb{B}_1$.
2. More generally, if $\{b_i, c_i : i < \omega\}$ are independent over $\mathbb{B}_1$, $\{a_i : i < \omega\} \subseteq \mathbb{B}_1$, $e_i = b_i \bigtriangleup c_i \bigtriangleup a_i$,e and $f_i = b_i \bigtriangleup c_i$ then each of $\{e_i : i < \omega\}$ and $\{f_i : i < \omega\}$ are independent over $\mathbb{B}_1$.

Summarising the argument

Theorem

We have 'corrected' M_* to an M which

1. is in $\boldsymbol{K}_1$,
2. satisfies Task A: so in $\boldsymbol{K}_2$,
3. satisfies task B: is P_0-maximal.

Corollary

There is an M' in $\boldsymbol{K}_2$ which is maximal.

Extend M as often you can. Since $|P_1| \leq 2^{|P_0|}$ in at most 2^λ steps you finish.

Further questions

From Paper

Extensions

1. Is there a $\kappa < \mu$, where μ is the first measurable, such that if a complete sentence has a maximal model in cardinality κ, it has maximal models in cardinalities cofinal in μ?
2. Is there a complete sentence that has maximal models cofinally in some κ with $\beth_{\omega_1} < \kappa < \mu$ where μ is the first measurable, but no larger models are maximal. Could the first inaccessible be such a κ?

Further questions

More generally

1. How important are Boolean algebras here?
   Could one use another a different variety of algebras?
   What are the important conditions on the independence relation?
   Could one use the Stone space of another theory?